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TOPOLOGICAL LOCALLY FINITE MV-ALGEBRAS AND RIEMANN SURFACES

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Abstract. It is known that any MV-algebra is a topological MV-algebra. For a locally finite MV-algebra A with some algebraic and topological conditions the product $A \times A$ becomes a compact Riemann surface (modulo conformal equivalence). Topologically, it is a torus.

1. Introduction

In this paper we consider a relationship between locally finite MV-algebras and Riemann surfaces understanding MV-algebras as topological MV-algebras. The idea was motivated by the work of Hoo where he pointed out that a locally finite MV-algebra may be Hausdorff and connected (one of the three possibities) [4]. This enables to make $A \times A$ first into a surface and, then leading to a compact Riemann surface $A \times A$ (up to a conformal mapping) as follows: Compactness means that also A ought to be compact. As a matter of fact, assuming A to be algebraically complete and infinite, we can keep $A \times A$ (up to homeomorphism) as a product of the real unit interval $[0,1] \times [0,1]$ endowed with the product topology of the relative usual topology on $[0,1]$. Then the interior of the square with vertices at 0, 1, $1+i$, i in the complex plane is the fundamental domain linked up with the Riemann surface $A \times A$.

2. Preliminaries

2.1. MV**-algebras.** An MV-algebra is a system $A = (A, \oplus, \odot, *, 0, 1)$ such that $(A, \oplus, 0)$ is an Abelian monoid, $x \oplus 1 = 1$, $x^{**} = x$, $0^* = 1$, $x \odot y = (x^* \oplus y^*)^*$, $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$ for all $x, y \in A$. By setting $x \vee y = (x \odot y^*) \oplus y$, $x \wedge y = (x \oplus y^*) \odot y$ and $x \leq y$ iff $x \wedge y = x$ for all $x, y \in A$, the system $L(A) = (A, \vee, \wedge, \leq, 0, 1)$ is a bounded distributive lattice with smallest element 0 and greatest element 1. Moreover, $(x \vee y)^* = x^* \wedge y^*$, $(x \wedge y)^* = x^* \vee y^*$ [7], p. 23. For MV-algebras we also refer to [2] and for lattices to [1]. An equivalent reformulation of an MV-algebra A is obtained by defining a binary operation $\to$ and a unary operation $\neg$ as follows: $(x, y) \mapsto x^* \oplus y = x \to y$ (implication) and $x \mapsto x^* = \neg x$ (negation) for all $x, y \in A$ [3], pp. 78–79.

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A lattice L is complete iff each of its subsets has a supremum and an infimum in L [1], p. 6, [3], p. 129. An MV-algebra A is complete iff its underlying lattice $L(A)$ is complete [3], p. 129. A set X is order-complete relative to an antisymmetric and transitive relation iff each non-void subset of X which has an upper bound has a supremum. Equivalently, they have infima [5], p. 14. In this paper, the above order coincides with the used lattice order. Since $L(A)$ is bounded, A is complete iff A is order-complete.

Let L be a lattice with smallest element 0. An atom of L is an element $a \in L$ such that $a > 0$ and whenever $x \in L$ and $x \leq a$ then either $x = 0$ or $x = a$ [3], p. 122. By an atom of an MV-algebra we mean an atom of the underlying lattice $L(A)$. A is atomic iff for every $0 \neq x \in A$ there is an atom $a \in A$ with $a \leq x$. A is atomless iff no element of A is an atom. [3], p. 132.

An MV-algebra A is linearly ordered iff for every $x, y \in A$ either $x \leq y$ or $y \leq x$ [2], p. 477. Let 0 and 1 be smallest and greatest elements in an MV-algebra A. Then A is locally finite iff every element $x \in A$ different from 0 has a finite order. This means that for every $0 \neq x \in A$ there is the least integer m such that $mx = x \oplus x \oplus \cdots \oplus x = 1$. Every locally finite MV-algebra is linearly ordered [2], pp. 476–477. Also, any locally finite MV-algebra is isomorphic to a subalgebra of the real unit interval $[0, 1]$ [3], p. 70. The relationship between complete and atomic or atomless locally finite MV-algebras are considered in [3], pp. 132–133.

