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A REDUCTION FOR ASYMPTOTIC TEICHMÜLLER SPACES

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To the memory of Professor Nobuyuki Suita.

Abstract. In this paper, we will introduce a new kind of isomorphism theorem (we call it the reducing theorem) for asymptotic Teichmüller spaces. Our isomorphism theorem is induced by (conformal) 2-surgery operations along simple closed loops on surfaces, and yields several interesting and pathological phenomena on the structures of asymptotic Teichmüller spaces.

1. Introduction

In this paper, we will give a new kind of isomorphism theorem for asymptotic Teichmüller spaces of Riemann surfaces. By definition, asymptotic Teichmüller spaces are recognized as the deformation space of ends of Riemann surfaces. Intuitively, one would think that asymptotic Teichmüller space admits "product structures" inherited from the structure of the end of corresponding Riemann surface, since each neighborhood of any end is deformed independently of the other ends. In this paper, we will give a certain concrete expression for this intuition.

To be more precise, let $AT(R)$ denote the asymptotic Teichmüller space of a Riemann surface R. Let c be a homotopically non-trivial simple closed curve on R. We apply a conformal 2-surgery to R along c (cf. §4.1 and see also Figure 2). Then, the resulting manifold consists of either two surfaces S_1 and S_2 when c is a separating loop, or one surface S_0 otherwise.

Main Theorem (Reducing Theorem). *One of the following holds:*

(1) *$AT(R)$ is biholomorphically equivalent to the product $AT(S_1) \times AT(S_2)$ if c is a separating loop, or*
(2) *$AT(R)$ is biholomorphically equivalent to $AT(S_0)$ otherwise.*

Namely, our reductions allow us to represent any asymptotic Teichmüller space as the direct product of two asymptotic Teichmüller spaces of "simpler" Riemann surfaces than given one. We will deal with the detail of our main theorem in §4.3 (Theorem 4.1).

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Usually (or from empirical observations), conformal 2-surgery operations do not seem to fit to the complex analytic theory of Teichmüller spaces. However, these operations adjust to the complex analytic theory of asymptotic Teichmüller spaces. One reason why the operations work in this theory is that we can always ignore deformations on any compact sets (rel the ideal boundaries).

Structures of asymptotic Teichmüller spaces. From our main theorem, we obtain several observations on the structures of asymptotic Teichmüller spaces. Indeed, the following corollaries will be discussed in §5.

Corollary 1. (Deformations are realized at ends.) *Let R be a Riemann surface and Z a regular domain in R. Suppose $C_1, ..., C_n$ are the common boundary curves of Z and $R - Z$, and $S_1, ..., S_m$ are the components of $R - \overline{Z}$, with capping disks along the boundary curves C_j. Then, $AT(R)$ is biholomorphically equivalent to the product $\prod_{i=1}^{m} AT(S_i)$.*

See §3 for the definition of regular domains. Corollary 1 immediately leads the following two results.

Corollary 2. *Let R be a Riemann surface of finite genus. Then there is a closed set E in $\widehat{\mathbf{C}}$ such that $AT(R)$ is biholomorphically equivalent to $AT(\widehat{\mathbf{C}} - E)$.*

Corollary 3. *Let R be a Riemann surface of topologically finite type. Then, $AT(R)$ is biholomorphically equivalent to the product space $AT(\mathbf{D})^m$, where m is the number of funnels of R and $\mathbf{D}$ is the unit disk in $\mathbf{C}$.*

The structures of asymptotic Teichmüller spaces seem to be well-behaved with the self-similarity of the ends as follows (see also §3 of [2]):

Corollary 4. *Let $C \subset \widehat{\mathbf{C}}$ be the middle thirds Cantor set and $\Omega = \widehat{\mathbf{C}} - C$. Then $AT(\Omega)$ is biholomorphically equivalent to the product space $AT(\Omega)^m$ for all $m \in \mathbf{N}$.*

Since $AT(\mathbf{D})$ is homogeneous (cf. [3]. See also [11]), by Corollary 3, we have

Corollary 5. *When R is of topologically finite type, the automorphism group of $AT(R)$ acts transitively on $AT(R)$.*

The main theorem and its corollaries intimate that asymptotic Teichmüller spaces have intriguing (or pathological) structures. The author hopes that some of the results in this paper, when used more carefully, will yield more informations about the structures of asymptotic Teichmüller spaces.

This paper is organized as follows: In §3, we will give modifications of quasiconformal mappings and quasiconformal isotopies, which are important for the proof of our main theorem. In §4 our main theorem will be restated and proved. In §5, we will prove corollaries stated above.

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2. Notation

2.1. Quasiconformal isotopies. Let R be a hyperbolic Riemann surface. Let Γ be the Fuchsian group acting on $\mathbf{D}$ with $\mathbf{D}/\Gamma = R$ and denote by $\Lambda(\Gamma)$ the limit set of Γ. Then $\overline{R} = (\overline{\mathbf{D}} - \Lambda(\Gamma))/\Gamma$ is an orbifold with interior $R = \mathbf{D}/\Gamma$ and boundary $(\partial\mathbf{D} - \Lambda(\Gamma))/\Gamma$. We say that $(\partial\mathbf{D} - \Lambda(\Gamma))/\Gamma$ is the *ideal boundary* of R, and denote it by $\partial^{id}R$. When R is not hyperbolic, we define $\partial^{id}R$ to be the set of its punctures (possibly empty). In any case, we set $\overline{R} = R \cup \partial^{id}R$. Notice that any quasiconformal mapping between two Riemann surfaces can be extended to their ideal boundaries (e.g. §8 of Chapter I in [8]).

Let R and S be Riemann surfaces and C a set in $\overline{R}$. We say that a continuous map $H\colon R \times [0,1] \to S$ is a *homotopy rel* C (which we denote by $H(p,t)$ or $H_t(x)$) if for each $t \in [0,1]$, H_t extends to $\overline{R}$ continuously and $H_t(p) = H_0(p)$ for all $(p,t) \in C \times [0,1]$. A *quasiconformal isotopy rel* C is a homotopy rel C with the additional property that there is a constant $K > 1$ such that $H_t\mid_R$ is a K-quasiconformal homeomorphism for all $t \in [0,1]$. Two quasiconformal mappings f and g from R to S are said to be *quasiconformally isotopic* (resp. *homotopic*) *rel* C if there is a quasiconformal isotopy (resp. a homotopy) H rel C with $H_0 = f$ and $H_1 = g$.

The following is due to Earle and McMullen.

Proposition 2.1. (Theorem 1.1 in [5]) *The following three conditions are equivalent for any two quasiconformal mappings f and g between hyperbolic Riemann surfaces R and S:*

(1) *f and g are quasiconformally isotopic rel $\partial^{id}R$;*
(2) *f and g are homotopic rel $\partial^{id}R$; and*
(3) *f and g have lifts to the universal cover $\mathbf{D}$ which agree on $\partial\mathbf{D}$.*

It is easy to check that (1) and (2) above are also equivalent even in the case when R and S are not hyperbolic.

2.2. Asymptotic Teichmüller spaces. Let R be a Riemann surface and $L^\infty(R)$ denote the space of bounded measurable $(-1,1)$-forms on R. We say that $\mu \in L^\infty(R)$ *vanishes at infinity* on R when for any $\varepsilon > 0$, there is a compact set $C \subset R$ such that $|\mu| < \varepsilon$ a.e. on $R - C$. We denote by $L_0^\infty(R)$ the closed subspace of $L^\infty(R)$ consisting of all $\mu \in L^\infty(R)$ vanishing at infinity. A quasiconformal mapping f on R is said to be *asymptotically conformal* if its complex dilatation vanishes at infinity. Set $\widehat{L}(R) = L^\infty(R)/L_0^\infty(R)$.

