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BMO ON SPACES OF HOMOGENEOUS TYPE: A DENSITY RESULT ON C-C SPACES

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Abstract. In the general setting of a space of homogeneous type endowed with an Ahlfors regular measure, we introduce the Banach spaces BMO and VMO defined through suitable cubes, and we prove that these spaces are topologically equivalent to the standard ones usually defined by means of balls. Through this fact we extend a known result of Sarason showing that C^∞ is locally dense in VMO in the setting of Carnot–Carathéodory metric spaces related to a family of free Hörmander vector fields $X_1, \dots, X_q$.

1. Introduction

The space of the functions with bounded mean oscillation BMO, is well known for its several applications in real analysis, harmonic analysis and partial differential equations. In particular, for regularity problems regarding solutions of partial differential equations, the subspace VMO of BMO plays a particular role. VMO is the space of the vanishing mean oscillation functions and it was introduced by Sarason in 1975 (see [27]). In regularity problems the importance of VMO consists in a density result due to Sarason: the space of smooth functions is dense in VMO. In this note we prove the analogous result in a more general setting than the euclidean one. First we introduce the classes BMO and VMO defined on spaces of homogeneous type endowed with an Ahlfors regular measure. Spaces of homogeneous type appear first in Coifman and Weiss (see [8]). A space of homogeneous type is a set with a quasimetric (that is a metric space with a weaker triangle property) endowed with a Borel measure with respect to which the ratio between the measure of any ball and the measure of the same ball with half radius is upper bounded by an absolute constant (doubling property). These spaces have been investigated since, in this context, classical results of real analysis such as Lebesgue theorem, Whitney type decompositions, boundedness of maximal operators, representation formulas, singular integrals, etc. are naturally settled. Particular spaces of homogeneous type

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are Carnot–Carathéodory metric spaces whose distance is generated by the sub-unit curves with respect to a system of free Hörmander vector fields $X_1, X_2, \ldots, X_q$. The main result of this paper is obtained adapting the original proof of Sarason to this new setting. In order to do this, we use a decomposition of a space of homogeneous type into "dyadic cubes" (see Christ [6, 7]; see also [11]) that allows us to employ a natural convolution operator in these C-C spaces. As in the classical setting, our density result has been used to solve L^p and BMO regularity problems of elliptic equations and systems of the type

$$-X_i^T\big(a^{ij}_{\alpha\beta}(x)\,X_j\,u^\beta\big) = g_\alpha - X_i^T f^i_\alpha(x)$$

with VMO coefficients, with respect to Carnot–Carathéodory metric (see [1, 5, 12, 2, 4, 13]).

2. BMO in spaces of homogeneous type

Let us begin by recalling the notion of space of homogeneous type.

Definition 2.1. A quasimetric d on a set S is a function $d\colon S \times S \to [0, +\infty[$ with the following properties

(qm$_1$) $d(x,y) = 0$ if and only if $x = y$;
(qm$_2$) $d(x,y) = d(y,x)$ $\forall\, x, y \in S$;
(qm$_3$) $\exists\, A_0 > 0$ such that $d(x,y) \le A_0\,[d(x,z) + d(z,y)]$ $\forall x, y, z \in S$.

A quasimetric defines a topology in which the balls $B(x,r) = \{y \in S : d(x,y) < r\}$ form a base. These balls may be not open in general; anyway, given a quasimetric d, it is easy to construct an equivalent quasimetric d' such that the d'-quasimetric balls are open (the existence of d' has been proved by using topological arguments in [21]): so we can assume that the quasimetric balls are open.

Definition 2.2. A space of homogeneous type (S, d, μ) is a set S with a quasimetric d and a Borel measure μ finite on bounded sets such that, for some absolute positive constant A_1, the following doubling property holds

$$\text{(D)} \qquad \mu(B(x,2r)) \le A_1\,\mu(B(x,r))$$

for all $x \in S$ and $r > 0$.

The number $Q = \log_2 A_1$ (where A_1 is the least number satisfying (D)) is called the *homogeneous dimension* of the space (S, d, μ).

It is well known that a space of homogeneous type (S, d, μ) satisfies the following equivalent properties:

(i) there exists an integer N such that for every $x \in S$ and for every $r > 0$, the ball $B(x,r)$ contains at most N points $x_1, x_2, \ldots, x_N$ with $d(x_i, x_j) \ge r/2$, for $i \ne j$;
(ii) there exists an integer N such that for every $x \in S$, for every $r > 0$ and for every $n \in \mathbf{N}$, the ball $B(x,r)$ contains at most N^n points $x_1, x_2, \ldots, x_{N^n}$ with $d(x_i, x_j) \ge r/2^n$, for $i \ne j$.

The equivalence of these two properties has been proved in [8]. We recall that a metric space S satisfying (i) or (ii) is usually called a *doubling metric space*; some other properties may be found in [19].

Definition 2.3. A Borel measure μ on a quasimetric space is said to be Ahlfors regular of dimension Q if there exist two absolute positive constants a and A such that for all $x \in S$ and $r > 0$ it results

$$a\, r^Q \le \mu(B(x,r)) \le A\, r^Q. \tag{A}$$

It is clear that (A) implies (D).

In the following, we shall assume that (S, d, μ) is a space of homogeneous type with μ Ahlfors regular measure; moreover, we assume that any open ball—and consequently the whole space—is a connected subset of S.

The first assumption is useful for Definitions 2.5 and 2.13 of VMO spaces; moreover, the two assumptions jointly simplify the proof of Proposition 2.14: each of them is satisfied in Carnot–Carathèodory metric spaces studied in Section 3.

If $E \subseteq S$ is a Borel set with positive measure and $f \in L^1(E)$, we denote by f_E the integral average $⨍_E f\, d\mu = \frac{1}{\mu(E)} \int_E f\, d\mu$.