Endow $[0, 1]$ with the Lukasiewicz structure by setting $x \oplus y = \min\{1, x + y\}$, $x \odot y = \max\{0, x + y - 1\}$ and $x^* = 1 - x$ for all $x, y \in [0, 1]$. Then the lattice order of an MV-algebra coincides with the natural order and $[0, 1]$ becomes a complete locally finite MV-algebra [7], p. 23. Binary and unary operations are defined by $(x, y) \mapsto \min\{1, 1 - x + y\}$ and $x \mapsto 1 - x$ [3], p. 78.

2.2. Riemann surfaces. Throughout this section, all the concepts are from complex analysis, referenced as [6] and [8].

Definition 1. A surface S is a connected Hausdorff space with a countable base for topology which is locally homeomorphic to the complex plane $\mathbf{C}$ (or $\mathbf{R}^2$).

Let S be a surface. Then for every point $p \in S$ there is an open neighborhood $U \subset S$ of p and a homeomorphism $h\colon U \to h(U)$. The mapping h is called a local parameter at p on S. The pair (U, h) is called a coordinate chart. An atlas of S is the collection $\{(U_i, h_i) \mid i \text{ is an index}\}$ of coordinate charts of S if $\bigcup\{U_i \mid i \text{ is an index}\} = S$. Let $h_1\colon U_1 \to h(U_1)$ and $h_2\colon U_2 \to h(U_2)$ be two local parameters on S such that $U_1 \cap U_2 \neq \emptyset$. A homeomorphism $h_2 h_1^{-1}\colon h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2)$ is called a parameter transformation.

An atlas is said to be complex analytic if all the parameter transformations are analytic homeomorphisms. The complex analytic atlases U^* and V^* are said to be equivalent if $U^* \cup V^*$ is a complex analytic atlas. An equivalence class of complex analytic atlases is called a conformal (a complex analytic) structure of the surface S. Let us define Riemann surfaces as follows [6], p. 130:

Definition 2. A surface with a conformal structure is a Riemann surface.

A smooth covering surface of a surface S is a pair (W, p), where W is a surface and $p\colon W \to S$ is a local homeomorphism. The mapping p is called a projection [6], p. 135.

Let S be a surface and (W, p) its smooth covering surface. A cover transformation $g\colon W \to W$ over S is a homeomorphism satisfying $pg = p$. All such homeomorphisms g form a covering group G of W over S [6], p. 138. It is known that for the smooth covering surface of a Riemann surface, cover transformations are conformal [6], p. 142.

Any surface S may have a universal covering surface D which is a simply connected surface with a projection $p\colon D \to S$ [8], p. 85. Roughly speaking, a surface S is simply connected if there are no holes on S. According to the Riemann mapping theorem [6], p. 143, we can normalize any universal covering surface of a Riemann surface to the unit disc, the complex plane, or the extended plane [6], p. 144. Let D be a universal covering surface of the Riemann surface S. Being conformal mappings, cover transformations $g\colon D \to D$ are Möbius transformations [6], p. 144, forming a covering group G of D over S.

Let G be a covering group of the universal covering surface D over S. If there exists $g \in G$ such that $g(z_1) = z_2$ for some $z_1, z_2 \in D$, then we say that the points z_1 and z_2 are equivalent under G. This means that the points z_1 and z_2 have the same projection on S. The connection between the Riemann surface S and the quotient surface D/G is well known [6], p. 144:

Proposition 1. [6] *Given an arbitrary Riemann surface S, let D be its universal covering surface, and G the covering group of D over S. Then S is conformally equivalent to the Riemann surface D/G.*

A subdomain of D is said to be a fundamental domain of G if it contains at most one point of every equivalence class of D/G and its closure in D meets every equivalence class [6], p. 149.

3. Topological locally finite MV-algebras

Hoo showed in [4] that any MV-algebra is a topological MV-algebra having then a topology. For locally finite MV-algebras he proved:

Proposition 2. [4] *The topology on a locally finite MV-algebra is one of the following types:*

(i) *Hausdorff and connected,*
(ii) *Hausdorff and totally disconnected,*
(iii) *the trivial topology.*

Let A be a locally finite MV-algebra. Making $A \times A$ into a surface, A should be connected and Hausdorff. Therefore we ignore the cases (ii) and (iii). Suppose that, in the case (i), A is infinite and complete as a complete lattice. Then it is not necessary to assume A to be Hausdorff since, by the following lemma, A has the order topology which implies Hausdorff.