The *asymptotic Teichmüller space* $AT(R)$ of R is, by definition, the space of the equivalence classes of quasiconformal mappings f from R onto a variable Riemann surface $f(R)$. Two mappings f from R to R_0 and g from R to R_1 are *equivalent* if there is an asymptotically conformal mapping $h\colon R_0 \to R_1$ such that $h\circ f$ and g are quasiconformally isotopic rel $\partial^{id}R$. We denote by $[f]$ the equivalence class of f in $AT(R)$. It is known that $AT(R)$ admits the natural structure of a complex Banach manifold (cf. [2]). The *Teichmüller space* $T(R)$ of R has the same definition with one exception. The mapping h has to be conformal. Since conformal mappings are asymptotically conformal, there is a canonical projection $T(R) \to AT(R)$.

If R is a Riemann surface of analytically finite type, $AT(R)$ consists of one point (cf [2]). Indeed, any quasiconformal mapping on R is isotopic (rel $\partial^{id}R$) to a quasiconformal mapping which is conformal outside a compact set in R (cf. Proposition 3.2).

2.3. Complex structure on $AT(R)$**.** Let R be a Riemann surface. Let $M(R)$ be the open unit ball in $L^\infty(R)$. Then there is a canonical projection $\Phi\colon M(R) \to T(R)$. Namely, $\Phi(\mu)$ is the equivalence class (in $T(R)$) of a quasiconformal mapping on R whose complex dilatation is μ. This projection is called the *Bers projection.*

Let $\widehat{M}(R)$ is the unit ball in $\widehat{L}(R)$. In [2], Earle, Gardiner and Lakic proved the existence of a holomorphic splitting submersion $\widehat{\Phi}_R\colon \widehat{M}(R) \to AT(R)$ with the following diagram is commutative:

$$\begin{array}{ccc} M(R) & \xrightarrow{\Phi} & T(R) \\ {\scriptstyle P_R}\big\downarrow & & \big\downarrow \\ \widehat{M}(R) & \xrightarrow{\widehat{\Phi}_R} & AT(R), \end{array}$$

where the vertical directions are canonical projections.

Lemma 2.1. *Let S_1, S_2 and R be Riemann surfaces (possibly $S_2 = \emptyset$). Let $\Psi\colon AT(S_1)\times AT(S_2) \to AT(R)$ be a map. Suppose that there is a* **C***-linear mapping $\mathscr{L}$ from $L^\infty(S_1) \times L^\infty(S_2)$ to $L^\infty(R)$ such that*

(1) *$\mathscr{L}(M(S_1) \times M(S_2)) \subset M(R)$,*
(2) *$\mathscr{L}(L_0^\infty(S_1) \times L_0^\infty(S_2)) \subset L_0^\infty(R)$, and*
(3) *Ψ and $\mathscr{L}$ satisfy the following commutative diagram:*

$$\begin{array}{ccc} M(S_1)\times M(S_2) & \xrightarrow{\mathscr{L}} & M(R) \\ {\scriptstyle (\widehat{\Phi}_{S_1}\circ P_{S_1})\times(\widehat{\Phi}_{S_2}\circ P_{S_2})}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\Phi}_R\circ P_R} \\ AT(S_1)\times AT(S_2) & \xrightarrow{\Psi} & AT(R). \end{array}$$

Then Ψ is a holomorphic mapping.

Proof. We deal only with the case where all $AT(S_1)$, $AT(S_2)$ and $AT(R_2)$ are not trivial. The other cases can be treated in a similar way.

By the assumption (2), $\mathscr{L}$ descends to a $\mathbf{C}$-linear mapping $\widehat{\mathscr{L}}$ from $\widehat{L}(S_1)\times\widehat{L}(S_2)$ to $\widehat{L}(R)$ satisfying the following commutative diagram

$$\begin{array}{ccccc} M(R_1) & \xrightarrow{P_{S_1}\times P_{S_2}} & \widehat{M}(S_1)\times\widehat{M}(S_2) & \xrightarrow{\widehat{\Phi}_{S_1}\times\widehat{\Phi}_{S_2}} & AT(S_1)\times AT(S_2) \\ {\scriptstyle\mathscr{L}}\big\downarrow & & {\scriptstyle\widehat{\mathscr{L}}}\big\downarrow & & {\scriptstyle\Psi}\big\downarrow \\ M(R) & \xrightarrow{P_R} & \widehat{M}(R) & \xrightarrow{\widehat{\Phi}_R} & AT(R). \end{array}$$

Since $\widehat{\Phi}_{S_i}$ is a holomorphic split submersion, by the implicit function theorem (cf. p. 89 of [12]), we get a neighborhood U_i of $[f_i]$ and a local holomorphic section $s_i\colon U_i \to \widehat{M}(S_i)$ with $\widehat{\Phi}_{S_i}\circ s_i = id$ on U_i. Therefore, Ψ satisfies

$$\begin{aligned}\Psi([g_1],[g_2]) &= \Psi(\widehat{\Phi}_{S_1}\circ s_1([g_1]), \widehat{\Phi}_{S_2}\circ s_2([g_2])) \\ &= \widehat{\Phi}_R\circ\widehat{\mathscr{L}}(s_1([g_1]), s_2([g_2])),\end{aligned}$$

for all $([g_1],[g_2])\in U_1\times U_2$. Thus Ψ is holomorphic at $([f_1],[f_2])$. □

3. Modifications of qc mappings and qc isotopies

The aim of this section is to prove Proposition 3.3, which tells us the existence of some kind of modifications of asymptotically conformal mappings on compact sets of Riemann surfaces. Our modification is described as follows: Given a compact set C and two quasiconformal mappings f and g which are mutually homotopic rel the ideal boundary, our modification allows us to find a quasiconformal mapping h (a modification of f) which coincides with f "at infinity" and is homotopic to g rel the ideal boundary and C. This modification adapts to the equivalence relation in the definition of asymptotic Teichmüller spaces and will be used at important parts of the proof of the well-definedness of our reductions.

3.1. Lemmas on qc and qc isotopies. This section collects three lemmas concerning quasiconformal mappings and quasiconformal isotopies. Since all of these follow from well-known facts, we state these lemmas without proofs.

We first note a distortion lemma for quasiconformal mappings which is deduced from the compactness of the set of normalized quasiconformal mappings (cf. Theorem 5.1 in p. 51 of [8]).

Lemma 3.1. *Let R be a hyperbolic Riemann surface and f a quasiconformal automorphism of R homotopic to the identity rel $\partial^{id}R$. Then, for any $p\in R$, the hyperbolic distance between p and $f(p)$ is bounded by a constant depending only on the maximal dilatation of f.*

The second lemma easily follows from (3) of Proposition 2.1.

Lemma 3.2. *Let R be a hyperbolic Riemann surface and $H\colon R\times[0,1]\to R$ a quasiconformal isotopy rel $\partial^{id}R$ with $H_1 = id$. Let $S\to R$ be a covering surface.*

Then there is a lift $\widetilde{H}\colon S\times[0,1]\to S$ *of* H *which is a quasiconformal isotopy rel* $\partial^{id}S$ *with* $\widetilde{H}_1 = id$.

