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ON PLANAR BELTRAMI EQUATIONS AND HÖLDER REGULARITY

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Abstract. We provide estimates for the Hölder exponent of solutions to the Beltrami equation $\overline{\partial} f = \mu \partial f + \nu \overline{\partial f}$, where the Beltrami coefficients μ, ν satisfy $\||\mu| + |\nu|\|_\infty < 1$ and $\Im(\nu) = 0$. Our estimates depend on the arguments of the Beltrami coefficients as well as on their moduli. Furthermore, we exhibit a class of mappings of the "angular stretching" type, on which our estimates are actually attained.

1. Introduction and statement of the main results

Let Ω be a bounded open subset of $\mathbf{R}^2$ and let $f \in W^{1,2}_{\rm loc}(\Omega, \mathbf{C})$ satisfy the Beltrami equation

$$\overline{\partial} f = \mu \partial f + \nu \overline{\partial f} \quad \text{a.e. in } \Omega, \tag{1}$$

where $\overline{\partial} = (\partial_1 + i\partial_2)/2$, $\partial = (\partial_1 - i\partial_2)/2$ and μ, ν, are bounded, measurable functions satisfying $\||\mu| + |\nu|\|_\infty < 1$. Equation (1) arises in the study of conformal mappings between domains equipped with measurable Riemannian structures, see [2]. By classical work of Morrey [10], it is well-known that solutions to (1) are Hölder continuous. More precisely, there exists $\alpha \in (0, 1)$ such that for every compact $T \Subset \Omega$ there exists $C_T > 0$ such that

$$\frac{|f(z) - f(z')|}{|z - z'|^\alpha} \le C_T \quad \forall z, z' \in T,\ z \ne z'.$$

Let

$$K_{\mu,\nu} = \frac{1 + |\mu| + |\nu|}{1 - |\mu| - |\nu|}$$

denote the distortion function. Then, the following estimate holds:

$$\alpha \ge \|K_{\mu,\nu}\|_\infty^{-1}. \tag{2}$$

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This estimate is sharp, in the sense that it reduces to an equality on the radial stretching

$$f(z) = |z|^{\alpha-1}z, \tag{3}$$

which satisfies (1) with $\mu(z) = -(1-\alpha)/(1+\alpha)z\bar{z}^{-1}$ and $\nu = 0$. There exists a wide literature concerning the regularity theory for (1), particularly in the degenerate case where $\||\mu| + |\nu|\|_\infty = 1$, or equivalently, when the distortion function $K_{\mu,\nu}$ is unbounded. See, e.g., [3, 6, 8, 9], and the references therein. See also [5], where an estimate of the constant C_T is given. Most of the results mentioned above provide estimates in terms of the distortion function $K_{\mu,\nu}$, and there is no loss of generality in assuming that $\nu = 0$. Indeed, the following "device of Morrey" may be used, as described in [4]: at points where $\partial f \neq 0$ we set $\widetilde{\mu} = \mu + \nu\overline{\partial f}/\partial f$; at points where $\partial f = 0$ we set $\widetilde{\mu} = 0$. Then f is a solution to $\overline{\partial} f = \widetilde{\mu}\partial f$ and $|\widetilde{\mu}| \leq |\mu| + |\nu|$. On the other hand, in this note we are interested in estimates which preserve the information contained in the arguments of the Beltrami coefficients μ, ν, in the spirit of the work of Andreian Cazacu [1] and of Reich and Walczak [12]. We restrict our attention to the case $\Im(\nu) = 0$. This assumption corresponds to assuming that the Riemannian metric in the target manifold is represented by a diagonal matrix-valued function. We will also show that our estimates are sharp, in the sense that they are attained in a class of mappings of the "angular stretching" type (see ansatz (8) below), which generalize the radial stretchings (3). It should be mentioned that such mappings also appear in Schatz [15], see also Gutlyanskiĭ and Ryazanov [7].

Our first result is the following.

Theorem 1. *Let $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{C})$ satisfy the Beltrami equation (1) with $\Im(\nu) = 0$. Then, f is α-Hölder continuous with $\alpha \geq \beta(\mu,\nu)$, where $\beta(\mu,\nu)$ is defined by*

$$\begin{aligned}\beta(\mu,\nu)^{-1} = \sup_{S_\rho(x)\subset\Omega} \inf_{\varphi,\psi\in\mathscr{B}_{x,\rho}} &\sqrt{\frac{\sup\varphi}{\inf\psi}} \\ &\left\{\frac{1}{|S_\rho(x)|}\int_{S_\rho(x)}\sqrt{\frac{\psi}{\varphi}}\frac{|1-\overline{n}^2\mu|^2-\nu^2}{\sqrt{1-(|\mu|+\nu)^2}\sqrt{1-(|\mu|-\nu)^2}}\,\mathrm{d}\sigma\right. \\ &\left.\cdot\left(\frac{4}{\pi}\arctan\left(\frac{\inf_{S_\rho(x)}\frac{(1-\nu)^2-|\mu|^2}{(1+\nu)^2-|\mu|^2}/\varphi\psi}{\sup_{S_\rho(x)}\frac{(1-\nu)^2-|\mu|^2}{(1+\nu)^2-|\mu|^2}/\varphi\psi}\right)^{1/4}\right)^{-1}\right\}.\end{aligned} \tag{4}$$

Here $S_\rho(x)$ denotes the circle centered at $x \in \Omega$ with radius $\rho > 0$, $\mathscr{B}_{x,\rho}$ denotes the set of positive functions in $L^\infty(S_\rho(x))$ which are bounded below away from zero, and n denotes complex number corresponding to the outer unit normal to $S_\rho(x)$.

Estimate (4) improves the classical estimate (2); a verification is provided in Section 3, Remark 1. In Theorem 2 below we will show that estimate (4) is sharp, in the sense that it reduces to an equality when μ, ν are of the special form

$$\mu(z) = -\mu_0(\arg z)z\overline{z}^{-1}, \quad \nu(z) = -\nu_0(\arg z)$$

and f is of the "angular stretching" form

$$f(z) = |z|^\alpha(\eta_1(\arg z) + i\eta_2(\arg z)),$$

for suitable choices of the bounded, 2π-periodic functions $\mu_0, \nu_0, \eta_1, \eta_2 \colon \mathbf{R} \to \mathbf{R}$. The following weaker form of estimate (4) is obtained by taking $\varphi = \psi = 1$.