Lemma 1. *Let A be a complete locally finite MV-algebra which is connected. Then, if A is infinite*

(1) *A has necessary the order topology,*
(2) *A has a countable base for the order topology,*
(3) *there exists an isomorphism $f\colon A \to [0,1]$ which is a homeomorphism.*
(4) *By [3], a locally finite MV-algebra A is finite iff A is isomorphic to the Lukasiewicz chain*

$$L_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \cdots, \frac{n-2}{n-1}, 1\}$$

for some integer $n \geq 2$.

Proof. In accordance to [3], the fourth assertion is proved, p. 71, A is a locally finite MV-algebra iff A is isomorphic to a subalgebra of $[0,1]$, p. 70, and all infinite subalgebras of $[0,1]$ are atomless, p. 132. By [3], p. 133, all locally finite, complete and atomless MV-algebras are isomorphic to $[0,1]$. An isomorphism $f\colon A \to [0,1]$ carries an MV-structure of A to the Lukasiewicz structure of $[0,1]$: Knowning that any locally finite MV-algebra is isomorphic to a subalgebra of the Lukasiewicz structure, say B, hence making $[0,1]$ and B isomorphic, we can endow $[0,1]$ with the Lukasiewicz structure.

Next the notation $a < b$ means that $a \leq b$ and $a \neq b$. The order in a locally finite MV-algebra A (as a linearly ordered MV-algebra) is said to have a gap if there are points a, $b \in A$ so that $a < b$ and there is no point $c \in A$ such that $a < c < b$ [5], p. 58. Since A is atomless as infinite, for $a < b$, $a, b \in A$, there exists $c \in A$ such that $a < c < b$, concluding with [2], p. 480. Hence there are no gaps on A. Summarizing, A is connected, infinite, complete (as a complete lattice) and there are no gaps on A. In fact, it is known that A is connected relative to the order topology iff A is complete (order-complete) and there are no gaps on A [5], p. 58. Being connected, A has necessary the order topology (the first assertion) with a subbase consisting of all sets of the form $\{x \mid 0 \leq x < a\}$ or $\{x \mid a < x \leq 1\}$ for some $a \in A$ where 0 and 1 are smallest and greatest elements in A.

Next we show that the isomorphism $f\colon A \to [0,1]$ is a homeomorphism: As an isomorphism, f is an order-preserving bijection [1], p. 3. For the subbase of A, image sets $\{f(x) \mid 0 \leq f(x) < f(a)\}$ and $\{f(x) \mid f(a) < f(x) \leq 1\}$ form a subbase of $[0,1]$ for the order topology. Then f maps a subbase of A to a subbase of $[0,1]$ and consequently, a base to a base. Being a bijection, f is a homeomorphism. The third assertion is proved.

Sets of the form $\{y \mid 0 \leq y < b\}$ and $\{y \mid b < y \leq 1\}$ for some $b \in [0,1] \cap Q$, constitute a countable subbase of $[0,1]$ meaning that $[0,1]$ has a countable base for the order topology. Consequently, A has such a base. This completes the second assertion. □

The following example illustrates the necessity of completeness of A:

Example 1. By [2], pp. 473–474, rational numbers of $[0,1]$ form a subalgebra of the Lukasiewicz structure which is not complete but, as (denumerably) infinite, atomless. In general, a locally finite MV-algebra A has only two MV-ideals, namely $\{0\}$ and A itself. By the proof of Proposition 2, given in [4], pp. 107–108, every connected component C of 0 is a closed ideal of A meaning that $C = \{0\}$ or $C = A$. In the first case, the component of any point x is a singleton, i.e. $\{x\}$, making A totally disconnected. In our example, we ignore this. In the second case, A is connected (however, not assume to have the trivial topology). As conclusion, since $[0,1] \cap Q$ is not isomorphic to the interval $[0,1]$, the assertion (3) of Lemma 1 is not satisfied due to A is not complete.

4. Product of two topological locally finite MV-algebras as a Riemann surface

4.1. The Riemann surface $A \times A$. Suppose that A is any topological MV-algebra. Let $p_1\colon A \times A \to A$, $p_1(a,b) = a$ and $p_2\colon A \times A \to A$, $p_2(a,b) = b$ be projections for a topology on A. If W_1 and W_2 are open neighborhoods of the points a and b, then $W = p_1^{-1}(W_1) \cap p_2^{-1}(W_2)$ is an open neighborhood of the point (a,b) for the product topology on $A \times A$.

Let A be a complete locally finite and infinite MV-algebra which is connected. According to Lemma 1, there is a homeomorphism $f\colon A \to [0,1]$.