We will use the third lemma to glue two quasiconformal isotopies along real analytic curves (see the proof of Propositions 3.1 and 3.2). Indeed, this lemma follows from the fact that any real analytic arc is removable with respect to quasiconformal mappings (cf. Theorem 8.3 of p. 45 in [8]).

Lemma 3.3. *Let* R *and* S *be Riemann surfaces and* $\{\gamma_i\}_{i=1}^N$ *(possibly* $N=\infty$*) a collection of real analytic simple closed curves on* R*. Suppose that every compact set on* R *intersect at most finitely many curves in* $\{\gamma_i\}_{i=1}^N$*. Then a homotopy* $H\colon R\times[0,1]\to S$ *rel* $\partial^{id}R$ *is a quasiconformal isotopy when for each* $t\in[0,1]$, H_t *is a homeomorphism from* R *to* S *and* K*-quasiconformal outside* $\bigcup_{i=1}^N\gamma_i$ *with some* $K\geq 1$.

3.2. Lemmas on subsurfaces. Let R be a Riemann surface and Z a subsurface of R. We say that Z is *incompressible* in R if the inclusion $Z\hookrightarrow R$ induces a monomorphism $\pi_1(Z)\to\pi_1(R)$.

Lemma 3.4. *Let* Z *be a subsurface of* R*. If* ∂Z *(in* R*) consists of homotopically non-trivial simple closed curves on* R*, then* Z *is incompressible in* R.

Proof. Fix $p\in Z$ and let c be a closed loop in Z with initial point p. Suppose that the homotopy class of c is trivial in R. We may assume that c is a simple closed curve. Then c bounds a disk D_c in R. We claim D_c is contained in Z. Otherwise, D_c intersects a component c_1 of $\partial Z\cap R$. Since $\partial D_c = c\subset Z$, by the connectivity of c_1 we deduce that $c_1\subset D_c$. Therefore, c_1 is homotopic to a point, which contradicts our assumption. □

A domain Z in a Riemann surface is said to be *regular* if (1) Z is relatively compact, (2) Z and $R-Z$ have a common boundary which is a 1-dimensional submanifold, and (3) all components of $R-Z$ are non-compact. It is known that any open Riemann surface admits a regular exhaustion. Namely, there is a family $\{Z_k\}_{k=1}^\infty$ of regular domains in R with $\overline{Z_k}\subset Z_{k+1}$ and $R=\bigcup_{k=1}^\infty Z_k$ (cf. [1]). We may suppose that all Z_k is incompressible in R. Indeed, when R is simply connected, R is either **C** or **D**. Thus, we can easily construct a regular exhaustion with the desired property. Suppose R is not simply connected. Then Z_k is not simply connected for sufficiently large k, since $R=\bigcup_{k=1}^\infty Z_k$. If a component c of ∂Z_k is homotopically trivial in R, by $\pi_1(Z_k)\neq\{1\}$, c bounds a closed disk outside Z_k. This contradicts to the definition of regular domains. Therefore, by Lemma 3.4, Z_k is incompressible for every sufficiently large k.

Lemma 3.5. *Let* Z *be a relatively compact subsurface in* R *such that* ∂Z *consists of homotopically non-trivial simple closed curves in* R*. Then* $R-Z$ *consists of finitely many components, and any component* Z' *of* $R-Z$ *is an incompressible surface in* R *such that* $\partial Z'\cap R$ *consists of finitely many homotopically non-trivial simple closed curves in* R.

Proof. Since Z is relatively compact, ∂Z consists of at most finitely many simple closed curves. For each component c of ∂Z, at most one component of $R - Z$ can share c with Z. Since the boundary of a component of $R - Z$ (in R) is contained in ∂Z, the number of components of $R - Z$ is less than the number of components of ∂Z.

Let Z' be a component of $R - Z$. Since any component of $\partial Z'$ is homotopically non-trivial in R, by Lemma 3.4, Z' is incompressible in R. □

Finally, we note

Lemma 3.6. *Let C be a compact set on a Riemann surface R with $\pi_1(R) \neq \{1\}$. Then there is a relatively compact subsurface Z of R such that $C \subset Z$ and ∂Z consists of finitely many homotopically non-trivial real analytic simple closed curves on R.*

Proof. Consider a sufficiently large regular domain which contains C. □

3.3. Modifications of qc mappings and qc isotopies on a compact set We give the first modification.

Proposition 3.1. *Let R and S be hyperbolic Riemann surfaces and f and g quasiconformal mappings from R to S which are quasiconformally isotopic rel $\partial^{id}R$. Let C be a compact set in R. Then there exist a quasiconformal mapping $h\colon R \to S$ and a quasiconformal isotopy $H\colon R \times [0,1] \to S$ rel $C \cup \partial^{id}R$ such that*

(1) *$H_0 = h$ and $H_1 = g$, and*
(2) *$h = f$ on $R - Z$ where Z is a regular domain of R which contains C.*

Proof. We may assume $S = R$ and $g = id$ if we consider $g^{-1} \circ f$ instead of f.

Let $G\colon R \times [0,1] \to R$ be a quasiconformal isotopy rel $\partial^{id}R$ with $G_0 = f$ and $G_1 = id$. Since $\mathscr{C}_0 := \bigcup_{0 \le t \le 1} G_t(C)$ is compact in R, there is a regular domain Z_0 in R containing $\mathscr{C}_0$ in its interior (cf. Lemma 3.6). Let us denote by $\{Z_i\}_{i=1}^m$ the components of $R - Z_0$.

Let Γ be the Fuchsian group acting on $\mathbf{D}$ with $\mathbf{D}/\Gamma = R$ and Γ_i a subgroup of Γ corresponding to $\pi_1(Z_i)$ for $i = 1, \cdots, m$. Let $R_i = \mathbf{D}/\Gamma_i$ and $\mathrm{pr}_i\colon R_i \to R$ the projection. Since Z_i is incompressible in R (cf. Lemma 3.5), there is an embedding $J_i\colon Z_i \to R_i$ satisfying $\mathrm{pr}_i \circ J_i(p) = p$ for all $p \in Z_i$. Furthermore, by definition, any component of $R_i - J_i(Z_i)$ is a funnel corresponding to some component in $\partial Z_i \cap R$. Therefore, there is a quasiconformal mapping $W_i\colon Z_i \to R_i$ such that $W_i = J_i$ outside a relatively compact neighborhood N_i of $\partial Z_i \cap R$ in Z_i (see Figure 1).

By Lemma 3.2, we have a lift $\widetilde{G}^i\colon R_i \times [0,1] \to R_i$ of G such that $(\widetilde{G}^i)_t(p) = p$ for all $p \in \partial^{id}R_i$ and $(\widehat{G}^i)_1 = id$ on R_i. Thus,

$$G^i(p,t) := W_i^{-1} \circ \widetilde{G}^i(W_i(p),t)\colon Z_i \times [0,1] \to Z_i$$

is a quasiconformal isotopy rel $\partial Z_i \cap R$ and satisfies $(G^i)_1(p) = p$ for all $p \in Z_i$ $(i = 1, \cdots, m)$. Furthermore, by Lemma 3.2 again, $(\widetilde{G}^i)_t(p) = (\widetilde{G}^i)_0(p)$ for $(p,t) \in \partial^{id}R_i \times [0,1]$. Therefore, G^i keeps fixing any point of $\partial^{id}R$ whose neighborhood is contained in Z_i.

