ON SOME QUATERNION EQUATIONS CONNECTED WITH FRESNEL'S WAVE SURFACE FOR BIAXAL CRYSTALS

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1. The ellipsoid of which the three semi-axes are usually denoted as a, b, c, in statements of the Fresnelian theory of the wave-surface in a biaxal crystal, being here represented by the equation,

$$S\rho\phi\rho = 1,$$

where the vector function ϕ has the distributive and other properties described by Sir W. R. H., in his Seventh Lecture on Quaternions, it follows from the physical principles, or hypotheses, of Fresnel, that a small displacement, $\delta\rho$, of a molecule of the ether in a crystal, gives rise to an elastic force, which may be denoted by $\phi^{-1}\,\delta\rho$. But if this displacement, $\delta\rho$, be (as is assumed) tangential to a wave-front in the medium, to which the vector μ is normal, and of which the tensor $T\mu$ denotes the slowness of propagation, so that μ may be called the INDEX-VECTOR, then the tangential component of the elastic force must admit of being represented by $\mu^{-2}\,\delta\rho$. Hence the normal component of the same force (supposed by Fresnel to be destroyed by the incompressibility of the ether) must admit of being denoted by the symbol,

$$(\phi^{-1} - \mu^{-2})\,\delta\rho;$$

which symbol must, therefore, admit of being equated to a vector of the form $\mu^{-1}\,\delta m$, δm being a small scalar. We are, therefore, at liberty to write the following symbolical expression for the displacement supposed by Fresnel to exist:

$$\delta\rho = (\phi^{-1} - \mu^{-2})^{-1}\mu^{-1}\,\delta m.$$

But it has been supposed that the displacement $\delta\rho$ is tangential to the wave, or perpendicular to μ; if therefore we write

$$\tau\,\delta m = \mu^{-1}\,\delta\rho, \quad \text{or} \quad \tau = \mu^{-1}(\phi^{-1} - \mu^{-2})^{-1}\mu^{-1},$$

then τ is at least a *vector*, *even* on the principles of Fresnel: while, on those of Mac Cullagh and of Neumann, it would have the direction of the *true* displacement, or vibration, within the crystal. And thus, *without any labour of calculation*, but simply by the *expressing* of the fundamental *conceptions* of Fresnel's theory in the LANGUAGE of Quaternions, Sir W. R. H. obtains an *Equation of the Index-surface*, under the following SYMBOLICAL FORM:—

$$0 = S\mu^{-1}(\phi^{-1} - \mu^{-2})^{-1}\mu^{-1}; \tag{a}$$

which is easily transformed into the following:—

$$1 = S\mu(\mu^2 - \phi)^{-1}\mu. \tag{a$'$}$$

He has also verified, that when he writes,

$$\phi = \alpha^{-1}S\,.\,\alpha^{-1} + \beta^{-1}S\,.\,\beta^{-1} + \gamma^{-1}S\,.\,\gamma^{-1},$$

α, β, γ, being three rectangular vectors, whereof the lengths are a, b, c, an easy quaternion *translation* enables him to pass from these last forms to certain others, although less concise ones, for the equation of the index surface, expressed in rectangular co-ordinates; one, at least, of which latter forms (he believes) was assigned by Fresnel himself.

2. To pass next to the *Equation of the Wave-surface*, let ρ be the vector of that surface; or the vector of Ray-velocity; or simply, the RAY-VECTOR. It is connected with the index vector μ (if this last vector be supposed to be measured in the direction of wave-propagation *itself*, and *not* in the *opposite* direction,) by the relations,

$$S\mu\rho = -1, \quad S\rho\,\delta\mu = 0;$$

with which may be combined their easy consequence,

$$S\mu\,\delta\rho = 0,$$

which assists to express the *reciprocity* of the two surfaces. Hence, by some *very unlaborious* (although, perhaps, *not obvious*) processes, depending on the published principles of the Quaternions, and especially on those of the Seventh Lecture, but in which it is found convenient to introduce an *auxiliary vector*,

$$\nu = (\mu^2 - \phi)^{-1}\mu,$$

(which may be considered to have both geometrical and physical significations,) Sir W. R. H. infers that ν is perpendicular to ρ; and also that it may be thus expressed as a function thereof:—

$$\nu = (\phi - \rho^{-2})^{-1}\rho^{-1}.$$

An immediate result is, that the "Equation of the Wave" may be *symbolically expressed* as follows:—

$$0 = S\rho^{-1}(\phi - \rho^{-2})\rho^{-1}; \tag{b}$$

or, by an easy transformation,—

$$1 = S\rho(\rho^2 - \phi^{-1})^{-1}\rho. \tag{b$'$}$$

Of these formulæ, likewise, the agreement with known results (including one of his own) has been verified by Sir W. R. H., who has also found that it is as easy to *return*, in the quaternion calculations, from the wave to the index-surface, as it had been to *pass* from the latter to the former: the only difference worth mentioning between the two processes being this, that when we interchange μ and ρ, in any one of the formulæ, we are at the same time to change the *symbol of operation* to the *inverse operational symbol*, ϕ^{-1}.