Lemma 2. *Mapping $h\colon A \times A \to [0,1] \times [0,1]$*

$$h(a,b) = (f(a), f(b))$$

is a homeomorphism.

Proof. Let $f\colon A \to [0,1]$ be a homeomorphism. Since f is bijective, h is bijective and since f is continuous, h is continuous (the components of h are continuous). Further, $f^{-1}\colon [0,1] \to A$ is a homeomorphism and so continuous. Hence $h^{-1}\colon [0,1] \times [0,1] \to A \times A$, $h^{-1}(x,y) = (f^{-1}(x), f^{-1}(y))$ is continuous. $\square$

Remark 1. The Lukasiewicz operations $[0,1] \times [0,1] \to [0,1]$, $(x,y) \mapsto \min\{1, 1 - x + y)$ and $[0,1] \to [0,1]$, $x \mapsto 1 - x$ are continuous in the order topology. Since $A \times A$ and $[0,1] \times [0,1]$ while A and $[0,1]$ are homeomorphic, the corresponding operations $A \times A \to A$, $(x,y) \mapsto x^* \oplus y$ and $A \to A$, $x \mapsto x^*$ are also continuous making A a topological MV-algebra for the order topology.

Remark 2. The order topology in the Lukasiewicz structure on $[0,1]$ generated by half-open sets of the form $[0,a)$ and $(a,1]$ for some $a \in [0,1]$ coinsides with the relative usual topology on $[0,1]$. Accordingly, the corresponding product topologies are the same. In the next proof of Proposition 3, this enables to pass from the Lukasiewicz structure to the ordinary algebra of the real axis without changing the topology.

Proposition 3. *Let A be a complete locally finite MV-algebra which is infinite and topologically connected. Then $A \times A$ is a Riemann surface.*

Proof. The following conditions are satisfied:

(i) Since A is Hausdorff and connected, $A \times A$ is Hausdorff and connected.

(ii) The product topology on $A \times A$ has a countable base concluding with Lemma 1.

(iii) Let $G = \langle z \mapsto z+1, z \mapsto z+i \rangle$ be a group generated by translations $z \mapsto z+1$ and $z \mapsto z+i$. The quotient space $\mathbf{C}/G$ consists of equivalence classes in the complex plane obtained by setting a relation $z_1 \sim z_2$ iff there exists $g \in G$ such that $g(z_1) = z_2$. The topology on $\mathbf{C}/G$ is the quotient topology:

$$V \in \mathbf{C}/G \text{ is open} \quad \text{iff} \quad \hat{f}^{-1}(V) \text{ is open in } \mathbf{C}$$

where $\hat{f} \colon \mathbf{C} \to \mathbf{C}/G$ is a canonical projection and $\mathbf{C}$ is equipped with the order topology on the plane. Identifying the square $[0,1] \times [0,1]$ in the plane $\mathbf{R}^2$ and the square P with vertices at 0, 1, $1+i$, i in the complex plane, the relative topology on P yields also a projection

$$\hat{f} \mid P \colon P \to \mathbf{C}/G,$$

the opposite sides of the square having the same image. As conclusion, by Lemma 2, there is a local homeomorphism

$$\hat{h} \colon A \times A \to \mathbf{C}/G.$$

Let $w \in A \times A$ and W an open neighborhood of w. Then $\hat{h}(W)$ is an open neighborhood of $\hat{h}(w)$ in $\mathbf{C}/G$, equivalently, $\hat{f}^{-1}(\hat{h}(W))$ is open in $\mathbf{C}$. We conclude that there is a local homeomorphism

$$(\hat{f}^{-1}\hat{h}) \colon A \times A \to \mathbf{C}, \quad W \mapsto (\hat{f}^{-1}\hat{h})(W)$$

for every open neighborhood W of w meaning that $A \times A$ is locally homeomorphic to $\mathbf{C}$.