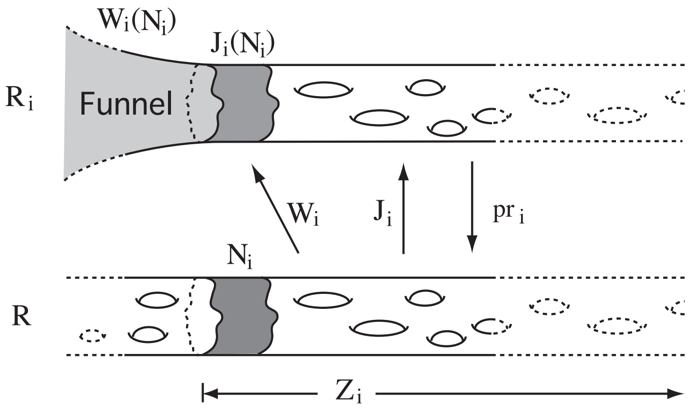

Figure 1. Projections.

We define a quasiconformal isotopy $H\colon R \times [0,1] \to R$ by

$$H(p,t) = \begin{cases} G^i(p,t) & p \in Z_i \quad (i = 1, \dots, m) \\ p & p \in Z_0. \end{cases}$$

Then $h := H_0$ and H have desired properties (cf. Lemma 3.3). Indeed, since Z_0 is relatively compact, any ideal boundary point has a neighborhood contained in some Z_i. Therefore, H is an isotopy rel $C \cup \partial^{id} R$ because $C \subset \mathscr{C}_0 \subset Z_0$. Since $H_1 = id$, h is quasiconformally isotopic to the identity mapping rel $C \cup \partial^{id} R$.

Finally, we check that h coincides with f outside a regular domain containing Z_0. Fix $i = 1, \cdots, m$. By Lemma 3.1 and Lemma 3.6, there is a regular domain Z such that any point $p \in Z_i - Z$ satisfies $f(p) \in Z_i - N_i$. By definition, $(\widetilde{G^i})_0$ is a lift of f to the covering space $\mathrm{pr}_i\colon R_i \to R$. Since J_i is the right-inverse of pr_i on Z_i, $J_i \circ f(p) = (\widetilde{G^i})_0 \circ J_i(p)$ for $p \in Z_i$ with $f(p) \in Z_i$. Thus, we conclude that

$$h(p) = H_0(p) = W_i^{-1} \circ (\widetilde{G^i})_0 \circ W_i(p) = f(p)$$

for all $p \in R - Z$, since $W_i = J_i$ on $Z_i - N_i$. □

3.4. Modifications around punctures. In this section, we give a modification of quasiconformal mapping around punctures. The modification we give here might be well-known, however, we will give a proof for the sake of completeness in Appendix (see §6).

Proposition 3.2. *Let R and S be Riemann surfaces and f a K-quasiconformal mapping from R onto S. Let $P = \{x_i\}_{i=1}^N$ (possibly $N = \infty$) be a set of punctures of R and C a compact set of R. Then there exist a quasiconformal isotopy $H\colon R \times [0,1] \to S$ rel $C \cup \partial^{id} R$ and a constant $K_1 = K_1(K) \geq 1$ such that*

(1) *$H_0 = f$ on R,*
(2) *H_t is K_1-quasiconformal for all $t \in [0,1]$, and*
(3) *H_1 is a conformal around all $p_i \in P$.*

3.5. Modifications of asymptotically conformal mappings. Combining Propositions 3.1 and 3.2, we conclude the following:

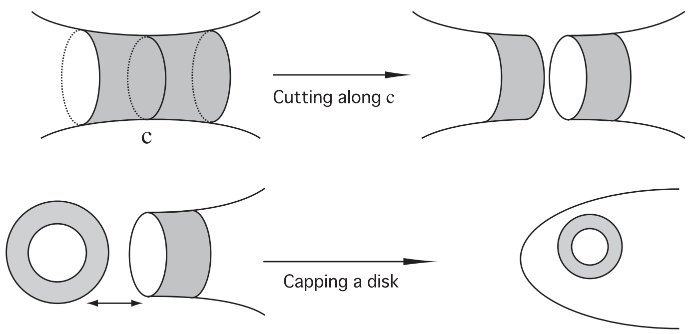

Figure 2. Conformal 2-surgery operation along c.

Proposition 3.3. *Let R be a Riemann surface and f and g quasiconformal mappings on R. Let C be a compact set in R. Then, if $[f] = [g]$ in $AT(R)$, there exist an asymptotically conformal mapping h of $f(R)$ to $g(R)$ and a quasiconformal isotopy $H\colon f(R)\times[0,1] \to g(R)$ rel $f(C)\cup\partial^{id} f(R)$ such that $H_0 = h$ and $H_1 = g\circ f^{-1}$ on $f(R)$.*

Proof. When R is compact, we define a quasiconformal isotopy $H\colon f(R) \times [0,1] \to g(R)$ by $H_t = g \circ f^{-1}$ $(=: h)$ for $(p,t) \in f(R) \times [0,1]$.

Suppose that R is open. Assume first that R is not hyperbolic. Then R is either $\mathbf{C}$ or $\mathbf{C} - \{0\}$. In any case, by Proposition 3.2, there is a quasiconformal isotopy $H\colon f(R) \times [0,1] \to g(R)$ rel $f(C) \cup \partial^{id} f(R)$ such that $H_1 = g \circ f^{-1}$ for $p \in R$ and $h := H_0$ is conformal around punctures. Such H and h have desired properties.

Next, we suppose that R is hyperbolic. Since $[f] = [g]$, there is an asymptotically conformal mapping h_0 from $f(R)$ and $g(R)$ such that h_0 and $g \circ f^{-1}$ are homotopic rel $\partial^{id} f(R)$. Then, by Proposition 3.1, we find a quasiconformal mapping h and a quasiconformal isotopy $H\colon f(R) \times [0,1] \to g(R)$ rel $f(C)$ such that

(1) $H_0 = h$ and $H_1 = g \circ f^{-1}$, and
(2) $h = h_0$ on $f(R) - Z$ where Z is a regular domain in $f(R)$ with $f(C) \subset Z$.

Since h_0 is asymptotically conformal, so is h. Thus, we have the assertion. □

4. Reductions of Asymptotic Teichmüller space by curves

In this section, we give the definition of our *reductions* for asymptotic Teichmüller spaces. Intuitively, our reduction is described as follows: Let R be a Riemann surface and c a homotopically non-trivial simple closed curve on R. Construct surfaces S_1 and S_2 from R by a conformal 2-surgery along c, that is, the surgery operation defined as cutting along c and then (conformally) capping the resulting boundary components with disks. (See Figure 2). Then, $AT(R)$ is biholomorphically equivalent to $AT(S_1) \times AT(S_2)$.

4.1. Conformal 2-surgery operations. Let R be a Riemann surface and c a homotopically non-trivial simple closed curve on R. Let A be a relatively compact annulus on R whose core curve coincides with c, where the *core curve* c of A is a closed curve defined as the preimage of the circle $\{|z| = 1/\sqrt{r}\}$ via a conformal mapping $\xi\colon A \to A(r) := \{1/r < |z| < 1\}$.

