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A NOTE OF THE TORELLI SPACES OF NON-ORIENTABLE COMPACT KLEIN SURFACES

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Abstract. The Torelli group of a compact surface consists of the homotopy classes of homeomorphisms that act like the identity on the first homology group of the surface. In this note we show that the Torelli group of a non-orientable compact surface acts without fixed points on the Teichmüller space of the surface.

1. Statement of results

The purpose of this paper is to extend the construction of Torelli spaces of compact Riemann surfaces to the case of compact non-orientable surfaces. We will show, by topological methods, that if a diffeomorphism of a Klein surface acts trivially in the first homology group, then it is homotopic to the identity. This allows us to define the Torelli spaces of Klein surfaces, and show that they are smooth real submanifolds of the Torelli spaces of compact Riemann surfaces.

We give a more precise statement of our results. Let Σ be a smooth compact non-orientable surface. The Teichmüller space $T(\Sigma)$ of Σ is defined as $T(\Sigma)=\mathscr{M}(\Sigma)/\operatorname{Diff}_0(\Sigma)$, where $\mathscr{M}(\Sigma)$ is the set of Klein surface structures on Σ that agree with the given smooth structure, and $\operatorname{Diff}_0(\Sigma)$ is the group of diffeomorphisms of Σ homotopic to the identity [7, p. 145]. The modular or mapping class group, $\operatorname{Mod}(\Sigma)=\operatorname{Diff}(\Sigma)/\operatorname{Diff}_0(\Sigma)$, acts on $T(\Sigma)$ by pull-back of structures. We define the Torelli group $U(\Sigma)$ as the subgroup of $\operatorname{Mod}(\Sigma)$ consisting of the mapping classes that act like the identity on $\mathrm{H}_1(\Sigma,\mathbf{Z})$. The parallel result to the following theorem is a classical fact in Riemann surfaces theory.

Theorem 3.2. *Let Σ be a smooth compact non-orientable surface of arithmetic genus $g\geq 2$. Let $f\in U(\Sigma)$, and suppose that there exists a Klein surface structure S on Σ such that $f\colon(\Sigma,S)\to(\Sigma,S)$ is dianalytic. Then $f=\mathrm{id}$. Therefore, the Torelli group $U(\Sigma)$ acts fixed-points free on $T(\Sigma)$, and the Torelli space $\operatorname{Tor}(\Sigma)=T(\Sigma)/U(\Sigma)$ is a smooth real manifold of dimension $3g-3$.*

Given a non-orientable surface Σ with a Klein surface structure S, there exists an unramified double cover (Σ^c,X), where Σ^c is an orientable surface of genus g, and X is a Riemann surface structure on Σ^c. The surface Σ^c has

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an involution σ, which is anti-holomorphic on the structure X, such that Σ is isomorphic to $\Sigma^c/\langle\sigma\rangle$. The mapping σ induces in a natural way involutions σ^* and $\tilde{\sigma}$ on the Teichmüller space $T(\Sigma^c)$ and the Torelli space $T(\Sigma^c)$, respectively. It is a classical result that $T(\Sigma)$ can be identified with the set of fixed points of σ^* on $T(\Sigma^c)$.

Proposition 3.4. *The Torelli space* $\mathrm{Tor}(\Sigma)$ *can be identified with the set of fixed points of* $\tilde{\sigma}$ *on* $\mathrm{Tor}(\Sigma^c)$.

2. Some general facts about Riemann and Klein surfaces

A *Klein surface* (or *dianalytic*) structure S on a surface without boundary Σ is given by a covering of Σ by open sets $\{U_i\}_i$, and a collection of homeomorphisms $z_i\colon U_i \to V_i$, where V_i are open subsets of $\mathbf{C}$, such that if $U_i \cap U_j \neq \emptyset$, then the mapping $z_i \circ z_j^{-1}$ is holomorphic or anti-holomorphic (conjugate of a holomorphic function) [1]. Klein surfaces are the generalisation of Riemann surfaces to the non-orientable case. We will write (Σ, S) or S for a Klein surface, depending on the context.