Definition 2.4. (BMO with balls) $BMO(S)$ is the set of classes of equivalence of functions f (with finite integral on bounded sets), modulo additive constants, such that each of the two following equivalent conditions is satisfied

$$\sup_{\substack{x\in S\\ r>0}} ⨍_{B(x,r)} |f - f_{B(x,r)}|\, d\mu < +\infty,$$

$$\sup_{\substack{x\in S\\ r>0}} \inf_{c\in\mathbf{R}} ⨍_{B(x,r)} |f - c\,|\, d\mu < +\infty.$$

We denote by $\|\cdot\|_{BMO(S)}$ each of the two equivalent norms above: according to the context it will be clear which of them we will refer to.

Definition 2.5. (VMO with balls) A function $f \in BMO(S)$ belongs to the space $VMO(S)$ if

$$M_0(f) = \lim_{a\to 0^+} M_a(f) = 0,$$

where

$$M_a(f) = \sup_{\substack{x\in S\\ 0<r\le a}} \inf_{c\in\mathbf{R}} ⨍_{B(x,r)} |f - c|\, d\mu.$$

The first results regarding the decomposition of a metric space with cubes appeared in [9] and [10]. In a more general setting than [9], Christ introduced a decomposition of a space of homogeneous type (S, d, μ) (see [6] and [7]): we thank R. L. Wheeden for pointing to us recently that a similar construction can be found in [28] (see also [29]). Actually, in [6] and [7], the following theorem has been proved:

Theorem 2.6. *For any $k \in \mathbf{Z}$ there exist a set, at most countable, I_k, and a family of subsets $Q^k_\alpha \subseteq S$, with $\alpha \in I_k$, such that*

(1) $\mu(S \setminus \bigcup_\alpha Q_\alpha^k) = 0\ \forall\, k \in \mathbf{Z}$;
(2) *for any* α, β, k, l *with* $l \geq k$, *either* $Q_\beta^l \subseteq Q_\alpha^k$ *or* $Q_\beta^l \cap Q_\alpha^k = \emptyset$;
(3) *for each* Q_α^{k+1} *there exists exactly one* Q_β^k *(parent of* Q_α^{k+1}*) such that* $Q_\alpha^{k+1} \subseteq Q_\beta^k$;
(4) *for each* Q_α^k *there exists at least one* Q_β^{k+1} *(child of* Q_α^k*) such that* $Q_\beta^{k+1} \subseteq Q_\alpha^k$.

These open subsets of the kind Q_α^k are called *dyadic cubes of generation* k due to the analogy between them and the standard euclidean dyadic cubes. To construct them, for any fixed real number $\delta \in]0,1[$ and for any integer k, Christ considers a maximal collection of points $z_\alpha^k \in S$ such that

$$d(z_\alpha^k, z_\beta^k) \geq \delta^k \quad \forall\, \alpha \neq \beta.$$

He orders the pairs (k,α) by constructing a tree with the following properties:

1) for each $k \in \mathbf{Z}$ and $x \in S$ there exists α such that $d(x, z_\alpha^k) < \delta^k$;
2) if $(k,\alpha) \leq (l,\beta)$ then $k \geq l$;
3) for each (k,α) and $l \leq k$ there exists an unique β such that $(k,\alpha) \leq (l,\beta)$;
4) if $(k,\alpha) \leq (k-1,\beta)$ then $d(z_\alpha^k, z_\beta^{k-1}) < \delta^{k-1}$;
5) if $(l,\beta) \leq (k,\alpha)$ then $d(z_\beta^l, z_\alpha^k) \leq 2A_0\,\delta^k$.

Then, for a suitable real number $a_0 \in \left]0, \frac{1}{2\,A_0}\right[$, he defines the open set Q_α^k as follows

$$Q_\alpha^k = \bigcup_{(l,\beta)\leq(k,\alpha)} B(z_\beta^l, a_0\,\delta^l).$$

We enunciate some other useful properties regarding such dyadic cubes:

(5) there exists $\varepsilon > 0$ such that if $Q_\alpha^{k+1} \subseteq Q_\beta^k$ then $\mu(Q_\alpha^{k+1}) \geq \varepsilon\,\mu(Q_\beta^k)$;
(6) there exists $c_1 > 1$ such that $\mathrm{diam}(Q_\alpha^k) \leq c_1\,\delta^k$;
(7) there exists $\widetilde{C} > 0$ such that for each (α,k) there exists $z_\alpha^k \in S$ such that $B(z_\alpha^k, a_0\delta^k) \subseteq Q_\alpha^k \subseteq B(z_\alpha^k, \widetilde{C}\delta^k)$.

In the sequel the following lemmas and definitions will be useful.

Lemma 2.7. *There exists an absolute positive constant* $\overline{C}$ *such that, for any integer* k, *if* $R \in]0,\delta^k]$, *then the number of dyadic cubes of generation* k *that intersect* $B(x,R)$ *is at most* $\overline{C}$.

Proof. Fix $k \in \mathbf{Z}$, $x \in S$, $0 < R \leq \delta^k$ and suppose that for some $\alpha \in I_k$ there exists $y \in B(x,R) \cap Q_\alpha^k$. So we can find $(l,\beta) \leq (k,\alpha)$ such that $y \in B(z_\beta^l, a_0\delta^l)$. Then $d(z_\alpha^k, x) \leq A_0 d(z_\alpha^k, y) + A_0 d(y,x) \leq A_0^2 d(z_\alpha^k, z_\beta^l) + A_0^2 d(z_\beta^l, y) + A_0 R \leq 2A_0^3\delta^k + \frac{A_0}{2}\delta^l + A_0 R \leq c'\delta^k$, with $c' = 2A_0^3 + A_0/2 + A_0$, from which z_α^k belongs to $B(x, c'\delta^k)$. From ii) and the maximality of the family $\{z_\alpha^k\}_{\alpha\in I_k}$ the thesis follows. □

Definition 2.8. Let Q' and Q'' be two dyadic cubes. We say that Q' is 1-step contiguous to Q'' if $\partial Q' \cap \partial Q'' \neq \emptyset$. Moreover we say that Q' is h-step contiguous $(h \geq 2)$ to Q'' if Q' is 1-step contiguous to some $(h-1)$-step dyadic cube contiguous to Q''.