Corollary 1. *Let $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{C})$ satisfy the Beltrami equation (1) with $\Im(\nu) = 0$. Then, f is α-Hölder continuous with*

$$\alpha \geq \left\{ \sup_{S_\rho(x)\subset\Omega} \frac{\frac{1}{|S_\rho(x)|}\int_{S_\rho(x)} \frac{|1-\overline{n}^2\mu|^2-\nu^2}{\sqrt{1-(|\mu|+\nu)^2}\sqrt{1-(|\mu|-\nu)^2}}\mathrm{d}\sigma}{\frac{4}{\pi}\arctan\left(\frac{\inf_{S_\rho(x)}\frac{(1-\nu)^2-|\mu|^2}{(1+\nu)^2-|\mu|^2}}{\sup_{S_\rho(x)}\frac{(1-\nu)^2-|\mu|^2}{(1+\nu)^2-|\mu|^2}}\right)^{1/4}} \right\}^{-1}. \tag{5}$$

This estimate is also sharp, in the sense that it actually reduces to an equality for suitable choices of μ, ν and f, but it does not contain estimate (2) as a special case. We now show that estimate (5) contains some known results for $\mu = 0$ and for $\nu = 0$ as special cases.

Special case $\nu = 0$. This case corresponds to assuming that the target domain is equipped with the standard Euclidean metric. In this special case, our estimate yields

$$\alpha \geq \left\{ \sup_{S_\rho(x)\subset\Omega} \frac{1}{|S_\rho(x)|}\int_{S_\rho(x)} \frac{|1-\overline{n}^2\mu|^2}{1-|\mu|^2}\,\mathrm{d}\sigma \right\}^{-1}, \tag{6}$$

which may also be obtained from the estimate in [13] for elliptic equations whose coefficient matrix has unit determinant. We note that the integrand function

$$\frac{|1-\overline{n}^2\mu|^2}{1-|\mu|^2} = \frac{|D_{\overline{n}}f|^2}{J_f} = K_{\mu,0} - 2\frac{|\mu| + \Re(\mu, n^2)}{1-|\mu|^2}$$

also appears in [12], in the study of the conformal modulus of rings.

Special case $\mu = 0$. This case corresponds to assuming that the metric on Ω is Euclidean. In this case, estimate (5) yields

$$\alpha \geq \sup_{S_\rho(x)\subset\Omega} \frac{4}{\pi}\arctan\left(\frac{\inf_{S_\rho(x)}\frac{1-\nu}{1+\nu}}{\sup_{S_\rho(x)}\frac{1-\nu}{1+\nu}}\right)^{1/2} \geq \frac{4}{\pi}\arctan\|K\|_\infty^{-1}, \tag{7}$$

which is a consequence of the sharp Hölder estimate obtained in Piccinini and Spagnolo [11] for isotropic elliptic equations.

In Theorem 2 below we assert that the equality $\alpha = \beta(\mu, \nu)$ may hold even when *both* $\mu \neq 0$ and $\nu \neq 0$. We denote by B the unit disk in $\mathbf{R}^2$.

Theorem 2. *For every $\tau \in [0, 1]$ there exist $\alpha_\tau > 0$, 2π-periodic functions $\eta_{\tau,1}, \eta_{\tau,2} \in W^{1,2}_{\mathrm{loc}}(\mathbf{R})$ and corresponding coefficients μ_τ, ν_τ, depending on the angular variable only, with the following properties:*

(i) *The mapping* $f_\tau \in W^{1,2}_{\rm loc}(B)$ *defined in* $B \setminus \{0\}$ *by*

$$f_\tau(z) = |z|^{\alpha_\tau}\,(\eta_{\tau,1}(\arg z) + i\eta_{\tau,2}(\arg z))$$

satisfies (1) *with* $\mu = \mu_\tau$ *and* $\nu = \nu_\tau$*;*

(ii) $\beta(\mu_\tau, \nu_\tau) = \alpha_\tau$*;*

(iii) $\mu_\tau = 0$ *if and only if* $\tau = 0$*;* $\nu_\tau = 0$ *if and only if* $\tau = 1$.

This note is organized as follows. In Section 2 we derive the basic properties of the mappings of the "angular stretching" form, which naturally appear in our problem. In Section 3 we provide the proofs of Theorem 1 and Theorem 2. Such proofs are based on the equivalence between Beltrami equations and elliptic equations, and on some results for elliptic equations from [14].

2. Angular stretchings

In order to prove Theorem 2 we need some properties for the special case where f is of the "angular stretching" form

$$f(z) = |z|^\alpha \phi(\arg z) = |z|^\alpha(\eta_1(\arg z) + i\eta_2(\arg z)), \tag{8}$$

where $\alpha \in \mathbf{R}$, $\phi\colon \mathbf{R} \to \mathbf{C}$ and $\eta_1, \eta_2\colon \mathbf{R} \to \mathbf{R}$ are 2π-periodic functions, and moreover f satisfies the Beltrami equation (1) with μ, ν of the special form

$$\mu(z) = -\mu_0(\arg z)\, z\bar{z}^{-1} \tag{9}$$

and

$$\nu(z) = -\nu_0(\arg z), \tag{10}$$

for some bounded, 2π-periodic functions $\mu_0, \nu_0\colon \mathbf{R} \to \mathbf{R}$ such that $\||\mu_0|+|\nu_0|\|_\infty < 1$. We assume $\alpha > 0$ and $\eta_1, \eta_2 \in W^{1,2}_{\rm loc}(\mathbf{R})$ so that $f \in W^{1,2}_{\rm loc}(\mathbf{C})$. We note that mappings of the form (8) generalize the radial stretchings (3). We also note that f is injective if and only if $|\phi(\theta)|^2 = \eta_1^2(\theta) + \eta_2^2(\theta) \neq 0$ for all $\theta \in \mathbf{R}$, η_1, η_2 have minimal period 2π and $\Im(\dot\phi\overline{\phi}) = \eta_1\dot\eta_2 - \dot\eta_1\eta_2 = (\eta_1^2 + \eta_2^2)(\mathrm{d}/\mathrm{d}\theta)\arg(\eta_1 + i\eta_2)$ has constant sign a.e.

