ON A NEW AND GENERAL METHOD OF INVERTING A LINEAR AND QUATERNION FUNCTION OF A QUATERNION

By

William Rowan Hamilton

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On a New and General Method of Inverting a Linear and Quaternion Function of a Quaternion.

Sir William Rowan Hamilton.

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Let a, b, c, d, e represent any five quaternions, and let the following notations be admitted, at least as temporary ones:—

$$ab - ba = [ab], \quad S[ab]c = (abc);$$

$$(abc) + [cb]\,Sa + [ac]\,Sb + [ba]\,Sc = [abc];$$

$$Sa[bcd] = (abcd);$$

then it is easily seen that

$$[ab] = -[ba]; \quad (abc) = -(bac) = (bca) = \&\text{c.};$$

$$[abc] = -[bac] = [bca] + \&\text{c.};$$

$$(abcd) = -(bacd) = (bcad) = \&\text{c.};$$

$$0 = [aa] = (aac) = [aac] = (aacd), \ \&\text{c.}$$

We have then these two Lemmas respecting Quaternions, which answer to two of the most continually occurring transformations of vector expressions:—

$$\begin{aligned} \text{I.}\ldots \quad & 0 = a(bcde) + b(cdea) + c(deab) + d(eabc) + e(abcd), \\ \text{or I}'\ldots \quad & e(abcd) = a(ebcd) + b(aecd) + c(abed) + d(abce); \\ \text{and II.}\ldots \quad & e(abcd) = [bcd]\,Sae - [cda]\,Sbe + [dab]\,Sce - [abc]\,Sde; \end{aligned}$$

as may be proved in various ways.

Assuming therefore *any four* quaternions a, b, c, d, which are *not* connected by the relation,

$$(abcd) = 0,$$

we can *deduce* from them four others, a', b', c', d', by the expressions,

$$a'(abcd) = f[bcd], \quad b'[abcd] = -f[cda], \ \&\text{c.},$$

where f is used as the characteristic of a linear or *distributive quaternion function* of a quaternion, of which the form is supposed to be given; and thus the *general form* of *such* a function comes to be represented by the expression,

$$\text{V} \ldots \quad r = fq = a'\,Saq + b'\,Sbq + c'\,Scq + d'\,Sdq;$$

involving *sixteen scalar constants*, namely those contained in $a'\,b'\,c'\,d'$.

The *Problem* is to *invert* this *function* f; and the *solution* of that problem is easily found, with the help of the new Lemmas I. and II., to be the following:—

$$\begin{aligned}\text{VI} \ldots \quad q(abcd)(a'b'c'd') &= (abcd)(a'b'c'd')f^{-1}r \\ &= [bcd](rb'c'd') + [cda](rc'd'a') + [dab](rd'a'b') + [abc](ra'b'c');\end{aligned}$$

of which solution the correctness can be verified, *à posteriori*, with the help of the same Lemmas.

Although the foregoing problem of *Inversion* had been *virtually* resolved by Sir W. R. H. many years ago, through a reduction of it to the corresponding problem respecting *vectors*, yet he hopes that, as regards the Calculus of *Quaternions*, the new solution will be considered to be an important step. He is, however, in possession of a general *method* for treating questions of this class, on which he may perhaps offer some remarks at the next meeting of the Academy.