Suppose that $R - c$ consists of two components R_1 and R_2. Let $A_i = A \cap R_i$ $(i = 1, 2)$ and $\xi_i\colon A_i \to A(r_i)$ a conformal mapping with $\xi_i(c) = \{|z| = 1/r_i\}$ (where we recognize c as the common boundary component of A_i and R_i). Then after gluing the unit disk along A_i to R_i by ξ_i, we obtain new surfaces $\mathrm{rd}(R_i; A)$ $(i = 1, 2)$, that is, $\mathrm{rd}(R_i; A) = R_i \cup_{\xi_i} \mathbf{D}$. When $R - c =: R_0$ is connected, we let $A - c = A_1 \cup A_2$ and define a new surface $\mathrm{rd}(R_0; A)$ by conformally gluing two copies of $\mathbf{D}$ along each A_1 and A_2. Namely, $\mathrm{rd}(R_0; A) = A_1 \cup_{\xi_1} R_0 \cup_{\xi_2} A_2$ where $\xi_i\colon A_i \to A(r_i)$ is a conformal mapping as above. In any case, we may recognize R_i as a subsurface of $\mathrm{rd}(R_i; A)$.

4.2. Quasiconformal mappings induced by 2-surgeries. Let f be a quasiconformal mapping on R and μ the complex dilatation of f. We then define a quasiconformal mapping $\mathrm{rd}(f)_{i,A}$ on $\mathrm{rd}(R_i; A)$ ($i = 1, 2$ when $R - c$ is disconnected, $i = 0$ otherwise) as follows:

Here we treat only the case when $R - c$ is disconnected. In the other case, we can construct a quasiconformal mapping in a similar way.

Fix $i = 1, 2$ and let f_i^A be a quasiconformal automorphism of $\mathbf{D}$ whose complex dilatation is

$$\frac{(f_i^A)_{\bar z}}{(f_i^A)_z} = \begin{cases} (\xi_i^{-1})^*\mu & \text{on } A(r_i) \\ 0 & \text{otherwise.} \end{cases}$$

Then, by definition,

$$\xi_i^f := f_i^A \circ \xi_i \circ f^{-1} \tag{4.1}$$

is conformal on $f(A_i) \subset f(R_i)$. Therefore, the restriction $f\,|_{R_i}$ extends as a quasiconformal mapping $\mathrm{rd}(f)_{i,A}$ from $\mathrm{rd}(R_i; A)$ onto $\mathrm{rd}(R_i; A)_f := f(R_i) \cup_{\xi_i^f} \mathbf{D}$ which satisfies the following commutative diagram:

$$\begin{array}{ccc} \mathrm{rd}(R_i; A) \supset A_i & \xrightarrow{\ \xi_i\ } & A(r_i)(\subset \mathbf{D}) \\ {\scriptstyle \mathrm{rd}(f)_{i,A}}\big\downarrow & & \big\downarrow{\scriptstyle f_i^A} \\ \mathrm{rd}(R_i; A)_f \supset f(A_i) & \xrightarrow{\ \xi_i^f\ } & \mathbf{D}. \end{array}$$

We note that the complex dilatation of $\mathrm{rd}(f)_{i,A}$ is

$$\begin{cases} (I_i)^*\mu & \text{on } R_i \\ 0 & \text{on } \mathrm{rd}(R_i; A) - R_i, \end{cases}$$

where $I_i\colon R_i \hookrightarrow R$ is the inclusion.

4.3. Reductions. A precise statement of our main theorem is given as follows:

Theorem 4.1. *Let R, c, and A as above and denote $S_i = \mathrm{rd}(R_i; A)$ for $i = 0, 1, 2$. Then it holds either*

(a) *if $R - c$ consists of two components, the mapping*

$$\mathscr{R}_c\colon AT(R) \ni [f] \mapsto ([\mathrm{rd}(f)_{1,A}], [\mathrm{rd}(f)_{2,A}]) \in AT(S_1) \times AT(S_2)$$

is well-defined and biholomorphic, or

(b) *if $R - c$ is connected, the mapping*

$$\mathscr{R}_c\colon AT(R) \ni [f] \mapsto [\mathrm{rd}(f)_{0,A}] \in AT(S_0)$$

is well-defined and biholomorphic.

Proof. We treat only the case where $R - c$ is disconnected The other case can be treated in a similar way.

First we suppose that both S_1 and S_2 are hyperbolic. We then define a $\mathbf{C}$-linear operator $\mathscr{L}\mathscr{R}_{c,i}\colon L^\infty(R) \to L^\infty(S_i)$ by

$$\mathscr{L}\mathscr{R}_{c,i}(\nu) = \begin{cases} (I_i)^*\nu & \text{on } R_i \\ 0 & \text{on } S_i - R_i. \end{cases}$$

Since A_i is relatively compact in R, the surgery operation constructing S_i from R do not effect the asymptotic behavior of any Beltrami differential on R. Hence $\mathscr{L}\mathscr{R}_{c,i}(L_0^\infty(R)) \subset L_0^\infty(S_i)$.

We want to show that $\mathscr{L}\mathscr{R}_{c,i}$ induces a holomorphic mapping $\mathscr{R}_{c,i}\colon AT(R) \to AT(S_i)$. To this end, we claim the following.

Claim 1. *Let $\mu, \nu \in M(R)$ and f and g quasiconformal mappings whose complex dilatations are μ and ν respectively. If $[f] = [g]$ in $AT(R)$, then $[\mathrm{rd}(f)_{i,A}] = [\mathrm{rd}(g)_{i,A}]$ in $AT(S_i)$.*

Proof. By Proposition 3.3, there exist an asymptotically conformal mapping h from $f(R)$ to $g(R)$ and a quasiconformal isotopy $H\colon f(R) \times [0,1] \to g(R)$ rel $f(A) \cup \partial^{id} f(R)$ with $H_0 = h$ and $H_1 = g \circ f^{-1}$. Especially, $h(f(c)) = g(c)$ and hence h maps $f(R_i)$ ($\subset \mathrm{rd}(R_i; A)_f$) onto $g(R_i)$ ($\subset \mathrm{rd}(R_i; A)_g$). Let $\xi_i\colon A_i \to A(r_i)$ be a conformal mapping. We define a conformal mapping ξ_i^f (resp. ξ_i^g) of $f(A_i)$ (resp. $g(A_i)$) into $\mathbf{D}$ and a quasiconformal mapping f^{A_i} (resp. g^{A_i}) of $\mathbf{D}$ satisfying the following commutative diagram (cf. Equation (4.1)):

$$\begin{array}{ccccc} f(A_i) & \xleftarrow{\mathrm{rd}(f)_{i,A}} & A_i & \xrightarrow{\mathrm{rd}(g)_{i,A}} & g(A_i) \\ \xi_i^f \big\downarrow & & \xi_i \big\downarrow & & \big\downarrow \xi_i^g \\ \mathbf{D} & \xleftarrow{f^{A_i}} & \mathbf{D} & \xrightarrow{g^{A_i}} & \mathbf{D}. \end{array}$$

Since $h = g \circ f^{-1}$ on A and $f(A_i)$ are relatively compact in $\mathrm{rd}(R_i; A)_f$, from the diagram above, we have an asymptotically conformal mapping $h_i\colon \mathrm{rd}(R_i; A)_f \to \mathrm{rd}(R_i; A)_g$ defined by

$$h_i = \begin{cases} h & \text{on } f(R_i) \\ g^{A_i} \circ (f^{A_i})^{-1} & \text{on } \mathbf{D}, \end{cases}$$

where we recall that $\mathrm{rd}(R_i; A)_f = f(R_i) \cup_{\xi_i^f} \mathbf{D}$ and $\mathrm{rd}(R_i; A)_g = g(R_i) \cup_{\xi_i^g} \mathbf{D}$.