We claim that

$$\begin{aligned}|Df|^2 &= \frac{|z|^{2(\alpha-1)}}{2}\left(\alpha^2|\phi|^2 + |\dot\phi|^2 + |\alpha^2\phi^2 + \dot\phi^2|\right)\\ &= \frac{|z|^{2(\alpha-1)}}{2}\left\{\alpha^2(\eta_1^2+\eta_2^2) + \dot\eta_1{}^2 + \dot\eta_2{}^2 + \sqrt{\mathscr{D}}\right\},\end{aligned} \tag{11}$$

where $|Df|$ denotes the operator norm of Df, and

$$\mathscr{D} = [\alpha^2(\eta_1^2+\eta_2^2) + \dot\eta_1{}^2 + \dot\eta_2{}^2]^2 - 4\alpha^2(\eta_1\dot\eta_2 - \dot\eta_1\eta_2)^2;$$

moreover

$$J_f = \alpha|z|^{2(\alpha-1)}\Im(\dot\phi\overline{\phi}) = \alpha|z|^{2(\alpha-1)}(\eta_1\dot\eta_2 - \dot\eta_1\eta_2). \tag{12}$$

To check (11)–(12) we use the well known formulae

$$|Df| = |f_z| + |f_{\bar z}|, \quad J_f = |f_z|^2 - |f_{\bar z}|^2.$$

We recall that in polar cooordinates $x = r\cos\theta$, $y = r\sin\theta$ we have

$$\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y) = \frac{e^{i\theta}}{2}\left(\partial_r + i\,\frac{\partial_\theta}{r}\right),$$
$$\partial = \frac{1}{2}(\partial_x - i\partial_y) = \frac{e^{-i\theta}}{2}\left(\partial_r - i\,\frac{\partial_\theta}{r}\right).$$

Hence,

$$f_z(z) = \frac{f(z)}{2z}\left(\alpha - i\frac{\dot\phi}{\phi}\right), \quad f_{\bar z}(z) = \frac{f(z)}{2\bar z}\left(\alpha + i\frac{\dot\phi}{\phi}\right)$$

and therefore

$$|f_z|^2 = \frac{|z|^{2(\alpha-1)}}{4}\left[\alpha^2|\phi|^2 + |\dot\phi|^2 + 2\alpha\Im(\dot\phi\overline{\phi})\right],$$
$$|f_{\bar z}|^2 = \frac{|z|^{2(\alpha-1)}}{4}\left[\alpha^2|\phi|^2 + |\dot\phi|^2 - 2\alpha\Im(\dot\phi\overline{\phi})\right].$$

Hence, (12) follows. To obtain (11) we finally observe that

$$f_z f_{\bar z} = \frac{|z|^{2(\alpha-1)}}{4}\left(\alpha^2\phi^2 + \dot\phi^2\right)$$

and

$$\left|\alpha^2\phi^2 + \dot\phi^2\right|^2 = \alpha^2|\phi|^4 + |\dot\phi|^4 + 2\alpha^2\Re(\dot\phi\overline{\phi})^2 = \mathscr{D}.$$

Therefore, at every point in $\mathbf{R}^2 \setminus \{0\}$ the distortion of f is given by

$$\frac{|Df|^2}{J_f} = \frac{\alpha|\phi|^2 + |\dot\phi|^2 + |\alpha^2\phi^2 + \dot\phi^2|}{2\alpha\Im(\dot\phi\overline{\phi})}$$
$$= \frac{\alpha^2(\eta_1^2 + \eta_2^2) + \dot\eta_1{}^2 + \dot\eta_2{}^2 + \sqrt{\mathscr{D}}}{2\alpha(\eta_1\dot\eta_2 - \dot\eta_1\eta_2)}.$$

In particular, f has bounded distortion if and only if

$$|\phi|^2 + |\dot\phi|^2 \le C\Im(\dot\phi\overline{\phi})$$

for some constant $C > 0$, or equivalently

$$\eta_1^2 + \eta_2^2 + \dot\eta_1^2 + \dot\eta_2^2 \le C(\eta_1\dot\eta_2 - \dot\eta_1\eta_2)$$

for some constant $C > 0$.

We use the following facts.

Proposition 1. *Suppose f is of the angular stretching form* (8) *and satisfies the Beltrami equation* (1) *with μ, ν given by* (9)–(10). *Then, (η_1, η_2) satisfies the system:*

$$\begin{cases} \dot\eta_1 = -\alpha k_2^{-1}\eta_2, \\ \dot\eta_2 = \alpha k_1\eta_1, \end{cases} \tag{13}$$

where $k_1, k_2 > 0$ *are defined by*

$$k_1 = \frac{1+\mu_0+\nu_0}{1-\mu_0-\nu_0}, \quad k_2 = \frac{1-\mu_0+\nu_0}{1+\mu_0-\nu_0}. \tag{14}$$

Conversely, if (η_1, η_2) *satisfies* (13) *for some* $\alpha > 0$ *and for some* 2π*-periodic functions* $k_1, k_2 > 0$ *bounded from above and from below away from zero, then* f *defined by* (8) *is a solution to* (1) *with* μ, ν *defined in* (9)–(10) *and* μ_0, ν_0 *given by*

$$\mu_0 = \frac{k_1 - k_2}{1+k_1+k_2+k_1k_2}, \quad \nu_0 = \frac{k_1k_2 - 1}{1+k_1+k_2+k_1k_2}. \tag{15}$$

Proof. In polar cooordinates $x = r\cos\theta$, $y = r\sin\theta$ we have

$$\begin{aligned}\bar\partial =& \frac{1}{2}(\partial_x + i\partial_y) = \frac{e^{i\theta}}{2}\left(\partial_r + i\,\frac{\partial_\theta}{r}\right),\\ \partial =& \frac{1}{2}(\partial_x - i\partial_y) = \frac{e^{-i\theta}}{2}\left(\partial_r - i\,\frac{\partial_\theta}{r}\right).\end{aligned}$$

Hence, (1) is equivalent to

$$(e^{i\theta} - \mu e^{-i\theta})f_r - \nu e^{i\theta}\overline{f_r} = -\frac{i}{r}\left[(e^{i\theta} + \mu e^{-i\theta})f_\theta - \nu e^{i\theta}\overline{f_\theta}\right].$$

In view of the form (9) of μ and of the form (10) of ν, the equation above is equivalent to

$$(1+\mu_0)f_r + \nu_0\overline{f_r} = -\frac{i}{r}[(1-\mu_0)f_\theta + \nu_0\overline{f_\theta}].$$

We compute

$$f_r = \alpha r^{\alpha-1}(\eta_1 + i\eta_2), \quad f_\theta = r^\alpha(\dot\eta_1 + i\dot\eta_2).$$

Substitution yields

$$\alpha(1+\mu_0+\nu_0)\eta_1 + i\alpha(1+\mu_0-\nu_0)\eta_2 = (1-\mu_0-\nu_0)\dot\eta_2 - i(1-\mu_0+\nu_0)\dot\eta_1. \tag{16}$$