A non-orientable surface Σ is homeomorphic to the connected sum of $p \geq 1$ real projective planes [2]. The integer p is called the topological genus of Σ. However, we will use the *arithmetic genus*, which is defined as $g = p - 1$ [7]. If $g = 2n$, the fundamental group of Σ has a presentation given by generators $c, a_1, \ldots, a_n, b_1, \ldots, b_n$, satisfying the relation $c^2 \prod_{j=1}^n [a_j, b_j] = 1$, where $[a, b] = aba^{-1}b^{-1}$. If the genus of Σ is odd, $g = 2n + 1$, we can choose generators of the fundamental group $c, d, a_1, \ldots, a_n, \ldots, b_1, \ldots, b_n$, that satisfy $c^2 d^2 \prod_{j=1}^n [a_j, b_j] = 1$.

Throughout this paper, all surfaces are assumed to be compact without boundary, of genus $g \geq 2$.

Given a compact non-orientable surface Σ of genus g, there exists a compact orientable surface Σ^c, of genus g, and a double covering map: $\pi\colon \Sigma^c \to \Sigma$ [2]. If Σ has a Klein surface structure S, then it is possible to give a Riemann surface structure to X. Moreover, there exist local coordinates z and w, on Σ^c and Σ, respectively, such that the mapping $w \circ \pi \circ z^{-1}$ is holomorphic. The pair (Σ^c, X) together with π is called the *complex double* [1] of (Σ, S). By an abuse of notation, we will refer to Σ^c as the complex double, when we are only interested in the topological aspects.

Let Y denote a compact orientable smooth surface, with a fixed orientation. The Teichmüller space $T(Y)$ is defined as $T(Y) = \mathscr{M}(Y)/\operatorname{Diff}_0(Y)$. Here $\mathscr{M}(Y)$ is the set of Riemann surface structures on M that agree with the given orientation and smooth structure [7]. The group $\operatorname{Diff}_0(Y)$ consists of the diffeomorphisms of Y that are homotopic to the identity. The *modular group* $\operatorname{Mod}(Y) = \operatorname{Diff}^+(Y)/\operatorname{Diff}_0(Y)$, consisting of homotopy classes of orientation preserving diffeomorphisms of Y, acts on $T(Y)$ by pull-back: if $[f] \in \operatorname{Mod}(Y)$ and $[X] \in T(Y)$, then $[f]^*([X])$ is defined as the class $[f^*(X)]$ of the Riemann surface structure that makes $f\colon \bigl(Y, f^*(X)\bigr) \to (Y, X)$ biholomorphic. It is a well-known result that $\mathrm{Mod}(Y)$ acts on $T(Y)$ with fixed points, corresponding to surfaces with automorphisms. A subgroup G of $\mathrm{Mod}(Y)$ is said to have the *Hurwitz–Serre* property [5] if for any element $[g] \in G$, such that there exists a point $[X]$ in $T(Y)$ for which $g\colon (Y, X) \to (Y, X)$ is biholomorphic, we have that $[g] = [\mathrm{id}]$. A group with this property acts fixed-points free on $T(Y)$. The *Torelli group* $U(Y)$ consists of those elements of $\mathrm{Mod}(Y)$ that act trivially on $\mathrm{H}_1(Y, \mathbf{Z})$; it is a classical fact that $U(Y)$ satisfies the Hurwitz–Serre property [3]. The quotient space $\mathrm{Tor}(Y) := T(Y)/U(Y)$ is called the Torelli space of Y.

Teichmüller spaces and modular groups of non-orientable surfaces are defined in a similar way, removing all the conditions that involve the orientability of the surface, and substituting Riemann surface structures by Klein surfaces.