The proof of the following lemma is similar to the one above.

Lemma 2.9. *There exists an absolute positive constant C' such that, for every dyadic cube Q_α^k, there exist at most C' dyadic cubes of the same generation k that are 1-step contiguous to Q_α^k.*

Now we are able to define "cubes" on our space of homogeneous type.

Definition 2.10. We call cube either a dyadic cube or the union of a given dyadic cube with its contiguous cubes of the same generation, up to some step $h \geq 1$.

Remark 2.11. In the euclidean setting we can construct these cubes using standard euclidean dyadic cubes, thus obtaining a family of cubes "dense", in some sense, in the family of all euclidean cubes. The analogy between the cubes in Definition 2.10 and the euclidean ones defined by glueing euclidean dyadic cubes, is useful for a geometric interpretation of Proposition 2.14 below.

We will denote by Q a generic cube of the space of homogeneous type (S, d, μ).

Definition 2.12. (BMO with cubes) $BMO_{\mathscr{C}}(S)$ is the set of classes of equivalence of functions f (with finite integral on bounded sets), modulo additive constants, such that each of the two following equivalent conditions is satisfied

$$\sup_Q \fint_Q |f - f_Q| \, d\mu < +\infty,$$

$$\sup_Q \inf_{c \in \mathbf{R}} \fint_Q |f - c| \, d\mu < +\infty.$$

As before we denote by $\|\cdot\|_{BMO_{\mathscr{C}}(S)}$ each of the two above equivalent norms; moreover, by standard arguments (see for instance [24]) it can be proved that the spaces $BMO(S)$ and $BMO_{\mathscr{C}}(S)$ are Banach spaces.

Definition 2.13. (VMO with cubes) A function $f \in BMO_{\mathscr{C}}(S)$ belongs to the space $VMO_{\mathscr{C}}(S)$ if

$$M_{\mathscr{C},0}(f) = \lim_{a \to 0^+} M_{\mathscr{C},a}(f) = 0,$$

where

$$M_{\mathscr{C},a}(f) = \sup_{\operatorname{diam}(Q) \leq a} \inf_{c \in \mathbf{R}} \fint_Q |f - c| \, d\mu.$$

Now we can show the equivalence between the spaces $BMO(S)$ and $BMO_{\mathscr{C}}(S)$.

Proposition 2.14. *Let (S, d, μ) be a space of homogeneous type with μ Ahlfors regular measure. Then there exists an absolute positive constant C such that*

$$\frac{1}{C} \|\cdot\|_{BMO(S)} \leq \|\cdot\|_{BMO_{\mathscr{C}}(S)} \leq C \|\cdot\|_{BMO(S)}. \tag{B}$$

Proof. Since the spaces $BMO(S)$ and $BMO_{\mathscr{C}}(S)$ are complete, it suffices to prove that $BMO_{\mathscr{C}}(S)$ is continuously embedded into $BMO(S)$.

Let us consider a ball $B(x_0, r)$: we have to construct two cubes Q' and Q'', such that $Q' \subseteq B(x_0, r) \subseteq Q''$. We stress that all set inclusions within this proof hold up to μ-negligible sets.

Let k be an integer such that $\delta^k A_0(a_0 + 2A_0^2 + A_0) < r \le \delta^{k-1} A_0(a_0 + 2A_0^2 + A_0)$. From property 1), there exists $\alpha \in I_k$ such that $d(x_0, z_\alpha^k) < \delta^k < r$, from which $z_\alpha^k \in B(x_0, r)$. It results $Q' = Q_\alpha^k \subseteq B(x_0, r)$. Indeed, let $x \in Q_\alpha^k$: there exists $(l, \beta) \le (k, \alpha)$ such that $x \in B(z_\beta^l, a_0\delta^l)$. Then

$$\begin{aligned} d(x, x_0) &\le A_0 d(x, z_\beta^l) + A_0^2 d(z_\beta^l, z_\alpha^k) + A_0^2 d(z_\alpha^k, x_0) \\ &< \delta^k A_0(a_0 + 2A_0^2 + A_0) < r. \end{aligned}$$

Now we construct a cube Q'' such that $B(x_0, r) \subseteq Q''$. From property 3), there exists a unique β such that $(k, \alpha) \le (k-1, \beta)$. If $B(x_0, r) \nsubseteq Q_\beta^{k-1}$, let Q_γ^{k-1} be a dyadic cube (different from Q_β^{k-1}) such that $Q_\gamma^{k-1} \cap B(x_0, r) \ne \emptyset$. Now we estimate the distance between z_γ^{k-1} and z_β^{k-1}. Let $x \in Q_\gamma^{k-1} \cap B(x_0, r)$, there exists $(l, \zeta) \le (k-1, \gamma)$ such that $x \in B(z_\zeta^l, a_0\delta^l)$. From properties 4) and 5), we have

$$\begin{aligned} d(z_\gamma^{k-1}, z_\beta^{k-1}) &\le A_0 d(z_\gamma^{k-1}, x) + A_0 d(x, z_\beta^{k-1}) \\ &\le A_0^2 d(z_\gamma^{k-1}, z_\zeta^l) + A_0^2 d(z_\zeta^l, x) + A_0^2 d(x, x_0) + A_0^2 d(x_0, z_\beta^{k-1}) \\ &\le 2A_0^3 \delta^{k-1} + A_0^2 a_0 \delta^l + A_0^2 r + A_0^3 d(x_0, z_\alpha^k) + A_0^3 d(z_\alpha^k, z_\beta^{k-1}) \\ &\le 2A_0^3 \delta^{k-1} + A_0^2 a_0 \delta^{k-1} + A_0^3(a_0 + 2A_0^2 + A_0)\delta^{k-1} \\ &\quad + A_0^4(a_0 + 2A_0^2 + A_0)\delta^{k-1} + A_0^3 \delta^{k-1} \\ &\le c(a_0, A_0)\delta^{k-1}. \end{aligned}$$

