Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 24, 1999, 105–108

ON V. I. SMIRNOV DOMAINS

Peter W. Jones and Stanislav K. Smirnov

Yale University, Department of Mathematics
New Haven, CT06520, U.S.A.; stas@math.yale.edu

Abstract. We show that complement of a non-V. I. Smirnov domain, coming from the Duren–Shapiro–Shields or Kahane construction must be a V. I. Smirnov domain. There is therefore a negative answer to the old question: need a complement of a non-V. I. Smirnov domain be a non-V. I. Smirnov domain itself?

The purpose of this note is to prove a general, elementary theorem that provides an answer to an old problem on V. I. Smirnov domains. Recall that if Γ is a rectifiable closed Jordan curve and Ω_+ is the bounded component complementary to Γ, Ω_+ is a V. I. Smirnov domain if F'_+ is an outer function. Here F_+ is any choice of conformal map of $\mathbf{D}$ onto Ω_+. The class of such domains was introduced by V. I. Smirnov in [S] in connection with some questions of approximation theory. Consult the expository paper [D] of P. Duren for their properties and further references.

Several authors (starting with P. Duren, H. S. Shapiro, and A. L. Shields, see [DSS]) in the 1960's found examples of Γ and Ω_+, with F'_+ a singular inner function,

$$F'_+ = \exp\{-(\mu + i\tilde{\mu})\} \tag{1}$$

where μ is a positive measure on the circle that is singular with respect to Lebesgue measure, $d\mu \perp d\theta$. When Condition (1) holds, Ω_+ is *not* of V. I. Smirnov type (existence of non-Smirnov domains was established earlier by M. V. Keldysh and M. A. Lavrentiev in [KM]). If we let Ω_- denote the unbounded domain complementary to Γ, we have a similar definition of V. I. Smirnov (and non-V. I. Smirnov) domains.

Theorem. *Let Γ, Ω_+, Ω_-, be as above. Then if F_+ satisfies Condition* (1), *F_- (the conformal map of $\{|z|>1\}$ to Ω_-) satisfies*

$$|F'_-(z)| \geq c > 0, \qquad |z| > 1. \tag{2}$$

Corollary. *If Ω_+ is a non-V. I. Smirnov domain satisfying Condition* (1), *Ω_- is a V. I. Smirnov domain.*

1991 Mathematics Subject Classification: Primary 30C20; Secondary 30C45, 30C85, 31A15.
The first author is supported by N.S.F. Grant No. DMS-9423746.

There is therefore a negative answer to the old question (see [T1], [T2] for dicsussion and references): *Does* Ω_+ *V. I. Smirnov imply* Ω_- *V. I. Smirnov*?

Proof of Theorem. Let $c_1 := |F_+'(0)|$ and let ω_+ denote harmonic measure for Ω_+ (on $\partial\Omega_+$) with respect to $F_+(0)$. Then by Condition (1),

$$d\omega_+ = \frac{1}{2\pi c_1}\, ds,$$

where $ds = d\mathscr{H}^1$ is one dimensional Hausdorff measure on Γ. An immediate consequence of this is the inequality

$$\omega_+\big(D(x,r)\big) \ge \frac{r}{\pi c_1}, \qquad x \in \Gamma, \tag{3}$$

where $D(x,r)$ is the disk centered at x with radius $r < \operatorname{diam}(\Gamma)$ (we use the notation $\omega_+(E) := \omega_+(E \cap \Gamma)$).

Now we invoke the result of C. Bishop, L. Carleson, J. Garnett, P. Jones (see [BCGJ])

$$\omega_+\big(D(x,r)\big)\omega_-(D(x,r)) \le c_2 r^2, \tag{4}$$

where ω_- is harmonic measure for Ω_- with respect (say) to ∞. Here the constant c_2 depends on $F_+(0)$, but not on $x \in \Gamma$, $r < \operatorname{diam}(\Gamma)$. This result is valid for harmonic measures on any two disjoint, simply connected domains Ω_+, Ω_-, and its proof is quite elementary. Now from (3) and (4) we obtain

$$\omega_-\big(D(x,r)\big) \le c_3 r, \qquad x \in \Gamma,\ r < \operatorname{diam}(\Gamma). \tag{5}$$

It is an easy exercise that Condition (5) implies Condition (2) (one can look at the harmonic measure of $D\big(x, 2\operatorname{dist}(x,\Gamma)\big)$), and that F_-' is an outer function. □

Remark. In certain cases, one can draw a stronger conclusion, than (2). For example, let μ come from Kahane's construction in [K]. Let $K_+ \subset S^1$ be the closed support of μ. Then $K_- := F_-^{-1}(F_+(K_+)) \subset S^1$ satisfies

$$\operatorname{Box\,Dimension}(K_-) < 1. \tag{6}$$

On the other hand, general results (the sharp version is due to N. Makarov, [M]) show that if a set $K \subset S^1$ has zero measure for the Hausdorff gauge function $t\sqrt{\log(1/t)\log\log\log(1/t)}$, then there is no Riemann mapping defined on $\mathbf{D}$ with the properties $F' \in H^1$ (the Hardy space),

$$F' = G\exp\{-(\mu + i\tilde{\mu})\},$$

where G is outer, $d\mu \perp d\theta$, and μ is supported on K.