Next we check that $[\mathrm{rd}(f)_{i,A}] = [\mathrm{rd}(g)_{i,A}]$. To this end, we show that $h_i \circ \mathrm{rd}(f)_{i,A}$ is homotopic to $\mathrm{rd}(g)_{i,A}$ rel $\partial^{id} S_i$. Indeed, since $H_t(p) = g \circ f^{-1}(p)$ for all $(p, t) \in$

$f(A_i) \times [0,1]$, H_t satisfies the equation

$$\xi_i^g \circ H_t \circ (\xi_i^f)^{-1}(p) = g^{A_i} \circ (f^{A_i})^{-1}(p)$$

for $(p,t) \in \xi_i^f(A_i) \times [0,1]$. Therefore, for each $t \in [0,1]$, the restriction of $H_t \mid_{f(R_i)}$ admits an extension $\widehat{H}_t$ from $\mathrm{rd}(R_i; A)_f$ to $\mathrm{rd}(R_i; A)_g$ defined by

$$\widehat{H}_t(p) = \begin{cases} H_t(p) & p \in f(R_i) \\ g^{A_i} \circ (f^{A_i})^{-1}(p) & p \in \mathbf{D}. \end{cases}$$

We notice that $\widehat{H}_0 = h_i$ and $\widehat{H}_1 = \mathrm{rd}(g)_{i,A} \circ (\mathrm{rd}(f)_{i,A})^{-1}$, since $H_0 = h$ and $H_1 = g \circ f^{-1}$ on $f(R)$. Thus, the mapping

$$G(p,t) := \widehat{H}_t \circ \mathrm{rd}(f)_{i,A}(p)\colon \mathrm{rd}(R_i; A) \times [0,1] \to \mathrm{rd}(R_i; A)_g$$

is a quasiconformal isotopy rel $\partial^{id} S_i$ with $G_0 = h_i \circ \mathrm{rd}(f)_{i,A}$ and $G_1 = \mathrm{rd}(g)_{i,A}$, which means $[\mathrm{rd}(f)_{i,A}] = [\mathrm{rd}(g)_{i,A}]$ in $AT(S_i)$. □

Let f be a quasiconformal mapping on R and μ the complex dilatation of f. By definition, the complex dilatation of $\mathrm{rd}(f)_{i,A}$ coincides with $\mathscr{L}\mathscr{R}_{c,i}(\mu)$. Since $\mathscr{L}\mathscr{R}_{c,i}$ is a $\mathbf{C}$-linear mapping with $\mathscr{L}\mathscr{R}_{c,i}(L_0^\infty(R)) \subset L_0^\infty(R_i)$, by Claim 1 and Lemma 2.1, we get a holomorphic mapping

$$\mathscr{R}_{c,i}\colon AT(R) \ni [f] \mapsto [\mathrm{rd}(f)_{i,A}] \in AT(S_i),$$

satisfying the following commutative diagram:

$$\begin{array}{ccc} M(R) \ni \mu & \xrightarrow{\mathscr{L}\mathscr{R}_{c,i}} & \mathscr{L}\mathscr{R}_{c,i}(\mu) \in M(S_i) \\ \downarrow & & \downarrow \\ AT(R) \ni [f] & \xrightarrow{\mathscr{R}_{c,i}} & [\mathrm{rd}(f)_{i,A}] \in AT(S_i). \end{array}$$

By definition, $\mathscr{R}_c([f]) = (\mathscr{R}_{c,1}([f]), \mathscr{R}_{c,2}([f]))$. Hence $\mathscr{R}_c$ is holomorphic.

We next check that $\mathscr{R}_c$ is biholomorphic. To show this, we will construct the inverse mapping of $\mathscr{R}_c$.

For the sake of simplicity, we set $D_i = (S_i - R_i) \cup A_i$, and abuse notations by recognizing A_i as a subset of both S_i and R respectively. Let $\xi_1\colon A_1 \to A(r_1)$ and $\xi_2\colon A_2 \to \{1/(r_1 r_2) < |z| < 1/r_1\}$ be conformal mappings with $\xi_1(c) = \xi_2(c) = \{|z| = 1/r_1\}$. After choosing ξ_1 in an appropriate way, we may assume R is biholomorphically equivalent to $R_1 \cup_{\xi_1} A \cup_{\xi_2} R_2$, where $A = \{1/(r_1 r_2) < |z| < 1\}$.

Let f_i be a quasiconformal mapping on S_i and μ_i the complex dilatation of f_i. We first construct a Riemann surface R_f and a quasiconformal mapping $f\colon R \to R_f$ as follow: Define a Beltrami differential μ_{12} on $A(1/r_1 r_2)$ by

$$\mu_{12}(z) = \begin{cases} (\xi_1^{-1})^* \mu_1(z) & z \in A(r_1) \\ (\xi_2^{-1})^* \mu_2(z) & z \in \{1/(r_1 r_2) < |z| < 1/r_1\}. \end{cases}$$

Let f^A be a quasiconformal mapping on A with complex dilatation μ_{12}. Then for $i = 1, 2$, there is a conformal embedding $\xi_i^f\colon f_i(A_i) \to f^A(A)$ such that $\xi_1^f(f(A_1)) \cap$

$\xi_2^f(f(A_2)) = \emptyset$ and $\xi_i^f \circ f_i = f^A \circ \xi_i$ $(i = 1, 2)$. Set

$$R_f = f_2(R_1) \cup_{\xi_1^f} \cup f^A(A) \cup_{\xi_2^f} f_2(R_2).$$

From the definitions of ξ_i^f, the mapping

$$f(p) = \begin{cases} f_i(p) & p \in R_i \\ f^A(p) & p \in A \end{cases}$$

is a quasiconformal mapping from R to R_f.

We then show the following claim.

Claim 2. *Let g_i be a quasiconformal mapping on S_i $(i = 1, 2)$. Let R_g and g be the Riemann surface and the quasiconformal mapping constructed from g_1 and g_2 as above. If $[f_i] = [g_i]$ in $AT(S_i)$ for $i = 1, 2$, then $[f] = [g]$ in $AT(R)$.*

Proof. By Proposition 3.3, there exist an asymptotically conformal mapping $h_i\colon f_i(S_i) \to g_i(S_i)$ and a quasiconformal isotopy $H^i\colon f_i(S_i) \times [0, 1] \to g_i(S_i)$ rel $f_i(D_i) \cup \partial^{id} f_i(S_i)$ such that $(H^i)_0 = h_i$ and $(H^i)_1 = g_i \circ f_i^{-1}$. Notice that $h_i(f_i(R_i)) = g_i(R_i)$ for $i = 1, 2$. Furthermore, h_i satisfies the following commutative diagram:

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & f_i(A_i) \\ \Big\| & & \Big\downarrow h_i \\ A_i & \xrightarrow{g_i} & g_i(A_i) \end{array}$$

Hence, we have an asymptotically conformal mapping $h\colon R_f \to R_g$ defined by

$$h(p) = \begin{cases} h_i(p) & p \in f_i(R_i) \\ g^A \circ (f^A)^{-1}(p) & p \in f(A). \end{cases}$$