3. Torelli groups of non-orientable surfaces

In this section, we first prove that the Torelli groups of compact non-orientable surfaces have the Hurwitz–Serre property. We do this by lifting homeomorphisms from the surface to its complex double, and then using the fact that the Torelli groups of Riemann surfaces satisfy the above mentioned property. We use this fact to show that the Torelli spaces of Klein surfaces are smooth manifolds, which can be embedded into the Torelli spaces of the complex doubles, in a situation similar to what happens between Teichmüller spaces.

Theorem 3.1. *Let Σ be a compact non-orientable surface, and let f be a homeomorphism of Σ. Assume that f acts trivially on $\mathrm{H}_1(\Sigma, \mathbf{Z})$. Let $\tilde{f}$ be the unique orientation lift of f to the complex double Σ^c. Then $\tilde{f}$ acts trivially on $\mathrm{H}_1(\Sigma^c, \mathbf{Z})$.*

Proof. Our proof will be divided in two cases, depending on whether the genus of Σ is even or odd; we will provide full details in the first case, and only a sketch of the second situation.

Let $f\colon \Sigma \to \Sigma$ be a homeomorphism, and let $\tilde{f}$ the the unique orientation preserving lift of f to the complex double, so that the following diagram is commutative [6]:

$$\begin{tikzcd} \Sigma^c \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & \Sigma^c \arrow[d, "\pi"] \\ \Sigma \arrow[r, "f"] & \Sigma. \end{tikzcd}$$

Let $f_\#$ and $\tilde{f}_\#$ denote the corresponding mappings induced by f and $\tilde{f}$ on the first homology groups of Σ and Σ^c respectively. By hypothesis we have that $f_\# = \mathrm{id}$, and we want to show that $\tilde{f}_\# = \mathrm{id}$.

We start by recalling how the surface Σ^c is constructed, from a topological point of view; the reader can find more details in [2]. Assume that Σ has genus $g = 2n$. By the presentation of the fundamental group of Σ given in Section 2, we can view this surface as a $(2n+2)$-polygon, with the sides identified by the relation of the fundamental group. The surface Σ^c is given by two polygons, with boundary relations:

$$c_1 c_2 \prod_{j=1}^{n} [a_{j,1}, b_{j,1}] = 1 \qquad \text{and} \qquad c_2 c_1 \prod_{j=1}^{n} [a_{j,2}, b_{j,2}] = 1.$$

To obtain a single relation, we find the value of c_2 on the right-hand side equation and substitute it on the left-hand side one (equivalently, we glue the polygons by the c_2 sides):

$$c_2 = \left(\prod_{j=1}^{n} [b_{n+1-j,2}, a_{n+1-j,2}] \right) c_1^{-1};$$

so

$$\begin{aligned} c_1 &\left(\prod_{j=1}^{n} [b_{n+1-j,2}, a_{n+1-j,2}] \right) c_1^{-1} \left(\prod_{j=1}^{n} [a_{j,1}, b_{j,2}] \right) \\ &= \left(\prod_{j=1}^{n} [c_1 b_{n+1-j,2} c_1^{-1}, c_1 a_{n+1-j,2} c_1^{-1}] \right) \left(\prod_{j=1}^{n} [a_{j,1}, b_{j,2}] \right) = 1. \end{aligned}$$

The above expression shows that Σ^c is a compact surface of genus g. We can take the following paths as generators of the fundamental group of Σ^c:

$$\begin{aligned} &\alpha_1 = c_1 b_{n,2} c_1^{-1}, \ldots, \alpha_n = c_1 b_{1,2} c_1^{-1}, \quad \alpha_{n+1} = a_{1,1}, \ldots, \alpha_{2n} = a_{n,1}, \\ &\beta_1 = c_1 a_{n,2} c_1^{-1}, \ldots, \beta_n = c_1 a_{1,2} c_1^{-1}, \quad \beta_{n+1} = b_{1,1}, \ldots, \beta_{2n} = b_{n,1}. \end{aligned}$$