Hence, (η_1, η_2) satisfies the system (13), with k_1, k_2 defined by (14). Conversely, suppose (η_1, η_2) satisfies (13) for some 2π-periodic functions $k_1, k_2 > 0$ bounded from above and from below away from zero and for some $\alpha > 0$. Then the functions μ_0, ν_0 such that (14) is satisfied are uniquely defined by (15) as the solutions to the linear system

$$\begin{aligned}(1+k_1)\mu_0 + (1+k_1)\nu_0 =& -1 + k_1,\\ -(1+k_2)\mu_0 + (1+k_2)\nu_0 =& -1 + k_2.\end{aligned}$$

It follows that (13) is equivalent to (16), with f defined by (8). □

We finally observe that if (η_1, η_2) is a solution of the system (13), then the Jacobian determinant of f is given by

$$r^{-2(\alpha-1)}J_f = \alpha^2(k_1\eta_1^2 + k_2^{-1}\eta_2^2)$$

and furthermore,

$$\frac{|Df|^2}{J_f} = \Big[2(k_1\eta_1^2 + k_2^{-1}\eta_2^2)\Big]^{-1}\Big[(1+k_1^2)\eta_1^2 + (1+k_2^{-2})\eta_2^2 + \sqrt{(1-k_1^2)^2\eta_1^4 + (1-k_2^{-2})^2\eta_2^4 + 2[(1-k_1k_2^{-1})^2 + (k_1 - k_2^{-1})^2]\eta_1^2\eta_2^2}\Big]. \tag{17}$$

We also note that system (13) implies that η_1 is a 2π-periodic solution to the weighted Sturm–Liouville equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(k_2\dot{\eta}_1) + \alpha^2 k_1\eta_1 = 0$$

and similarly η_2 satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}(k_1^{-1}\dot{\eta}_2) + \alpha^2 k_2^{-1}\eta_2 = 0.$$

Special case $\nu = 0$. The results described in Proposition 1 take a particularly simple form when $\nu = 0$, which is equivalent to $k_1 = k_2^{-1} = k$. It should be mentioned that solutions to the Beltrami equation (1) with $\nu = 0$ and μ depending on $\theta = \arg z$ only have been considered in [15], see also [7]. In this case, the normalized homeomorphic solution admits the representation

$$f(z) = |z|^\alpha \exp\left\{i\alpha\int_0^\theta \frac{1-\mu(\theta')e^{-2i\theta'}}{1+\mu(\theta')e^{-2i\theta'}}\,\mathrm{d}\theta'\right\},$$

where

$$\alpha = 2\pi\left(\int_0^{2\pi}\frac{1-\mu(\theta')e^{-2i\theta'}}{1+\mu(\theta')e^{-2i\theta'}}\,\mathrm{d}\theta'\right)^{-1}.$$

Under our additional assumption $\mu(\theta) = -\mu_0(\theta)e^{2i\theta}$, we have

$$\frac{1-\mu(\theta')e^{-2i\theta'}}{1+\mu(\theta')e^{-2i\theta'}} = \frac{1+\mu_0(\theta')}{1-\mu_0(\theta')} = k(\theta')$$

and therefore we obtain the representation $f(z) = |z|^\alpha \exp\{i\alpha\int_0^\theta k\}$. On the other hand, a direct proof may be as follows. If $k_1 = k_2^{-1} = k$, system (13) reduces to

$$\begin{cases}\dot{\eta}_1 = -\alpha k\eta_2,\\ \dot{\eta}_2 = \alpha k\eta_1,\end{cases} \tag{18}$$

which may be explicitly solved. Indeed, from (18) we derive $\dot{\eta}_1\eta_1 + \dot{\eta}_2\eta_2 = 0$ and therefore $\eta_1^2+\eta_2^2$ is constant. By linearity we may assume $\eta_1^2+\eta_2^2 \equiv 1$. Hence, there exists a funtion $h(\theta)$ such that $\eta_1(\theta) = \cos h(\theta)$ and $\eta_2(\theta) = \sin h(\theta)$. By (18) we conclude that up to an additive constant $h(\theta) = \alpha\int_0^\theta k$, and therefore we obtain that $f(z) = |z|^\alpha\exp\{i\alpha\int_0^\theta k\}$. In view of the 2π-periodicity of η_1, η_2 we also obtain that $\alpha = 2\pi n(\int_0^{2\pi} k)^{-1}$ for some $n \in \mathbf{N}$. From equation (17) we derive, for every $z \neq 0$:

$$\frac{|Df|^2}{J_f} = \frac{1+k^2+|1-k^2|}{2k} = \max\{k, k^{-1}\}.$$

Since $k \geq 1$ if and only if $\mu_0 \geq 0$, the expression above implies the known fact

$$\frac{|Df|^2}{J_f} = \frac{1+|\mu|}{1-|\mu|} = K_{\mu,0}.$$

3. Proofs

We first of all check that estimate (4) in Theorem 1 actually improves the classical estimate (2).

Remark 1. *The following estimate holds:*

$$\beta(\mu,\nu) \geq \|K_{\mu,\nu}\|_\infty^{-1},$$

where $\beta(\mu,\nu)$ is the quantity defined in Theorem 1.

Proof. Recall from Section 1 that $K_{\mu,\nu} = (1+|\mu|+|\nu|)/(1-|\mu|-|\nu|)$. For every $S_\rho(x) \subset \Omega$, we choose

$$\varphi = \frac{|1-\overline{n}^2\mu|^2-\nu^2}{(1+\nu)^2-|\mu|^2}\bigg|_{S_\rho(x)}, \quad \psi = \frac{(1-\nu)^2-|\mu|^2}{|1-\overline{n}^2\mu|^2-\nu^2}\bigg|_{S_\rho(x)}.$$

We have that

$$\sup\varphi \leq \sup\frac{(1+|\mu|)^2-\nu^2}{(1+\nu)^2-|\mu|^2} = \sup\frac{1+|\mu|-\nu}{1-|\mu|+\nu} \leq \|K_{\mu,\nu}\|_\infty,$$
$$\inf\psi \geq \inf\frac{(1-\nu)^2-|\mu|^2}{(1+|\mu|)^2-\nu^2} = \inf\frac{1-|\mu|-\nu}{1+|\mu|+\nu} \geq \|K_{\mu,\nu}\|_\infty^{-1}$$

and therefore

$$\frac{\sup\varphi}{\inf\psi} \leq \|K_{\mu,\nu}\|_\infty^2.$$

Moreover,

$$\varphi\psi = \frac{(1-\nu)^2-|\mu|^2}{(1+\nu)^2-|\mu|^2}\bigg|_{S_\rho(x)}.$$

In view of the elementary identity

$$[(1-\nu)^2-|\mu|^2][(1+\nu)^2-|\mu|^2] = [1-(|\mu|+\nu)^2][1-(|\mu|-\nu)^2]$$

we finally obtain

$$\frac{\psi}{\varphi} = \frac{(1-(|\mu|+\nu)^2)(1-(|\mu|-\nu)^2)}{(|1-\overline{n}^2\mu|^2-\nu^2)^2}\bigg|_{S_\rho(x)}.$$