Then the points like z_γ^{k-1} (centers of dyadic cubes of generation $k-1$ that intersect $B(x_0, r)$) belong to a ball centered in z_β^{k-1}. S is a space of homogeneous type (see ii)) so there exists an absolute number m (depending only on S) of points z_γ^{k-1} such that $Q_\gamma^{k-1} \cap B(x_0, r) \ne \emptyset$ for any γ. Since $B(x_0, r)$ is connected, we can find an integer $s \le m$, the maximum step of contiguity of all such cubes with respect to Q_β^{k-1} : define Q'' as the union of Q_β^{k-1} with its contiguous cubes of generation $k-1$, up to the step s. According to Definition 2.10 Q'' is a cube and it is the union $\bigcup_{\gamma=1}^t Q_\gamma^{k-1}$, where $t \le C' + C'^{\,2} + \cdots + C'^{\,m}$ (C' is the constant in Lemma 2.9). So, for $c \in \mathbf{R}$, we have

$$\begin{aligned} \fint_{B(x_0,r)} |f - c| d\mu &\le \frac{1}{\mu(B(x_0, r))} \int_{Q''} |f - c|\, d\mu \\ &\le \frac{1}{\mu(B(x_0, r))} \sum_{\gamma=1}^t \mu(Q_\gamma^{k-1}) \fint_{Q''} |f - c|\, d\mu \\ &\le \sum_{\gamma=1}^t \frac{\mu(Q_\gamma^{k-1})}{\mu(Q_\alpha^k)} \fint_{Q''} |f - c|\, d\mu. \end{aligned}$$

Now we observe that, from property (7) and Ahlfors regularity, we can find an absolute positive constant $\bar{c}$ such that $\mu(Q_\gamma^{k-1})/\mu(Q_\alpha^{k-1}) \leq \bar{c}$ for all $\gamma = 1, 2, \ldots, t$; moreover from property (5), we have that $\sum_{\gamma=1}^{t} \mu(Q_\gamma^{k-1})/\mu(Q_\alpha^{k}) \leq t\,\bar{c}/\varepsilon$. So it follows that, choosing $c = f_{Q''}$, if f is in $BMO_{\mathscr{C}}(S)$, then f is in $BMO(S)$ and the left inequality of (B) is proved. $\square$

It is not difficult to verify that $VMO(S)$ (respectively $VMO_{\mathscr{C}}(S)$) is a closed subspace of $BMO(S)$ (respectively $BMO_{\mathscr{C}}(S)$), so the following proposition holds.

Proposition 2.15. *Let (S, d, μ) be a space of homogeneous type with μ Ahlfors regular measure, then*

$$VMO(S) = VMO_{\mathscr{C}}(S).$$

Remark 2.16. We stress that in this setting the space $BMO_{\mathscr{C}}(S)$ is smaller than the dyadic $BMO(S)$, in analogy with the euclidean dyadic BMO (see [16]). Proposition 2.14 is just the analogous of a property much more easy to verify in the euclidean setting. Indeed, it is simple to prove that every euclidean cube can be filled up (respectively covered) by a finite union of euclidean dyadic cubes, in such a way that both the ratio between the measure of the covering union and the measure of the given cube, and the ratio between the measure of the cube and the measure of the enclosed union are bounded by an absolute positive constant. This fact proves that, in the euclidean case, BMO equals $BMO_{\mathscr{C}}$, where, as noted in Remark 2.11, the last one is made up by Definition 2.10 related to standard euclidean dyadic cubes.

3. Carnot–Carathéodory spaces: A density result

In this section we prove that the class VMO is locally the closure of C^∞ in the space BMO, with respect to the Carnot–Carathéodory metric induced by a finite set of free Hörmander vector fields.

We recall some preliminary facts about a particular class of nilpotent Lie groups: for more details we refer, for instance, to [15, 30, 14] and to [31] for general facts about Lie groups and Lie Algebras.

Let $X_1, \ldots, X_q$ be generators of the free real Lie algebra $\mathfrak{g}_{q,s}$. For every $d \in \mathbf{N}$ and every multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $1 \leq \alpha_i \leq q$, we set $d = |\,\alpha\,|$ and denote by X_α the commutator of length d $\left[X_{\alpha_1}, [X_{\alpha_2}, \ldots, [X_{\alpha_{d-1}}, X_{\alpha_d}] \ldots]\right]$. Then there exists a finite set A such that $\{X_\alpha\}_{\alpha \in A}$ is a base for the underlying vector space V of $\mathfrak{g}_{q,s}$. Writing explicitly $V = \bigoplus_{i=1}^{s} V_i$, if $N = \mathrm{Card}(A)$, we can assume $A = \{1, 2, \ldots, N\}$ so that if, for any $i = 1, \ldots, s$, we set $d_i = \dim(V_i)$, one has $d_1 + \cdots + d_s = N$. More precisely $X_1, \ldots, X_q$ span V_1 as a real vector space, so that $d_1 = q$, while $V_i = [V_1, V_{i-1}]$ for $i = 2, \ldots, s$, being zero every further commutator. Let $\mathscr{G}$ be the connected and simply connected Lie group associated to $\mathfrak{g}_{q,s}$. By the property of the global diffeomorphism $\exp\colon \mathfrak{g}_{q,s} \to \mathscr{G}$ and the Baker–Campbell–Hausdorff formula we can multiply two N-tuples of exponential coordinates—of the first kind—of elements of $\mathscr{G}$, so that we can identify $\mathscr{G}$ with $(\mathbf{R}^N, \cdot)$, where "$\,\cdot\,$" is a polynomial law group. Moreover, it is possible to endow $\mathbf{R}^N$ with a group of automorphisms