Since H_i keeps fixing D_i pointwise for each $i = 1, 2$, these isotopies are extended as a quasiconformal isotopy connecting h and $g \circ f^{-1}$ rel $\partial^{id} R_f$ (cf. Lemma 3.3). This means that $[f] = [g]$. □

Let f_1, f_2, and f as above. From Claim 2, the mapping

$$\Psi\colon AT(S_1) \times AT(S_2) \ni ([f_1], [f_2]) \mapsto [f] \in AT(R)$$

is well-defined. Next, we check that Ψ is holomorphic. To this end, we define a **C**-linear mapping $\mathscr{L}\colon L^\infty(S_1) \times L^\infty(S_2) \to L^\infty(R)$ by

$$\mathscr{L}(\mu_1, \mu_2) = \begin{cases} (I_1)_*(\mu_1) & \text{on } I_1(R_1) \\ (I_2)_*(\mu_2) & \text{on } I_2(R_2), \end{cases}$$

where $I_i\colon R_i \hookrightarrow R$ is the inclusion. By definition, Ψ and $\mathscr{L}$ satisfy

$$\begin{array}{ccc} M(S_1) \times M(S_2) & \xrightarrow{\mathscr{L}} & M(R) \\ {\scriptstyle \widehat{\Phi}_{S_1} \circ P_{S_1} \times \widehat{\Phi}_{S_2} \circ P_{S_2}} \Big\downarrow & & \Big\downarrow {\scriptstyle \widehat{\Phi}_R \circ P_R} \\ AT(S_1) \times AT(S_2) & \xrightarrow{\Psi} & AT(R). \end{array}$$

Since $\mathscr{L}(L_0^\infty(S_1)\times L_0^\infty(S_2))\subset L_0^\infty(R)$ and $\mathscr{L}(M(S_1)\times M(S_2))\subset M(R)$, by Lemma 2.1, Ψ is holomorphic.

We claim that Ψ is the inverse of $\mathscr{R}_c$. Indeed, this follows from $\mathscr{L}\circ(\mathscr{L}\mathscr{R}_{c,1}\times\mathscr{L}\mathscr{R}_{c,2})=id$ on $M(R)$ and $(\mathscr{L}\mathscr{R}_{c,1}\times\mathscr{L}\mathscr{R}_{c,2})\circ\mathscr{L}=id$ on $M(S_1)\times M(S_2)$. Consequently, $\mathscr{R}_c$ is biholomorphic.

Finally, we note for the case where either S_1 or S_2 is not hyperbolic. In this case, we can see that Claims 1 and 2 do work since Proposition 3.3 is available to all Riemann surfaces. Thus, we conclude the assertion. □

5. Structures of asymptotic Teichmüller spaces

We give proofs of corollaries stated in §1.

Proof of Corollary 1. Suppose that S_i is originated from a component S_i' of $R-\overline{Z}$. Fix points $x_0\in Z$ and $x_i\in S_i'$ for $i=1,\dots m$. For each $i=1,\cdots,m$, we connect x_0 and x_i by a path η_i. Since x_0 and x_i are contained in the different components of $R-\bigcup_{j=1}^n C_j$, we may assume that each η_i intersect only one curve C_{j_i} which is a common boundary component of Z and S_i'.

Let $I=\{1,\dots,n\}$, $I_1=\{j_1,\dots,j_m\}$ and $I_2=I-I_1$. Then, $R-\bigcup_{j\in I_2}C_j$ is connected because so is $\bigcup_{j=1}^n\eta_j$. Hence, by induction on the cardinality of I_2 and applying (b) of Theorem 4.1, we have that $AT(R)$ is biholomorphically equivalent to $AT(R_0)$, where R_0 is the resulting surface after operating the conformal 2-surgery along C_j for $j\in I_2$.

By definition, each C_{j_i} $(i=1,\dots,m)$ is recognized as a curve on R_0, and S_i is the resulting surface after operating the conformal 2-surgery along C_{j_i} to a component of $R_0-C_{j_i}$ which contains S_i'. Hence, by induction on the cardinality of I_1 and applying (a) of Theorem 4.1, we conclude that $AT(R_0)$ (and hence $AT(R)$) is biholomorphically equivalent to the product $\prod_{i=1}^m AT(S_i)$, which is what we desired.

Proofs of Corollaries 2 and 3. Suppose a Riemann surface R is of finite genus. Then, we may assume that $R=S-E$, where S is a compact Riemann surface and E a closed set of S. Take a simply connected domain D in S which contains E and let $Z:=S-D\subset R$. Then, by applying Corollary 1 to the regular domain Z of R, We have that $AT(R)$ is biholomorphically equivalent to $AT(R_0)$, where R_0 is the resulting surface of the conformal 2-surgery operation along ∂D to $D-E$. Since D is simply connected, R_0 is of genus 0, which implies Corollary 2 holds.

In the case of Corollary 3, we may assume that $R=S-P\cup E$ where S is a compact surface, P consists of points of S and E is the union of closed disks in S. Let D be a simply connected domain in S which contains P and $D\cap E=\emptyset$. Then, by applying the conformal 2-surgery operation along ∂D, we may assume that R has no puncture.

If $E=\emptyset$ we have nothing to do. If $E\neq\emptyset$, R is a hyperbolic surface and let Z be the Nielsen convex core of R. Then, Z is a regular domain and $R-Z$ is the union of the funnels of R. Thus applying Corollary 1 to Z, we conclude the desired result.

Proof of Corollary 3. It suffices to prove the case of $m = 2$. The middle thirds Cantor set C is obtained inductively as follows: We first remove from an open interval $(-1/3, 1/3)$ from $I_0 = [-1, 1]$ and at the n^{th} step remove the middle thirds of the remaining intervals.

Let $c = \{\mathrm{Re} z = 0\} \cup \{\infty\}$ and $A = \{z \mid |z+1|/r < |z+1| < r|z-1|\}$ for $r > 1$. Then c is a separating curve in $\Omega = \widehat{\mathbf{C}} - C$. Let $\Omega_\pm = \{z \in \Omega \mid (-1)^\pm \mathrm{Re} z > 0\}$ and $C_\pm = C \cap \{(-1)^\pm \mathrm{Re} z > 0\}$. Then we can check that $\mathrm{rd}(\Omega_\pm; A) = \widehat{\mathbf{C}} - C_\pm$. Since $z \mapsto 3z + (-1)^\pm 2$ is a conformal mapping from $\mathrm{rd}(\Omega_\pm; A)$ to Ω, $AT(\mathrm{rd}(\Omega_\pm; A))$ is biholomorphic to $AT(\Omega)$. Thus, by (a) of Theorem 4.1, we conclude

$$AT(\Omega) \cong AT(\mathrm{rd}(\Omega_+; A)) \times AT(\mathrm{rd}(\Omega_-; A)) \cong AT(\Omega) \times AT(\Omega).$$

6. Appendix: Proof of Proposition 3.2

In this appendix, we give a proof of Proposition 3.2. To do this, we begin with the following.