Consequently, inserting into (4), we find that for every $S_\rho(x) \subset \Omega$:

$$\inf_{\varphi,\psi\in\mathscr{B}_{x,\rho}} \sqrt{\frac{\sup\varphi}{\inf\psi}} \Bigg\{ \frac{1}{|S_\rho(x)|} \int_{S_\rho(x)} \sqrt{\frac{\psi}{\varphi}} \frac{|1-\overline{n}^2\mu|^2-\nu^2}{\sqrt{1-(|\mu|+\nu)^2}\sqrt{1-(|\mu|-\nu)^2}}\, \mathrm{d}\sigma$$

$$\cdot \left(\frac{4}{\pi} \arctan \left(\frac{\inf_{S_\rho(x)} \frac{(1-\nu)^2-|\mu|^2}{(1+\nu)^2-|\mu|^2}/\varphi\psi}{\sup_{S_\rho(x)} \frac{(1-\nu)^2-|\mu|^2}{(1+\nu)^2-|\mu|^2}/\varphi\psi} \right)^{1/4} \right)^{-1} \Bigg\} \le \|K_{\mu,\nu}\|_\infty.$$

Consequently,

$$\beta(\mu,\nu)^{-1} \le \|K_{\mu,\nu}\|_\infty,$$

and the asserted estimate is verified. □

We use some results in [14] for solutions to the elliptic divergence form equation

$$\operatorname{div}(A\nabla\cdot) = 0 \quad \text{in } \Omega \tag{19}$$

where A is a bounded and symmetric matrix-valued function. More precisely, let

$$J(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

For every $M > 1$, let

$$c = c(M,\tau) = \frac{2}{1+M^{-\tau}}, \quad d = d(M,\tau) = \frac{4}{\pi}\arctan M^{-(1-\tau)/2}.$$

Note that when $\tau = 0$ we have $d = 4\pi^{-1}\arctan M^{-1/2}$ and $c = 1$, and when $\tau = 1$ we have $d = 1$ and $c = 2/(1+M^{-1})$. We define the intervals

$$I_1 = [0, \frac{c\pi}{2}), \quad I_2 = [\frac{c\pi}{2}, \pi), \quad I_3 = [\pi, \pi + \frac{c\pi}{2}), \quad I_4 = [\pi + \frac{c\pi}{2}, 2\pi).$$

Let $\Theta_{\tau,1}, \Theta_{\tau,2} \colon \mathbf{R} \to \mathbf{R}$ be the 2π-periodic Lipschitz functions defined in $[0, 2\pi)$ by

$$\Theta_{\tau,1}(\theta) = \begin{cases} \sin[d(c^{-1}\theta - \pi/4)], & \theta \in I_1, \\ M^{-(1-\tau)/2}\cos[d(c^{-1}M^\tau(\theta - c\pi/2) - \pi/4)], & \theta \in I_2, \\ -\sin[d(c^{-1}(\theta-\pi) - \pi/4)], & \theta \in I_3, \\ -M^{-(1-\tau)/2}\cos[d(c^{-1}M^\tau(\theta - \pi - c\pi/2) - \pi/4)], & \theta \in I_4, \end{cases}$$

and

$$\Theta_{\tau,2}(\theta) = \begin{cases} -\cos[d(c^{-1}\theta - \pi/4)], & \theta \in I_1, \\ M^{(1-\tau)/2}\sin[d(c^{-1}M^\tau(\theta - c\pi/2) - \pi/4)], & \theta \in I_2, \\ \cos[d(c^{-1}(\theta-\pi) - \pi/4)], & \theta \in I_3, \\ -M^{(1-\tau)/2}\sin[d(c^{-1}M^\tau(\theta - \pi - c\pi/2) - \pi/4)], & \theta \in I_4. \end{cases}$$

The following facts were established in [14] and will be used in the sequel.

Theorem 3. ([14]) *The following estimates hold.*

(i) *Let $w \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a weak solution to* (19). *Then, w is α-Hölder continuous with $\alpha \geq \gamma(A)$, where*

$$\gamma(A) = \left(\sup_{S_\rho(x)\subset\Omega} \inf_{\varphi,\psi\in\mathscr{B}_{x,\rho}} \sqrt{\frac{\sup\varphi}{\inf\psi}} \frac{\frac{1}{|S_\rho(x)|}\int_{S_\rho(x)} \sqrt{\frac{\psi}{\varphi}} \frac{\langle n, A\, n\rangle}{\sqrt{\det A}}}{\frac{4}{\pi}\arctan\left(\frac{\inf_{S_\rho(x)}\det A/\varphi\psi}{\sup_{S_\rho(x)}\det A/\varphi\psi}\right)^{1/4}} \right)^{-1} \tag{20}$$

and where n denotes the outer unit normal.

(ii) *For every $\tau \in [0,1]$ let A_τ be the symmetric matrix-valued function defined for every $z \neq 0$ by*

$$A_\tau(z) = (k_{\tau,1}(\arg z) - k_{\tau,2}(\arg z))\frac{z \otimes z}{|z|^2} + k_{\tau,2}(\arg z)\mathbf{I}, \tag{21}$$

where $k_{\tau,1}, k_{\tau,2}$ piecewise constant, 2π-periodic functions defined by

$$k_{\tau,1}(\theta) = \begin{cases} 1, & \text{if } \theta \in I_1 \cup I_3, \\ M, & \text{if } \theta \in I_2 \cup I_4, \end{cases} \tag{22}$$

and

$$k_{\tau,2}(\theta) = \begin{cases} 1, & \text{if } \theta \in I_1 \cup I_3, \\ M^{1-2\tau}, & \text{if } \theta \in I_2 \cup I_4. \end{cases} \tag{23}$$

There exists $M_0 > 1$ such that

$$\gamma(A_\tau) = \frac{d}{c},$$

for every $M \in (1, M_0^{1/\tau})$, if $\tau > 0$, and with no restriction on M if $\tau = 0$. Furthermore, the function $u_\tau = |z|^{d/c}\Theta_1(\arg z)$ is a weak solution to (19) *with $A = A_\tau$.*

We note that the matrix A_τ may be equivalently written in the form

$$\begin{aligned} A_\tau(z) &= \begin{bmatrix} k_{\tau,1}\cos^2\theta + k_{\tau,2}\sin^2\theta & (k_{\tau,1} - k_{\tau,2})\sin\theta\cos\theta \\ (k_{\tau,1} - k_{\tau,2})\sin\theta\cos\theta & k_{\tau,1}\sin^2\theta + k_{\tau,2}\cos^2\theta \end{bmatrix} \\ &= JK_\tau J^T \end{aligned}$$

where $K_\tau = \mathrm{diag}(k_{\tau,1}, k_{\tau,2})$.