To verify (6) one can argue as in the following sketch. One first proves that for all $e^{i\theta} \in K_+$ and $\frac{1}{2} \le r < 1$, there is $R \in (r, \frac{1}{2} + \frac{1}{2}r)$ such that

$$|SF_+(z)| \ge \varepsilon(1 - |z|)^{-2}, \qquad z := \mathrm{R}e^{i\theta},$$

where S denotes the Schwarzian derivative. Here $\varepsilon > 0$ is independent of Γ and θ. An easy normal families argument then yields

$$\beta_\Gamma(x, r) \ge \delta, \qquad x \in F_+(K_+),\ r < \operatorname{diam}(\Gamma). \tag{7}$$

Here β is the usual measure of "deviation from flatness" for the set $\Gamma \cap D(x, r)$ (see e.g. [J]). Then estimates on harmonic measure (similar to those proving Condition (4), but we get a stronger result because of the "twisting", provided by Condition (7)) yield

$$\omega_+\bigl(D(x,r)\bigr)\omega_-\bigl(D(x,r)\bigr) \le cr^{2+\alpha}, \qquad x \in F_+(K_+).$$

Fortunately the argument for this inequality is given in the paper [R] of S. Rohde. S. Rohde proves his result for $K_+ = S^1$, but one sees easily, that his argument will work in our case.

Combining the last inequality with (3) one obtains

$$\omega_-\bigl(D(x,r)\bigr) \le cr^{1+\alpha}, \qquad x \in F_+(K_+). \tag{8}$$

Now cover $F_+(K_+)$ by disks of harmonic measure comparable to 2^{-n}. Applying the Besikovitch covering lemma and pulling back by F_- yields

$$\text{Box Dimension}\,\bigl(F_-^{-1}\bigl(F_+(K_+)\bigr)\bigr) < \frac{1}{1+\alpha}.$$

Here one must use that Γ has finite length. We remark that applications of estimates like (8) are the main point of S. Rohde's paper.

References

[BCGJ] Bishop, C., L. Carleson, J. Garnett, and P. Jones: Harmonic measures supported on curves. - Pacific J. Math. 138, 1989, 233–236.

[D] Duren, P.L.: Smirnov domains. - Zap. Nauchn. Sem. LOMI 170, 1989, 95–101; Issled. Linein. Oper. Teorii Funktsii. 17; see also J. Soviet Math. 67, 1993, 167–170.

[DSS] Duren, P., H.S. Shapiro, and A.L. Shields: Singular measures and domains not of Smirnov type. - Duke Math. J. 33, 1966, 247–254.

[J] Jones, P.W.: Rectifiable sets and the traveling salesman problem. - Invent. Math. 102, 1990, 1–15.

[K] Kahane, J.P.: Trois notes sur les ensembles parfait linéares. - Enseign. Math. 15, 1969, 185–192.

[KL] Keldysh, M.V., and M.A. Lavrentiev: Sur la représentation conforme des domaines limités par des courbes rectifiables. - Ann. Sci. École Norm. Sup. 54, 1937, 1–38.

[M] Makarov, N.G.: Size of the set of singular points on the boundary of a non-Smirnov domain. - Zap. Nauchn. Sem. LOMI 170, 1989, 176–183 (Russian); Issled. Linein. Oper. Teorii Funktsii. 17; English transl.: J. Soviet Math. 67, 1993, 212–216.

[R] Rohde, S.: On conformal welding and quasicircles. - Michigan Math. J. 38, 1991, 111–116.

[S] Smirnov, V.I.: Sur la théorie des polynômes orthogonaux à la variable complexe. - Zh. Leningr. Fiz.-Mat. Ob-va 2, 1928, 155–179.

[T1] Tumarkin, G.Ts.: Boundary conditions for conformal mappings of certain classes of domains. - In: Some Problems in Modern Function Theory, Proc. Conf. Modern Problems of Geometric Theory of Functions, Inst. Mat., Akad. Sci. USSR, Novosibirsk, 1976, 149–160 (Russian); Akad. Nauk SSSR Sibirsk. Otdel. Inst. Mat., Novosibirsk, 1976.

[T2] Tumarkin, G.Ts.: Some problems concerning classes of domains determined by properties of Cauchy type integrals. - In: Linear and Complex Analysis Problem Book 3, Lecture Notes in Math. 1573, Springer-Verlag, Berlin, 1994, 411–413.

Received 29 April 1997