These loops satisfy $\prod_{j=1}^{n} [\alpha_j, \beta_j] = 1$. Let $\mathscr{B}$ and $\mathscr{B}^c$ denote the basis in homology induced by the above two sets of generators of the fundamental groups of Σ and Σ^c, respectively. By an abuse of notation, we will use the same letters for paths of the fundamental groups and the corresponding classes in homology. It is not difficult to see that $\mathscr{B}^c$ is a symplectic basis of $\mathrm{H}_1(\Sigma^c, \mathbf{Z})$; that is, its intersection matrix is

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the identity matrix of order g. The covering map π has the following associated matrix for its action on homology, with respect to the two bases given above, ordered as $\mathscr{B} = \{\{c\}, \{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}\}$ and $\mathscr{B}^c = \{\{\alpha_1, \ldots, \alpha_n\}, \{\alpha_{n+1}, \ldots, \alpha_{2n}\}, \{\beta_1, \ldots, \beta_n\}, \{\beta_{n+1}, \ldots, \beta_{2n}\}\}$:

$$\pi_{\#} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & K & 0 \\ K & 0 & 0 & I \end{pmatrix}$$

where K is the matrix

$$K = \begin{pmatrix} 0 & \dots & \dots & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

The symmetry σ maps $a_{j,1}$ (respectively $b_{j,1}$) to $a_{j,2}$ (respectively $b_{j,2}$); therefore, its action on $\mathrm{H}_1(\Sigma^c, \mathbf{Z})$ is given by the matrix K (of order $2g$). Let $\tilde{f}_\# = (A_{ij})_{i,j=1}^4$ be the matrix associated to the mapping $\tilde{f}$. Then $\tilde{f}_\#$ must satisfy

$$\text{(3.1)} \qquad \text{(a)}\ \pi_\# \tilde{f}_\# = \pi_\#, \qquad \text{(b)}\ \tilde{f}_\# \sigma_\# = \sigma_\# \tilde{f}_\#, \qquad \text{(c)}\ \tilde{f}_\#^t J \tilde{f}_\# = J.$$

Equation (a) of (3.1) is due to the fact that f acts trivially on $\mathrm{H}_1(\Sigma, \mathbf{Z})$. The uniqueness of the orientation-preserving lift gives us (b) above. In (c), which is true for any orientation-preserving homeomorphism of Σ^c [4, Theorem N13, p. 178], $\tilde{f}_\#^t$ denotes the transpose matrix. Equation (3.1a) is equivalent to the following:

$$\begin{cases} A_{21} + KA_{31} = 0 & KA_{11} + A_{41} = K \\ A_{22} + KA_{32} = \mathrm{I} & KA_{12} + A_{42} = 0 \\ A_{23} + KA_{33} = K & KA_{13} + A_{43} = 0 \\ A_{24} + KA_{34} = 0 & KA_{14} + A_{44} = \mathrm{I}. \end{cases}$$

Therefore, we have that the matrix $\tilde{f}_\#$ is given by

$$\tilde{f}_\# = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ -KA_{21} & K - KA_{22} & I - KA_{23} & -KA_{24} \\ K - KA_{11} & -KA_{12} & -KA_{13} & I - KA_{14} \end{pmatrix}.$$

From (3.1)(b) we obtain:

$$\text{(3.2)} \qquad \begin{cases} A_{14}K = K(K - KA_{11}) & A_{24}K = K(-KA_{21}) \\ A_{13}K = K(-KA_{12}) & A_{23}K = K(K - KA_{22}) \\ A_{12}K = K(KA_{13}) & A_{22}K = K(\mathrm{I} - KA_{23}) \\ A_{11}K = K(\mathrm{I} - KA_{14}) & A_{21}K = K(-KA_{24}). \end{cases}$$