The following equivalence between Beltrami equations and elliptic equations of the form (19) is well-known. See, e.g., [2, 16].

Lemma 1. *Let $g \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{C})$ satisfy the Beltrami equation*

$$\overline{\partial} g = \mu\partial g + \nu\overline{\partial g} \quad \text{in } \Omega, \tag{24}$$

where $\mu, \nu \in L^\infty(\Omega, \mathbf{C})$ satisfy $|\mu| + |\nu| \leq \kappa < 1$ a.e. in Ω. Let $B_{\mu,\nu}$ be the bounded matrix-valued function defined in terms of the Beltrami coefficients μ, ν by

$$B_{\mu,\nu} = \frac{1}{\Delta_1}\left(\begin{bmatrix} |1-\mu|^2 & -2\Im(\mu-\nu) \\ -2\Im(\mu+\nu) & |1+\mu|^2 \end{bmatrix} - |\nu|^2\mathbf{I}\right),$$

where $\Delta_1 = |1+\nu|^2 - |\mu|^2$ *and let* $\widetilde{B}_{\mu,\nu}$ *be defined by*

$$\widetilde{B}_{\mu,\nu} = \frac{1}{\Delta_2}\left(\begin{bmatrix} |1-\mu|^2 & -2\Im(\mu+\nu) \\ -2\Im(\mu-\nu) & |1+\mu|^2 \end{bmatrix} - |\nu|^2\mathbf{I}\right),$$

where $\Delta_2 = |1-\nu|^2 - |\mu|^2$. *Then* $\Re(g)$ *is a weak solution tor the elliptic equation* (19) *with* $A = B_{\mu,\nu}$ *and* $\Im(g)$ *is a weak solution tor* (19) *with* $A = \widetilde{B}_{\mu,\nu}$.

Proof. Setting $z = x+iy = (x,y)^T$, $g(z) = u(x,y) + iv(x,y)$, we have:

$$\overline{\partial} g = \frac{1}{2}\begin{bmatrix} u_x - v_y \\ u_y + v_x \end{bmatrix}, \quad \partial g = \frac{1}{2}\begin{bmatrix} u_x + v_y \\ -u_y + v_x \end{bmatrix}.$$

Setting

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

for every z we have

$$Qz = \begin{bmatrix} -y \\ x \end{bmatrix} = iz, \quad Rz = \begin{bmatrix} x \\ -y \end{bmatrix} = \overline{z}.$$

Hence, we can write

$$\overline{\partial} g = \frac{1}{2}\left(\nabla u + Q\nabla v\right), \quad \partial g = \frac{1}{2}R\left(\nabla u - Q\nabla v\right).$$

Setting

$$M = \begin{bmatrix} \Re(\mu) & -\Im(\mu) \\ \Im(\mu) & \Re(\mu) \end{bmatrix}, \quad N = \begin{bmatrix} \Re(\nu) & -\Im(\nu) \\ \Im(\nu) & \Re(\nu) \end{bmatrix},$$

equation (24) may be written in the form:

$$\nabla u + Q\nabla v = MR\left(\nabla u - Q\nabla v\right) + N\left(\nabla u - Q\nabla v\right).$$

It follows that

$$\left(I - MR - N\right)\nabla u = -\left(I + MR + N\right)Q\nabla v$$

and consequently u satisfies

$$\left(I + MR + N\right)^{-1}\left(I - MR - N\right)\nabla u = -Q\nabla v$$

and v satisfies

$$-Q\left(I - MR - N\right)^{-1}\left(I + MR + N\right)Q\nabla v = Q\nabla u.$$

By direct computation,

$$\begin{aligned} B_{\mu,\nu} &= \left(I + MR + N\right)^{-1}\left(I - MR - N\right), \\ \widetilde{B}_{\mu,\nu} &= -Q\left(I - MR - N\right)^{-1}\left(I + MR + N\right)Q = -QB_{-\mu,-\nu}Q. \end{aligned}$$

Now the conclusion follows observing that $\operatorname{div}(Q\nabla\cdot) = 0$. □

For every matrix A let

$$\widehat{A} = \frac{A}{\det A}.$$

Lemma 1 implies the following correspondence.

Lemma 2. *Let* $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{C})$ *be a solution to* (1) *with* $\Im(\nu) = 0$ *and let* $A_{\mu,\nu}$ *be defined by*

$$A_{\mu,\nu} = \frac{1}{\Delta}\left(\begin{bmatrix} |1-\mu|^2 & -2\Im(\mu) \\ -2\Im(\mu) & |1+\mu|^2 \end{bmatrix} - \nu^2 \mathbf{I}\right), \tag{25}$$

where $\Delta = (1 + |\mu| + \nu)(1 - |\mu| + \nu)$. *Then,* $\Re(f)$ *satisfies* (19) *with* $A = A_{\mu,\nu}$ *and* $\Im(f)$ *satisfies* (19) *with* $A = \widehat{A}_{\mu,\nu}$.

Proof. In view of Lemma 1, we need only check that when $\Im(\nu) = 0$ we have

$$\widetilde{B}_{\mu,\nu} = \frac{B_{\mu,\nu}}{\det B_{\mu,\nu}} = \widehat{B}_{\mu,\nu}. \tag{26}$$

Let

$$\Gamma_{\mu,\nu} = \begin{bmatrix} |1-\mu|^2 - \nu^2 & -2\Im(\mu) \\ -2\Im(\mu) & |1+\mu|^2 - \nu^2 \end{bmatrix}.$$

Then

$$B_{\mu,\nu} = \frac{\Gamma_{\mu,\nu}}{\Delta_1}, \quad \widetilde{B}_{\mu,\nu} = \frac{\Gamma_{\mu,\nu}}{\Delta_2}$$

with $\Delta_1 = (1+\nu)^2 - |\mu|^2 = (1 + \nu + |\mu|)(1 + \nu - |\mu|)$ and $\Delta_2 = (1-\nu)^2 - |\mu|^2 = (1 - \nu + |\mu|)(1 - \nu - |\mu|)$. On the other hand,

$$\det \Gamma_{\mu,\nu} = (1 + |\mu| + \nu)(1 + |\mu| - \nu)(1 - |\mu| + \nu)(1 - |\mu| - \nu)$$

and therefore $\Delta_2 = \det \Gamma_{\mu,\nu}/\Delta_1$. It follows that

$$\widetilde{B}_{\mu,\nu} = \frac{\Gamma_{\mu,\nu}}{\Delta_2} = \frac{\Delta_1}{\det \Gamma_{\mu,\nu}}\Gamma_{\mu,\nu} = \frac{\Delta_1^2}{\det \Gamma_{\mu,\nu}}\frac{\Gamma_{\mu,\nu}}{\Delta_1} = \frac{B_{\mu,\nu}}{\det B_{\mu,\nu}},$$

and (26) is established. □

The following lemma states that the function $\gamma(A)$ defined in (20) attains the same value on A and $\widehat{A}$.