Lemma 6.1. *Let $f\colon \mathbf{D} \to \mathbf{D}$ be a K-quasiconformal mapping with $f(0) = 0$. Then there exist a quasiconformal isotopy $H\colon \mathbf{D} \times [0, 1] \to \mathbf{D}$ rel $\{0\} \cup \partial\mathbf{D}$ and constants $K_1 \geq 1$ and $\delta_1 \in (0, 1)$ such that*

(1) *$H_0 = f$ on $\overline{\mathbf{D}}$,*
(2) *H_t is K_1-quasiconformal for all $t \in [0, 1]$, and*
(3) *$H_1(z) = z$ on $\{|z| < \delta_1\}$.*

Moreover, the constants K_1 and δ_1 are dependent only on K.

Proof. By conjugating by the reflection in $\partial\mathbf{D}$, we consider f as a quasiconformal mapping on $\widehat{\mathbf{C}}$ with $f(0) = 0$ and $f(\infty) = \infty$.

Fix $\varepsilon_1 \in (0, 1)$ and let $w_1 = f(\varepsilon_1)$. Then $|w_1| \leq 16\varepsilon_1^{1/K}$ by Mori's theorem (cf. Theorem 4.16 of [7]). Consider a quasiconformal mapping g defined by $g(z) = f(\varepsilon_1 z)/w_1$. Since g is normalized (that is, g fixes $\{0, 1, \infty\}$ pointwise), there is a holomorphic family of injections $H^1\colon \widehat{\mathbf{C}} \times \mathbf{D} \to \widehat{\mathbf{C}}$ such that

(a) $(H^1)_0 = id$
(b) $(H^1)_\lambda$ is normalized $(1 + |\lambda|)/(1 - |\lambda|)$-quasiconformal for all $\lambda \in \mathbf{D}$, and
(c) $(H^1)_{k_1^2} = g$ where $k_1 = k_1(K)$ with $K \leq (1 + k_1^2)/(1 - k_1^2)$.

Set $H^2(z, \lambda) = H^1(z, k_1\lambda)$. Then H^2 is a holomorphic family of injections such that for $\lambda \in \mathbf{D}$, $(H^2)_\lambda$ is a normalized $K_1' := (1 + k_1)/(1 - k_1)$-quasiconformal mapping with $H_2(z, k_1) = g(z)$. By the compactness normalized K_1'-quasiconformal mappings, there is a constant $M_0 = M_0(k_1) > 0$ such that

$$\left|(H^2)_\lambda(z)\right| \leq M_0$$

for all $z \in \mathbf{D}$ and $\lambda \in \mathbf{D}$ (see also [9]).

Let $w(\lambda) = \varepsilon_1 \exp\{\lambda \mathrm{Log}(w_1/\varepsilon_1)/k_1^2\}$, where $\mathrm{Log}(w_1/\varepsilon_1)$ denotes the principal value. By definition, w satisfies $w(0) = \varepsilon_1$, $w(k_1^2) = w_1$, and $w(\lambda) \neq 0$ for all $\lambda \in \mathbf{D}$. Furthermore, when $|\lambda| < k_1$,

$$|w(\lambda)| \leq \exp\left(\frac{2\pi + 4\log 2}{k_1^2}\right) \cdot \exp\left(\frac{(1-k_1)^2}{1+k_1^2}\log\varepsilon_1\right) = o(1),$$

as $\varepsilon_1 \to 0$. Let $H^3(z,\lambda) = w(k_1\lambda) \cdot H^2(z/\varepsilon_1, \lambda)$. Notice that $(H^3)_0(z) = w(0)z/\varepsilon_1 = z$ and $(H^3)_{k_1}(z) = w_1 \cdot g(z/\varepsilon_1) = f(z)$. Furthermore, it holds

$$\left|(H^3)_\lambda(z)\right| \leq |w(k_1\lambda)| M_0 = o(1) \quad (\varepsilon_1 \to 0),$$

for all $(z,\lambda) \in \{|z| \leq \varepsilon_1\} \times \mathbf{D}$.

Choose $\varepsilon_1 = \varepsilon_1(k_1) > 0$ to satisfy $|w(k_1\lambda)|M_0 \leq 1/2$ for all $\lambda \in \mathbf{D}$. Then we have a holomorphic motion H^4 of $f(\{|z| \leq \varepsilon_1\}) \cup \partial\mathbf{D}$ defined by

$$H^4(z,\lambda) = \begin{cases} H^3\left(f^{-1}(z), (k_1-\lambda)/(1-k_1\lambda)\right) & \text{if } z \in f(\{|z| \leq \varepsilon_1\}) \\ z & \text{if } z \in \partial\mathbf{D}. \end{cases}$$

By the improved λ-lemma (cf. [13]), H^4 is extended as a holomorphic motion of $\widehat{\mathbf{C}}$ (we use the same symbol H^4 to denote the extension, for simplicity). Let $\delta_1 > 0$ so that $\{|z| \leq \delta_1\} \subset f(\{|z| \leq \varepsilon_1\})$. Since f is K-quasiconformal on $\mathbf{D}$ with $f(0) = 0$, we may choose such δ_1 so that $\delta_1 \leq (\varepsilon_1/16)^K$ by Mori's theorem again.

Thus, from the argument above, we can check that $H(z,t) := f \circ H^4(z, k_1 t)$, $K_1 := KK_1' \geq 1$ and $\delta_1 > 0$ have the desired properties. □

Suppose R is hyperbolic. The *ε-thin neighborhood* U of a puncture x of R is, by definition, an open set consisting of points $p \in R$ with the property that some loop circling around x with initial point p has length less than ε. By virtue of Margulis lemma (cf. e.g. [10]), there is a universal constant $\varepsilon_{mar} > 0$, called the *Margulis constant*, such that the ε_{mar}-thin neighborhood of any puncture of R does not intersect those of any other punctures. Notice that every ideal boundary point of R has a neighborhood which is disjoint from the ε_{mar}-thin neighborhood of any puncture of R (for instance, we can check this with the Nielsen convex core of R).

Proof of Proposition 3.2. If $P = \emptyset$, we have nothing to do.

Suppose first that R is hyperbolic. We choose $\varepsilon < \varepsilon_{mar}$ so that for any i, the ε-thin neighborhood U_i of x_i is disjoint from C. Fix i and let $\xi_1 \colon U_i \cup \{x_i\} \to \mathbf{D}$ and $\xi_2 \colon f(U_i) \cup \{f(x_i)\} \to \mathbf{D}$ be conformal mappings with $\xi_1(x_i) = \xi_2(f(x_i)) = 0$. Then $f_i := \xi_2 \circ f \circ \xi_1^{-1}$ is a K-quasiconformal mapping of $\mathbf{D}$ with $f_i(0) = 0$. Hence, by Lemma 6.1, there exist $\delta_1 > 0$ and a quasiconformal isotopy $H^i \colon \mathbf{D} \times [0,1] \to \mathbf{D}$ such that $(H^i)_0 = f_i$ on $\mathbf{D}$ and $(H^i) = id$ on $\{|z| \leq \delta_1\}$.

We define a quasiconformal isotopy $H \colon R \times [0,1] \to S$ by

$$H(p,t) = \begin{cases} f(p) & p \in R - \bigcup_{i=1}^N U_i \\ \xi_2^{-1} \circ H^i(\xi_1(p), t) & p \in U_i. \end{cases}$$

(cf. Lemma 3.3). We can check that H has desired properties.

Next, we assume that R is not hyperbolic. Then, R is either $\mathbf{C}$ or $\mathbf{C} - \{0\}$. By fixing a suitable number of points outside C, we recognize R as the Riemann sphere with three points specified. Thus, from the case of hyperbolic surfaces, we conclude the assertion. □

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