Looking at the first row of the equation (c) of (3.1) we get the following identities, after using (3.2) to simplify the result:

$$\begin{cases} A_{21}^t K - KA_{21} = 0 \\ A_{11}^t K - KA_{22} = 0 \\ -2I + A_{11}^t + KA_{22}K = 0 \end{cases}$$

Solving these equations we obtain $A_{21} = 0$ and $A_{11} = I$, which imply $A_{24} = A_{14} = 0$. By a similar set of equations given by the second row of the relation (c) of (3.1) we get $A_{22} = I$ and $A_{12} = 0$. By (3.2) we have $A_{23} = A_{13} = 0$, which shows that $\widetilde{f}_{\#}$ is the identity matrix.

For the case of odd genus $g = 2n+1$, we will use the presentation of the fundamental group of Σ given in Section 2. The complex double Σ^c is homeomorphic to two polygons with the following boundary relations:

$$c_1c_2d_1d_2\prod_{j=1}^{n}[a_{j,1}, b_{j,2}] = 1 \qquad \text{and} \qquad c_2c_1d_2d_1\prod_{j=1}^{n}[a_{j,2}, b_{j,1}] = 1.$$

Calculating as in the previous situation, we have

$$d_2 = c_1^{-1}c_2^{-1}\left(\prod_{j=1}^{n}[b_{n+1-j,2}, a_{n+1-j,2}]\right)d_1^{-1},$$

which can be substituted into the first boundary equation of Σ^c to get

$$\begin{aligned} &c_1c_2c_1^{-1}c_2^{-1}\left(\prod_{j=1}^{n}[b_{n+1-j,2}, a_{n+1-j,2}]\right)\left(\prod_{j=1}^{n}[a_{j,1}, b_{j,1}]\right) \\ &= c_1c_2d_1c_1^{-1}c_2^{-1}d_1^{-1}\left(\prod_{j=1}^{n}[d_1b_{n+1-j,2}d_1^{-1}, d_1a_{n+1-j,2}d_1^{-1}]\right)\left(\prod_{j=1}^{n}[a_{j,1}, b_{j,1}]\right) = 1. \end{aligned}$$

We therefore obtain the following set of generators of the fundamental group of Σ^c:

$$\alpha_1 = c_1d_1^{-1},\ \alpha_2 = d_1b_{n,2}d_1^{-1}, \dots, \alpha_{n+1} = d_1b_{1,2}d_1^{-1},\ \alpha_{n+2} = a_{1,1}, \dots, \alpha_{2n+1} = a_{n,1},$$

$$\beta_1 = d_1c_2,\ \beta_2 = d_1a_{n,2}d_1^{-1}, \dots, \beta_{n+1} = d_1a_{1,2}d_1^{-1},\ \beta_{n+2} = b_{1,1}, \dots, \beta_{2n+1} = b_{n,1},$$

which satisfy the relation $\prod_{j=1}^{2n+1}[\alpha_j, \beta_j] = 1$. Although the basis $\{\alpha_1, \dots, \alpha_{2n+1}, \beta_1, \dots, \beta_{2n+1}\}$ is symplectic, computations are easier if we arrange the generators in the following way $\mathscr{B}^c = \{\alpha_1, \beta_1, \alpha_2, \dots, \alpha_{2n+1}, \beta_2, \dots, \beta_{2n+1}\}$, whose intersection matrix is:

$$\begin{pmatrix} N & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 & 0 \\ 0 & -I & 0 & 0 & 0 \end{pmatrix}.$$