3. From the expression (b), by the introduction of two auxiliary and constant vectors, ι, κ, such that (as in the Lecture above cited) the following identity holds good:—

$$S\rho\phi\rho = \left(\frac{T(\iota\rho + \rho\kappa)}{\kappa^2 - \iota^2}\right)^2,$$

Sir W. R. H. has lately succeeded in deducing, in a new way, a less symbolical, but more developed, *quaternion form* for the Equation of the Wave, which he communicated in 1849 to a few scientific friends, and which he wishes to be allowed to put on record here: namely the equation,

$$(\kappa^2 - \iota^2)^2 = \{S(\iota - \kappa)\rho\}^2 + (TV\iota\rho \pm TV\kappa\rho)^2; \tag{c}$$

which exhibits the *physical property* of the two vectors, ι, κ, as *lines of single ray-velocity*; and is also adapted to *express*, and even to *suggest*, certain *conical cusps* and *circular ridges* on the Biaxal Wave, discussed many years ago.

In the course of a recent correspondence, on the subject of the quaternions, with Peter G. Tait, Esq., Professor of Mathematics in the Queen's College, Belfast, Sir W. R. Hamilton has learned that Professor Tait has independently arrived at this last form (c) of the Equation of Fresnel's Wave; and he hopes that the *method* employed by Mr. Tait will soon be, through some channel, made public. In the meantime he desires to add, for himself, that he is not to be understood as here offering any *opinion* of his own on the rival merits of any *physical hypotheses* which have been proposed respecting the *directions* of the *vibrations* in a crystal, or other things therewith connected; but merely as *applying* the Calculus of Quaternions, considered as a Mathematical Organ, to the *statement* and *combination* of a few of those hypotheses, especially as bearing on the Wave.

ON CERTAIN EQUATIONS IN QUATERNIONS, CONNECTED WITH THE THEORY OF FRESNEL'S WAVE SURFACE.

[Monday, May 9, 1859.]

If $S\rho\phi\rho = 1$ be the *equation of an ellipsoid* (or, indeed, of any other central surface of the second order), then the *identity*,

$$\rho^{-1}V\rho\phi\rho = \phi\rho - \rho^{-1} = (\phi - \rho^{-2})\rho,$$

proves that the vector $\sigma = \phi\rho - \rho^{-1}$, is perpendicular at once to ρ and to $V\rho\phi\rho$. But $V\rho\phi$ has the direction of a line tangent to the surface, which is also perpendicular to the semidiameter ρ, because $\phi\rho$ has the direction of the normal to the surface, at the end of that semidiameter. Hence σ is *normal* to the *plane* of the *section*, whereof ρ is (not merely a *semidiameter*, but) a *semiaxis*; the other semiaxis having the direction of $V\rho\phi\rho$. But $\rho = (\phi - \rho^{-2})^{-1}\sigma$; ρ and $\perp \sigma$;

$$\therefore \quad 0 = S\sigma(\phi - \rho^{-2})^{-1}\sigma; \tag{1}$$

and this last formula, which (when developed either by the $\alpha\,\beta\,\gamma$ or by the $\iota\,\kappa$ form of ϕ), is found to lead to a *quadratic equation*, relatively to ρ^2 (or to $T\rho^2$), must, therefore, give, in general, the *two* scalar values of the *square* of a *semiaxis* of the *section* perpendicular to σ, when the *direction* of this normal σ, or of the plane itself, is *given.*

Suppose now that the normal σ is erected *at the centre* of the ellipsoid, and that its *length* is made equal to the length of one of the semiaxes ρ of the section, we shall have, of course, $T\sigma = T\rho$, and we may write

$$0 = S\sigma(\phi - \sigma^{-2})^{-1}\sigma, \tag{2}$$

as the equation of the *locus* of the extremity of σ: that is, according to Fresnel, of the *wave surface.* But this is just the form (b), when we write ρ for σ.