Lemma 3. *For any matrix valued function* A *we have*

$$\gamma(A) = \gamma(\widehat{A})$$

where $\gamma(A)$ *is the quantity defined in* (20).

Proof. We have $\det \widehat{A} = (\det A)^{-1}$, and therefore

$$\frac{\widehat{A}}{\sqrt{\det \widehat{A}}} = \frac{A}{\sqrt{\det A}}. \tag{27}$$

Furthermore, for every $S \subset \Omega$ and for every $\varphi, \psi \in L^\infty(S)$,

$$\frac{\sup \varphi}{\inf \psi} = \frac{\sup \psi^{-1}}{\inf \varphi^{-1}}$$

and

$$\inf_S \frac{\det \widehat{A}}{\varphi\psi} = \frac{1}{\sup_S(\varphi\psi \det A)}, \quad \sup_S \frac{\det \widehat{A}}{\varphi\psi} = \frac{1}{\inf_S(\varphi\psi \det A)}.$$

Hence,

$$\frac{\inf_S \det \widehat{A}/(\varphi\psi)}{\sup_S \det \widehat{A}/(\varphi\psi)} = \frac{\inf_S \det A/(\varphi^{-1}\psi^{-1})}{\sup_S \det A/(\varphi^{-1}\psi^{-1})}. \tag{28}$$

It follows from (27) and (28) that for any function $F\colon \mathbf{R} \to \mathbf{R}$

$$\begin{aligned}
&\sqrt{\frac{\sup \varphi}{\inf \psi}} \frac{1}{|S_\rho(x)|} \int_{S_\rho(x)} \sqrt{\frac{\psi}{\varphi}} \frac{\langle n, \widehat{A}n\rangle}{\sqrt{\det \widehat{A}}}\, F\left(\frac{\inf_{S_\rho(x)} \frac{\det \widehat{A}}{\varphi\psi}}{\sup_{S_\rho(x)} \frac{\det \widehat{A}}{\varphi\psi}}\right) \\
&= \sqrt{\frac{\sup \psi^{-1}}{\inf \varphi^{-1}}} \frac{1}{|S_\rho(x)|} \int_{S_\rho(x)} \sqrt{\frac{\varphi^{-1}}{\psi^{-1}}} \frac{\langle n, An\rangle}{\sqrt{\det A}}\, F\left(\frac{\inf_{S_\rho(x)} \frac{\det A}{\varphi^{-1}\psi^{-1}}}{\sup_{S_\rho(x)} \frac{\det A}{\varphi^{-1}\psi^{-1}}}\right).
\end{aligned}$$

Now the statement follows by taking $F(t) = (4\pi^{-1} \arctan t^{1/4})^{-1}$ and observing that $\varphi^{-1} \in \mathscr{B}_{x,\rho}$ whenever $\varphi \in \mathscr{B}_{x,\rho}$. □

At this point, we can provide the proof of Theorem 1.

Proof of Theorem 1. In view of Lemma 2, Lemma 3 and Theorem 3, $\Re(g)$ and $\Im(g)$ are α-Hölder continuous with $\alpha \ge \gamma(A_{\mu,\nu})$, where $A_{\mu,\nu}$ is the matrix defined in (25). Setting $\xi = x + \rho e^{it}$, $t \in \mathbf{R}$ for every $\xi \in S_\rho(x) \subset \Omega$, we have $n(\xi) = e^{it}$. We recall that $\Delta = (1 + |\mu| + \nu)(1 - |\mu| + \nu) = (1+\nu)^2 - |\mu|^2$. Hence, we compute

$$\begin{aligned}
&\Delta\, \langle n(\xi), A_{\mu,\nu}(\xi) n(\xi)\rangle = \Delta \langle e^{it}, A_{\mu,\nu}(\xi) e^{it}\rangle \\
&= \Delta\, (a_{11}\cos^2 t + 2a_{12} \sin t \cos t + a_{22} \sin^2 t) \\
&= 1 + |\mu|^2 - \nu^2 - 2(\Re(\mu)\cos 2t + \Im(\mu) \sin 2t) = |1 - \overline{n}^2\mu|^2 - \nu^2.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\Delta^2 \det A_{\mu,\nu} &= (|1-\mu|^2 - \nu^2)(|1+\mu|^2 - \nu^2) - 4\Im(\mu)^2 \\
&= (1 + |\mu|^2 - \nu^2)^2 - 4|\mu|^2 = ((1-|\mu|)^2 - \nu^2)((1+|\mu|)^2 - \nu^2) \\
&= (1 - |\mu| + \nu)(1 - |\mu| - \nu)(1 + |\mu| + \nu)(1 + |\mu| - \nu) \\
&= (1 - (|\mu| - \nu)^2)(1 - (|\mu| + \nu)^2)
\end{aligned}$$

and therefore

$$\frac{\langle n, A_{\mu,\nu} n\rangle}{\sqrt{\det A_{\mu,\nu}}} = \frac{\Delta\langle n, A_{\mu,\nu} n\rangle}{\sqrt{\Delta^2 \det A_{\mu,\nu}}} = \frac{|1 - \overline{n}^2\mu|^2 - \nu^2}{\sqrt{(1 - (|\mu| - \nu)^2)(1 - (|\mu| + \nu)^2)}}.$$

Finally, recalling the definition of Δ, we derive

$$\det A_{\mu,\nu} = \frac{(1 + |\mu| - \nu)(1 - |\mu| - \nu)}{(1 + |\mu| + \nu)(1 - |\mu| + \nu)} = \frac{(1-\nu)^2 - |\mu|^2}{(1+\nu)^2 - |\mu|^2}.$$