Here

$$N = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

With respect to the basis $\mathscr{B} = \{\{c,d\},\{a_1,\ldots,a_n\},\{b_1,\ldots,b_n\}\}$ and $\mathscr{B}^c = \{\{\alpha_1,\beta_1\},\{\alpha_2,\ldots,\alpha_{n+1}\},\{\alpha_{n+2},\ldots,\alpha_{2n+1}\},\{\beta_2,\ldots,\beta_{n+1}\},\{\beta_{n+2},\ldots,\beta_{2n+1}\}\}$, we have that the action of σ on the first homology group is given by

$$\sigma_{\#} = \begin{pmatrix} M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & O & K \\ 0 & 0 & 0 & K & 0 \\ 0 & 0 & K & 0 & 0 \\ 0 & K & 0 & 0 & 0 \end{pmatrix},$$

where

$$M = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}.$$

To see how M is obtained, observe that in homology we have $\alpha_1 = c_1 - d_1$ and $\beta_1 = d_1 + c_2$. Therefore, $\sigma(\alpha_1) = c_2 - d_2$ and $\sigma(\beta_1) = d_2 + c_1$. Substituting the value of d_2 obtained previously we get the matrix M. Similarly, we have that the covering map $\pi\colon \Sigma^c \to \Sigma$ has the following associated matrix for its action on homology:

$$\pi_{\#} = \begin{pmatrix} L & 0 & 0 & 0 & 0 \\ 0 & 0 & I & K & 0 \\ 0 & K & 0 & 0 & I \end{pmatrix},$$

where

$$L = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The rest of the proof is similar to the even genus case. □

A mapping $f\colon (\Sigma_1, S_1) \to (\Sigma_2, S_2)$ between Klein surfaces is called *dianalytic* if there exist local coordinates z_j in Σ_j, $j = 1,2$, such that the mapping $z_2 \circ f \circ z_1^{-1}$ is holomorphic or anti-holomorphic. The above computations show that the Torelli group $U(\Sigma)$ of a compact non-orientable Klein surface satisfies the Hurwitz–Serre property.

Theorem 3.2. *Let Σ be a compact non-orientable surface of genus $g \geq 2$. Let $[f] \in U(\Sigma)$, and suppose that there exists a Klein surface structure S on Σ such that $f\colon (\Sigma, S) \to (\Sigma, S)$ is dianalytic. Then $f = \mathrm{id}$.*

Corollary 3.3. *The Torelli space $\mathrm{Tor}(\Sigma) = T(\Sigma)/U(\Sigma)$ is a smooth real manifold of dimension $3g-3$.*

Proof of the theorem. Since f is dianalytic on the Klein surface (Σ, S), the orientation preserving lift $\tilde{f}$ is biholomorphic on the Riemann surface (Σ^c, X). By the previous result we have that $\tilde{f}$ acts like the identity on $\mathrm{H}_1(\Sigma^c, \mathrm{Z})$. But then, by the classical theory of Riemann surfaces [3], we have that $\tilde{f} = \mathrm{id}_{\Sigma^c}$, which implies that $f = \mathrm{id}_{\Sigma}$. □

The involution σ induces in a natural way a symmetry σ^* on the Teichmüller space $T(\Sigma^c)$. It is clear that σ^* descends to a symmetry $\tilde{\sigma}$ of $\mathrm{Tor}(\Sigma^c)$.

Proposition 3.4. *The Torelli space* $\mathrm{Tor}(\Sigma)$ *can be identified with the set of fixed points of* $\tilde{\sigma}$ *in* $\mathrm{Tor}(\Sigma^c)$.

Proof. The proof follows immediately from the definition of Torelli spaces. In fact, we have that two points S_1 and S_2 of $\mathscr{M}(\Sigma)$ project to the same point in $\mathrm{Tor}(\Sigma)$ *if and only if* there exists a dianalytic mapping $h\colon (\Sigma, S_1) \to (\Sigma, S_2)$ such that $h_{\#}\colon \mathrm{H}_1(\Sigma, \mathbf{Z}) \to \mathrm{H}_1(\Sigma, \mathbf{Z})$ is the identity. The rest of the proof is similar to the fact that $T(\Sigma)$ can be identified with the set of fixed points of σ^* in $T(\Sigma^c)$; see [7] for more details. □

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