$\{\delta_\lambda\}_{\lambda>0}$, called *dilations*, that we are going to describe. If $V_1 = \mathrm{span}\{X_j\}_{1\le j\le d_1}$ and $V_i = \mathrm{span}\{X_j\}_{d_1+\cdots+d_{i-1}+1\le j\le d_1+\cdots+d_i}$ for $i = 2,\dots,s$, it is enough to define, for $\lambda > 0$, an automorphism γ_λ of the Lie algebra $\mathfrak{g}_{q,s}$ on the generators by the position $\gamma_\lambda(X_j) = \lambda^i X_j$ for $j = 1,\dots,N$, where $i = 1,2,\dots,s$ is such that $X_j \in V_i$; thus, the position $\delta_\lambda = \exp\circ\gamma_\lambda\circ\exp^{-1}$ defines, for $\lambda > 0$, an automorphism on the Lie group $\mathbf{R}^N$ satisfying the following property: for any $y = (y_j)_{1\le j\le N} \in \mathbf{R}^N$ it results $\delta_\lambda(y) = (\lambda^i y_j)_{1\le j\le N} \in \mathbf{R}^N$ where $i = 1$ if $j = 1,\dots,d_1$ and otherwise $i = 2,\dots,s$ is such that $d_1+\cdots+d_{i-1}+1 \le j \le d_1+\cdots+d_i$. We recall that actually in the Lie group $\mathbf{R}^N$ one has $\xi\cdot\eta = \xi+\eta+\mathscr{Q}(\xi,\eta)$, where $\mathscr{Q} = (\mathscr{Q}_1,\dots,\mathscr{Q}_N)$ is a homogeneous polynomial vector function such that $\mathscr{Q}_1 = \cdots = \mathscr{Q}_{d_1} = 0$, $\mathscr{Q}_j$ has degree i with respect to any dilation δ_λ and depends only on the first $d_1+\cdots+d_{i-1}$ coordinates, for any $d_1+\cdots+d_{i-1}+1 \le j \le d_1+\cdots+d_i$, and for any $i = 2,\dots,s$. With respect to such group product the identity element is exactly 0 and the inverse ξ^{-1} of any $\xi \in \mathbf{R}^N$ is exactly $-\xi$. So $\mathbf{R}^N$ comes to be a *homogeneous group* in sense of Folland and Stein, more recently called *Carnot group*. Denoting by $|\cdot|$ the euclidean norm, we can introduce in $\mathbf{R}^N$, endowed with the above Lie group structure, a *homogeneous norm* $||\cdot||$ by setting, for every $\xi \in \mathbf{R}^N$, $\|\xi\| = \lambda \Leftrightarrow |\delta_{\frac{1}{\lambda}}(\xi)| = 1$ if $\xi \neq 0$ and $\|0\| = 0$ (note that the function $[0,+\infty[\ni \lambda \to |\delta_\lambda(\xi)| \in [0,+\infty[$ is strictly increasing and goes to infinity with λ, for any $\xi \neq 0$). This norm results a C^∞ function outside the origin and it follows that the law $\mathbf{R}^N \ni (\xi,\eta) \to \|\eta^{-1}\cdot\xi\| \in [0,+\infty[$, defines a quasimetric in $\mathbf{R}^N$. If τ_ξ denotes either a left or a right translation on the Lie group $\mathbf{R}^N$ then, according to the polinomial form of the group law recalled before, the matrix associated to $d\tau_\xi$ is lower triangular with ones on the diagonal so that the Lebesgue measure $\mathscr{L}^N$ is the bi-invariant Haar measure. Moreover, for any fixed dilation δ_λ it is clear that $J_{\delta_\lambda} = \mathrm{diag}\Big(\underbrace{\lambda^1,\dots,\lambda^1}_{d_1},\underbrace{\lambda^2,\dots,\lambda^2}_{d_2},\dots,\underbrace{\lambda^s,\dots,\lambda^s}_{d_s}\Big)$ so that, setting $Q = \sum_{i=1}^s i\,d_i$, $\det J_{\delta_\lambda} = \lambda^Q$. It follows that, for every $\xi \in \mathbf{R}^N$, $\lambda > 0$ and every Lebesgue measurable subset E, it results $\mathscr{L}^N\big(\delta_\lambda(\xi\cdot E)\big) = \mathscr{L}^N\big(\delta_\lambda(E\cdot\xi)\big) = \lambda^Q\,\mathscr{L}^N(E)$.

Now we denote by X a family $\{X_1, X_2,\dots,X_q\}$ of C^∞ real vector fields: without lost of generality we can assume that these vector fields are defined on the whole space $\mathbf{R}^N$.

The family X satisfies Hörmander condition of step s at some point $\xi_0 \in \mathbf{R}^N$ if, for any fixed set A of indexes as above, $\{X_j(\xi_0)\}_{j\in A}$ spans $\mathbf{R}^N$ as vector space. Moreover we say that the vector fields $X_1, X_2,\dots,X_q$ are free up to order s at ξ_0 if $\dim V = N$.