It follows from (i), (ii) and (iii) that $A \times A$ is a surface. We still show that the surface $A \times A$ has a conformal structure:

(iv) Since $\hat{h} \colon A \times A \to \mathbf{C}/G$ is a local homeomorphism, $A \times A$ is a smooth covering surface of the Riemann surface $\mathbf{C}/G$. By [6], p. 142, $A \times A$ has a unique conformal structure obtained by lifting the conformal structure of $\mathbf{C}/G$. □

4.2. Geometric interpretation of the Riemann surface $A \times A$. According to the Riemann mapping theorem [6], p. 143–144, every Riemann surface admits as its universal covering surface the unit disc, the complex plane, or the extended plane. In our case, we obtain

Lemma 3. *The complex plane $\mathbf{C}$ is the normalized universal covering surface of the Riemann surface $A \times A$.*

Proof. By the proof of Proposition 3, there is a continuous mapping $\hat{h} \colon A \times A \to \mathbf{C}/G$ (as a local homeomorphism). By [6], p. 145, based on the Riemann mapping theorem, it is possible to choose a common universal covering surface D of the Riemann surfaces $A \times A$ and $\mathbf{C}/G$ such that D is the unit disc, the complex plane, or the extended plane. Therefore, as the universal covering surface of $\mathbf{C}/G$, $\mathbf{C}$ is conformally equivalent to D. Necessary, $D = \mathbf{C}$. $\square$

Lemma 4. *The group $G =< z \mapsto z+1, z \mapsto z+i >$ generated by mappings $z \mapsto z+1$ and $z \mapsto z+i$ is a covering group of the universal covering surface $\mathbf{C}$ over $A \times A$.*

Proof. Denote by A_{mn} the square in the complex plane with vertices at (m, ni), $(m+1, ni)$, $(m+1, (n+1)i)$ and $(m, (n+1)i)$ for any $m, n \in \mathbf{Z}$. Let us define

$$\bar{g}_{klmn} \colon A_{mn} \to A_{(m+k)(n+l)}, \quad \bar{g}_{klmn} = w_1^k w_2^l$$

for some $k, l \in \mathbf{Z}$ and any $m, n \in \mathbf{Z}$, where $w_1(z) = z+1$ and $w_2(z) = z+i$. Further, let

$$\bar{g}_{kl} \colon \mathbf{C} \to \mathbf{C}, \quad \bar{g}_{kl} \mid A_{mn} = \bar{g}_{klmn}$$

Clearly, $\bar{g}_{kl}$ is a continuous bijection with a continuous inverse $\bar{g}_{kl}^{-1} = w_2^{-l} w_1^{-k}$. Hence $\bar{g}_{kl}$ is a homeomorphism. In another way, being Möbius transformations, w_1 and w_2 as homeomorphisms implies $\bar{g}_{kl}$ to be a homeomorphism. For $z \in A_{mn} \subset \mathbf{C}$ and the projection $p = \hat{h}^{-1}\hat{f} \colon \mathbf{C} \to A \times A$ we obtain

$$\begin{aligned} p(\bar{g}_{kl}(z)) &= (\hat{h}^{-1}\hat{f})(\bar{g}_{kl}(z)) = (\hat{h}^{-1}\hat{f})(\bar{g}_{klmn}(z)) = \hat{h}^{-1}(\hat{f}(\bar{g}_{klmn}(z))) \\ &= \hat{h}^{-1}(\hat{f}(z+k+li)) = \hat{h}^{-1}(\hat{f}(z)) = (\hat{h}^{-1}\hat{f})(z) = p(z). \end{aligned}$$

The mappings $\bar{g}_{kl} \colon \mathbf{C} \to \mathbf{C}$ are cover transformations and form the covering group G of $\mathbf{C}$ over $A \times A$. $\square$

Proposition 4. *Let us assume*

(i) *A is a complete locally finite MV-algebra,*
(ii) *A is infinite,*
(iii) *A is connected as a topological MV-algebra.*

Then $A \times A$ is a Riemann surface which is conformally equivalent to a compact Riemann surface. Topologically, it is a torus.

Proof. The structure of the proof is partly based on [6], p. 150. By Proposition 3, and Lemmas 3 and 4, $A \times A$ is a Riemann surface having $\mathbf{C}$ as its normalized universal covering surface with the covering group $G =< z \mapsto z+1, z \mapsto z+i >$. It follows from Proposition 1 that $A \times A$ is conformally equivalent to the Riemann surface $\mathbf{C}/G$. A fundamental domain is the interior of the square P with vertices at 0, 1, $1+i$, i. The canonical projection $\mathbf{C} \to \mathbf{C}/G$ maps the compact closure of P onto $\mathbf{C}/G$. As the image of this continuous mapping, $\mathbf{C}/G$ is compact making $A \times A$ conformally equivalent to a compact Riemann surface. The opposite sides of the closure clP are equivalent under G. Identifying them, $\mathbf{C}/G$ becomes a torus. $\square$

Corollary 1. *$A \times A$ is compact Hausdorff.*

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