With such a family X we can introduce in $\mathbf{R}^N$ the Carnot–Carathéodory metric (see for instance [18]). A Lipschitz continuous curve $\gamma\colon [0,T] \to \mathbf{R}^N$ is said to be X-*subunit* if there exists a measurable vector function $h = (h_1,\dots,h_q)\colon [0,T] \to \mathbf{R}^q$ such that $\dot\gamma(t) = \sum_{i=1}^q h_i(t)X_i(\gamma(t))$ for a.e. $t \in [0,T]$ and $|h|_\infty \le 1$. Set

$$d_X(x,y) = \inf\Big\{T \ge 0 \,|\, \exists\,\gamma : [0,T] \to \mathbf{R}^N, X-\text{subunit}, \gamma(0) = x,\, \gamma(T) = y\Big\}.$$

From now on we assume that the vector fields $X_1, X_2, \dots, X_q$ satisfy Hörmander condition of step s and are free up to the same order at any point $\xi \in \mathbf{R}^N$.

Under these assumptions it can be shown that the above position defines a metric on $\mathbf{R}^N$, usually called the Carnot–Carathéodory distance (briefly C-C metric) associated to the family X. In the sequel we shall denote by $B(\xi, r)$ a C-C ball centered in $\xi \in \mathbf{R}^N$ with radius $r > 0$.

Next theorem (see [25, 26]) establishes a correspondence between some neighborhoods of the points ξ of a given compact set W endowed with the C-C metric induced by a family of free Hörmander vector fields, and a neighborhood of the origin $0 \in \mathbf{R}^N$, endowed with a Carnot group structure; this correspondence also makes possible to introduce a quasimetric on W, in terms of the distance between the corresponding points of the neighborhood of the origin of $\mathbf{R}^N$. Actually, this corrispondence imitates the standard one between the points of a real Lie algebra and its (connected and simply connected) Lie group, based on the property of the exponential mapping and the induced Malcev's coordinates of the first kind on the group. By means of this "local" coordinates, we will able to define locally a suitable convolution modeled to the Carnot–Carathéodory metric.

Theorem 3.1. *Let $X = \{X_1, X_2, \dots, X_q\}$ be a family of $C^\infty(\mathbf{R}^N)$ real vector fields satisfying Hörmander condition of step s and free up to the same order at $\xi_0 \in \mathbf{R}^N$. Then, for any fixed set A of indexes as above, there exist open neighborhoods U of 0, V and W of ξ_0, $W \Subset V$, such that for any fixed $\xi \in V$, the mapping*

$$U \ni y \to \eta = \exp\left(\sum_{j=1}^{N} y_j X_j\right) \xi \in V$$

is invertible, and calling $y = \Theta_\xi(\eta)$ its inverse, it results:

a) *$\Theta_\xi | V$ is a diffeomorphism onto the image for every $\xi \in V$;*
b) *$U \subseteq \Theta_\xi(V)$ for every $\xi \in W$;*
c) *$\Theta : V \times V \to \mathbf{R}^N$ defined by $\Theta(\xi, \eta) = \Theta_\xi(\eta)$ is $C^\infty(V \times V)$;*
d) *if we set, for any $\xi, \eta \in V$, $\rho(\xi, \eta) = \|\Theta(\xi, \eta)\|$, it results $\Theta(\xi, \eta) = \Theta(\eta, \xi)^{-1} = -\Theta(\eta, \xi)$ and there exists a positive constant c such that*

$$\rho(\xi, \eta) \le c\,(\rho(\xi, \zeta) + \rho(\zeta, \eta)),$$

whenever $\rho(\xi, \zeta), \rho(\zeta, \eta) \le 1$.

Clearly we can assume that the neighborhood V is compactly contained in $\mathbf{R}^N$.

The topology induced on $\mathbf{R}^N$ by the C-C metric associated to the family X and the Euclidean topology are the same, nevertheless the C-C metric and the Euclidean one are not equivalent: indeed, for any bounded subset E there exists a positive constant C depending on X and E such that $\frac{1}{C}\,|\xi - \eta| \le d_X(\xi, \eta) \le C\,|\xi - \eta|^{1/s}$, for any $\xi, \eta \in E$. Moreover, Lebesgue measure is locally doubling with respect to d_X; actually, for any bounded subset E there exists $\overline{R} > 0$ such that $\mathscr{L}^N(B) \approx r^Q$ for any C-C ball B with center in E and radius $r \in]0, \overline{R}]$. From the doubling property and the local equivalence between d_X and ρ of d) in Theorem 3.1 (see [25, 26]), it follows that the BMO and VMO spaces defined over the two space of homogeneous type $(V, d_X, \mathscr{L}^N)$ and $(V, \rho, \mathscr{L}^N)$ coincide.

In the sequel we shall assume $(V, d, \mathscr{L}^N)$ as our space of homogeneous type, where d is equivalently either d_X or ρ.

At last we need to recall a relevant structure property of C-C balls, known as *Ball–Box Theorem* (see [17, 20, 22]), that we state in a suitable form.

Theorem 3.2. *Let $X = \{X_1, X_2, \ldots, X_q\}$ be a family of $C^\infty(\mathbf{R}^N)$ real vector fields satisfying Hörmander condition of step s and free up to the same order at any point $\xi \in \mathbf{R}^N$. Set, for $r > 0$,*

$$\begin{aligned}\mathrm{Box}(r) = \Big\{ y = &(y_1, \ldots, y_N) \in \mathbf{R}^N : \\ &|y_j| \le r \text{ if } 1 \le j \le d_1, \\ &|y_j| \le r^i \text{ if } d_1 + \cdots + d_{i-1} + 1 \le j \le d_1 + \cdots + d_i \\ &\text{and for any } i = 2, \ldots, s \Big\}.\end{aligned}$$

Then for any bounded subset E, if $R > 0$, there exist σ_1, $\sigma_2 \in]0, 1[$, $\sigma_1 < \sigma_2$, such that, for every $\xi \in E$ and $r \in]0, R[$, it results

$$B(\xi, \sigma_1 r) \subseteq \left\{ \eta \in \mathbf{R}^N : \eta = \exp\left(\sum_{j=1}^{N} y_j X_j \right) \xi : \ y \in \mathrm{Box}(\sigma_2 r) \right\} \subseteq B(\xi, \sigma_2 r).$$

Now we are going to introduce a suitable convolution on the neighborhood W : according to the notation of Theorem 3.1, we state first the following lemma which, thanks to the Ball–Box theorem, geometrically says that the ball $B(\xi, \varepsilon)$ with $\xi \in W$, looks like a box in V and, through the diffeomorphism $V \Supset B(\xi, \varepsilon) \ni \eta \to \Theta_\xi(\eta) \in \mathbf{R}^N$, is mapped exactly, whatever $\xi \in W$ is chosen, into a suitable ball of radius $s > 0$, centered in the identity of the Carnot group $\mathbf{R}^N$, that we shall denote by $B(0, s)$. The easy proof, based on the properties of the map Θ, is omitted.

Lemma 3.3. *There exist $\overline{\varepsilon} > 0$ small enough such that, for all $\varepsilon \in]0, \overline{\varepsilon}]$, it results $B(\xi, \varepsilon) \Subset V$ for every $\xi \in W$ and there exists a positive constant ϑ for which $B(0, \vartheta\varepsilon) = \Theta(\xi, B(\xi, \varepsilon))$ for every $\xi \in W$.*

So we can define, for $y \in \mathbf{R}^N$ and $\varepsilon > 0$,

$$\varphi(y) = \begin{cases} 0 & \text{if} \quad \|y\| \ge 1 \\ c \exp\left(\frac{1}{\|y\|^2 - 1}\right) & \text{if} \quad \|y\| < 1 \end{cases},$$

where the constant $c > 0$ is such that $\int_{B(0,1)} \varphi(y)\, dy = 1$, and

$$\varphi_\varepsilon(y) = \frac{1}{(\vartheta\varepsilon)^Q}\, \varphi\big(\delta_{\frac{1}{\vartheta\varepsilon}}(y)\big).$$

Clearly $\int_{B(0,\vartheta\varepsilon)} \varphi_\varepsilon(y)\, dy = 1$. Denote by J_ξ the Jacobian of the map Θ_ξ.

Definition 3.4. If $f \in L^1_{\mathrm{loc}}(V)$ we set, for any $\xi \in W$ and $\varepsilon > 0$ small enough,

$$f_\varepsilon(\xi) = \int_{B(0,\vartheta\varepsilon)} f(\Theta_\xi^{-1}(y))\varphi_\varepsilon(y)\,dy = \int_{B(\xi,\varepsilon)} f(\eta)\varphi_\varepsilon(\Theta_\xi(\eta))J_\xi(\eta)\,d\eta.$$

The convolution-type operator clearly behaves like the euclidean one, as shown in the following lemma.

Lemma 3.5. *If $f \in L^1_{\mathrm{loc}}(V)$, then*

(a) $f_\varepsilon \in C^\infty(W)$;

(b) $f_\varepsilon \to f$ *a.e. as* $\varepsilon \to 0$; *moreover, if* $f \in C(W)$ *then* $f_\varepsilon \rightrightarrows f$ *in any* $W' \Subset W$;

(c) *if* $1 \le p < \infty$, $f \in L^p_{\mathrm{loc}}(W)$ *and* $W' \Subset W$, *then for* $\varepsilon > 0$ *small enough it results* $\|f_\varepsilon\|_{L^p(W')} \le \|f\|_{L^p(W)}$ *and* $f_\varepsilon \to f$ *in* $L^p_{\mathrm{loc}}(W')$;

(d) *if* $f \in BMO(V)$ *then* $f_\varepsilon \in BMO(W)$, *moreover, for* $\varepsilon > 0$ *small enough it results*

$$\|f_\varepsilon\|_{BMO(W)} \le \widetilde{c}\,\|f\|_{BMO(V)}$$

where $\widetilde{c}$ is an absolute positive constant.

Proof. We prove (d) since the other proofs are quite standard. Let $B(\xi_0, r)$ be a ball centered in $\xi_0 \in W$, and $c \in \mathbf{R}$; for $\varepsilon > 0$ small enough it results

$$(\mathrm{I}) \qquad \int_{B(\xi_0,r)} |f_\varepsilon(\xi) - c|\,d\xi \le \int_{B(\xi_0,r)} \int_{B(0,\vartheta\varepsilon)} |f(\Theta_\xi^{-1}(y)) - c|\varphi_\varepsilon(y)\,dy\,d\xi$$
$$= \int_{B(0,\vartheta\varepsilon)} \varphi_\varepsilon(y) \int_{B(\xi_0,r)} |f(\Theta_\xi^{-1}(y)) - c|\,d\xi\,dy.$$

Applying Theorem 3.2 to the set $E = B(\xi_0, r)$, there exists $\sigma \in]0,1[$ such that $\exp\left(\sum_{j=1}^N y_j X_j\right)\xi \in B(\xi, \sigma r) \subseteq B(\xi, r)$, for any $\xi \in B(\xi_0, r)$ and for any $y \in \mathrm{Box}\,(\sigma r)$. By the very definition of homogeneous norm, it is possible to choose $\varepsilon > 0$ small enough such that $B(0, \vartheta\varepsilon) \subseteq \mathrm{Box}\,(\sigma r)$. Let $\kappa > 1$ be an absolute positive constant—independent of ξ_0—such that $B(\xi, r) \subset B(\xi_0, \kappa r)$, for any $\xi \in B(\xi_0, r)$. Finally observe that the inverse function of the map $B(\xi_0, r) \ni \xi \to \eta = \Theta_\xi^{-1}(y) \in B(\xi_0, \kappa r)$, has a uniformly bounded jacobian with respect to $y \in B(0, \vartheta\varepsilon)$. So one can find an absolute positive constant $\widetilde{c}$ such that (I) yields us

$$\int_{B(\xi_0,r)} |f_\varepsilon(\xi) - c|\,d\xi \le \widetilde{c} \int_{B(0,\vartheta\varepsilon)} \varphi_\varepsilon(y) \int_{B(\xi_0,\kappa r)} |f(\eta) - c|\,d\eta\,dy,$$

from which the thesis follows. □

Last property allows us to extend to the setting of these Carnot–Carathéodory metric spaces the density result first proved by Sarason in the Euclidean setting; first we need the two following lemmas. We thank Marco Bramanti for useful hints in the proof of the second one.

Lemma 3.6. *There exists an absolute positive constant K such that, if $a > 0$, for all $f \in BMO(V)$ there exists a function $g \in C^\infty(W)$ such that*

$$\|f - g\|_{BMO_{\mathscr{C}}(W)} \le K\, M_{\mathscr{C},a}(f).$$

Proof. Fix $f \in BMO_{\mathscr{C}}(W)$, $a > 0$, and $l > M_{\mathscr{C},a}(f)$. Let $\overline{k}$ be an integer to be fixed later and let h be the step function assuming the value $f_{Q^{\overline{k}}_\alpha}$ on $Q^{\overline{k}}_\alpha$. Now we estimate $\|f-h\|_{BMO_{\mathscr{C}}(W)}$. Let Q be a cube made up around a dyadic cube of some generation k; taking $k' \geq \max\{k, \overline{k}\}$ we can also write $Q = \bigcup_{\alpha=1}^m Q^{k'}_\alpha$. It results

$$⨍_Q |f-h-(f-h)_Q|\,dx \leq 2 ⨍_Q |f-h|\,dx = \frac{2}{|Q|}\sum_{\alpha=1}^m \int_{Q^{k'}_\alpha} |f-f_{Q^{k'}_\alpha}|\,dx \leq 2\,l.$$

According to Lemma 3.3, for any $\varepsilon \leq \delta^{k'}$ small enough, let us consider the function h_ε. If $\xi \in W$, then, up to a Lebesgue negligible set, ξ belongs to some $Q^{k'}_\alpha$. By Lemma 2.7 there exist at most $\overline{C}$ dyadic cubes of generation k' that intersect the ball $B(\xi,\varepsilon)$. Let $s \leq \overline{C}$ the maximum step of contiguity of all such cubes with respect to $Q^{k'}_\alpha$; define Q' as the union of $Q^{k'}_\alpha$ with its contiguous cubes up to the step s: namely $Q' = \bigcup_{\beta=1}^t Q^{k'}_\beta$, where $t \leq c = C' + C'^{\,2} + \cdots + C'^{\,\overline{C}}$; moreover $\operatorname{diam}(Q') \leq p(C',\overline{C})\, c_1\, \delta^{k'}$, where $p(C',\overline{C})$ is a polynomial depending only on the embraced constants. Choosing so $\overline{k}$ such that $p(C',\overline{C})\, c_1\, \delta^{\overline{k}} < a$, we have, for any $\beta = 1, 2, \ldots, t$,

$$|f_{Q^{k'}_\beta} - f_{Q'}| \leq ⨍_{Q^{k'}_\beta} |f - f_{Q'}|\,d\xi \leq \frac{|Q'|}{|Q^{k'}_\beta|} ⨍_{Q'} |f - f_{Q'}|\,d\xi \leq \overline{c}\,c\,l,$$

where $\overline{c}$ is the constant as in Proposition 2.14. So, for any $\beta_1, \beta_2 = 1, 2, \ldots, t$, it results

$$|f_{Q^{k'}_{\beta_1}} - f_{Q^{k'}_{\beta_2}}| \leq |f_{Q^{k'}_{\beta_1}} - f_{Q'}| + |f_{Q'} - f_{Q^{k'}_{\beta_2}}| \leq 2\,\overline{c}\,c\,l,$$

from which

$$|h(\xi) - h_\varepsilon(\xi)| \leq \int_{B(\xi,\varepsilon)} |h(\xi) - h(\eta)| \varphi_\varepsilon(\Theta_\xi(\eta)) J_\xi(\eta)\,d\eta \leq 2\,\overline{c}\,c\,l.$$

Now we can estimate the $\|f - h_\varepsilon\|_{BMO_{\mathscr{C}}(W)}$:

$$\begin{aligned}\|f-h_\varepsilon\|_{BMO_{\mathscr{C}}(W)} &\leq \|f-h\|_{BMO_{\mathscr{C}}(W)} + \|h-h_\varepsilon\|_{BMO_{\mathscr{C}}(W)} \leq \\ &\leq 2l + 2\|h-h_\varepsilon\|_\infty \leq K\,l,\end{aligned}$$

where $K = 2(1 + \overline{c}\,c)$. □

Lemma 3.7. *Let $\Omega'' \Subset \Omega' \subseteq \mathbf{R}^N$ be open subsets. Then there exists an absolute positive constant K such that, if $a > 0$ and $f \in BMO(\Omega')$, then there exists a function $g \in C^\infty(\Omega'')$ such that*

$$\|f-g\|_{BMO_{\mathscr{C}}(\Omega'')} \leq K\,M_{\mathscr{C},a}(f).$$

Proof. Since $\overline{\Omega''}$ is compact, it is a finite union of suitable balls B_i; using a partition of unity related to these balls we can construct a function $g \in C^\infty(\Omega'')$ and, arguing as in Lemma 4.4 of [3], we can control the $BMO(\Omega'')$ norm of $f - g$ with the norm $BMO(B_i)$. □

So our density result follows.

Theorem 3.8. *Let $\Omega'' \Subset \Omega' \subseteq \mathbf{R}^N$ be open subsets, and $f \in VMO(\Omega')$. Then there exists a sequence $\{f_n\}$ in $C^\infty(\Omega'')$ such that $f_n \to f$ in $BMO(\Omega'')$. Moreover $f_n \to f$ a.e. in Ω''.*

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