Inserting the expressions above into (20), we obtain (4). □

We now turn to the proof of Theorem 2. We let $\mu_{0,\tau}, \nu_{0,\tau} \colon \mathbf{R} \to \mathbf{R}$ be the bounded, piecewise constant, 2π-periodic functions defined in $[0, 2\pi)$ by

$$\mu_{0,\tau}(\theta) = \begin{cases} 0, & \text{if } \theta \in I_1 \cup I_3, \\ (M - M^{1-2\tau})/(1 + M + M^{1-2\tau} + M^{2(1-\tau)}), & \text{if } \theta \in I_2 \cup I_4, \end{cases}$$

and

$$\nu_{0,\tau}(\theta) = \begin{cases} 0, & \text{if } \theta \in I_1 \cup I_3, \\ (M^{2(1-\tau)} - 1)/(1 + M + M^{1-2\tau} + M^{2(1-\tau)}), & \text{if } \theta \in I_2 \cup I_4, \end{cases}$$

and we set

$$\mu_\tau(z) = -\mu_{0,\tau}(\arg z)\, z\overline{z}^{-1}, \quad \nu_\tau(z) = -\nu_{0,\tau}(\arg z).$$

The following holds.

Proposition 2. *Let B the unit disk in $\mathbf{R}^2$ and let $f_\tau \in W^{1,2}(B, \mathbf{C})$ be defined in $B \setminus \{0\}$ by*

$$f_\tau(z) = |z|^{d/c} \left(\Theta_{\tau,1}(\arg z) + i\Theta_{\tau,2}(\arg z) \right).$$

Then f_τ satisfies (1) with $\mu = \mu_\tau$ and $\nu = \nu_\tau$. Furthermore, there exists $M_0 > 1$ such that

$$\beta(\mu_\tau, \nu_\tau) = \frac{d}{c},$$

for every $M \in (1, M_0^{1/\tau})$ if $\tau > 0$ and with no restriction on M if $\tau = 0$.

In order to prove Proposition 2, we first need a lemma.

Lemma 4. *Suppose μ, ν are of the form (9)–(10) and let k_1, k_2 be the corresponding functions defined in (14). Then $A_{\mu,\nu}$ as defined in (25) is given by*

$$\begin{aligned} A_{\mu,\nu}(z) &= J(\arg z) \begin{bmatrix} k_1(\arg z) & 0 \\ 0 & k_2(\arg z) \end{bmatrix} J^*(\arg z) \\ &= \begin{bmatrix} k_1 \cos^2\theta + k_2 \sin^2\theta & (k_1 - k_2)\sin\theta\cos\theta \\ (k_1 - k_2)\sin\theta\cos\theta & k_1 \sin^2\theta + k_2\cos^2\theta \end{bmatrix} \\ &= (k_1 - k_2)\frac{z \otimes z}{|z|^2} + k_2 \mathbf{I}. \end{aligned}$$

Proof. The assumptions (9)–(10) on μ, ν imply that

$$\Delta(z) = (1 + \mu_0(\theta) - \nu_0(\theta))(1 - \mu_0(\theta) - \nu_0(\theta)).$$

and

$$\mu(z) = -\mu_0(\theta)\,(\cos 2\theta + i \sin 2\theta).$$

Hence,

$$\begin{aligned}\Delta(A_{\mu,\nu})_{11} &= |1-\mu|^2-\nu^2 = 1+2\mu_0\cos 2\theta+\mu_0^2-\nu_0^2\\ &= [(1+\mu_0)^2-\nu_0^2]\cos^2\theta+[(1-\mu_0)^2-\nu_0^2]\sin^2\theta,\\ \Delta(A_{\mu,\nu})_{22} &= |1+\mu|^2-\nu^2\\ &= [(1-\mu_0)^2-\nu_0^2]\cos^2\theta+[(1+\mu_0)^2-\nu_0^2]\sin^2\theta,\\ \Delta(A_{\mu,\nu})_{12} &= -2\Im(\mu)\\ &= 4\mu_0\sin\theta\cos\theta.\end{aligned}$$

Dividing by Δ and observing that

$$\begin{aligned}\frac{(1+\mu_0)^2-\nu_0^2}{\Delta} &= \frac{1+\mu_0+\nu_0}{1-\mu_0-\nu_0} = k_1,\\ \frac{(1-\mu_0)^2-\nu_0^2}{\Delta} &= \frac{1-\mu_0+\nu_0}{1+\mu_0-\nu_0} = k_2,\\ \frac{4\mu_0}{\Delta} &= k_1-k_2,\end{aligned}$$

we obtain the asserted expression for $A_{\mu,\nu}$. □

Proof of Proposition 2. By direct check, $(\Theta_{\tau,1},\Theta_{\tau,2})$ satisfies (13) with $k_1=k_{\tau,1}$, $k_2=k_{\tau,2}$ as defined in (22)–(23), respectively, and $\alpha_\tau=d/c$. Hence, in view of Proposition 1, f_τ satisfies (1) with $\mu=\mu_\tau$ and $\nu=\nu_\tau$. In view of Lemma 2 and Lemma 4, $\Re(f_\tau)$ satisfies equation (19) with $A=A_\tau$ defined in (21) and $\Im(f_\tau)$ satisfies equation (19) with $A=\widehat{A_\tau}$. By Theorem 2–(ii), $\Re(f_\tau)$ and $\Im(f_\tau)$ are Hölder continuous with exponent exactly $\beta(\mu_\tau,\nu_\tau)=\gamma(A_\tau)=\gamma(\widehat{A_\tau})$ whenever $M\in(0,M_0^{1/\tau})$ if $\tau>0$ and with no restriction on M if $\tau=0$. Thus, Proposition 2 is established. □

Proof of Theorem 2. The proof is a direct consequence of Proposition 2. □

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References

[1] Andreian Cazacu, C.: Sur les transformations pseudo-analytiques. - Rev. Math. Pures Appl. 2, 1957, 383–397 (in Stoïlow anniversary volume).

[2] Astala, K., T. Iwaniec, and G. Martin: Elliptic partial differential equations and quasiconformal mappings in the plane. - In preparation.

[3] Astala, K., T. Iwaniec, P. Koskela, and G. Martin: Mappings of *BMO*-bounded distortion. - Math. Ann. 317, 2000, 703–726.

[4] Bers, L., and L. Nirenberg: On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications. - In: Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, 111–140. Edizioni Cremonese, Roma, 1955.

[5] Finn, R., and J. Serrin: On the Hölder continuity of quasiconformal mappings and elliptic mappings. - Trans. Amer. Math. Soc. 89, 1958, 1–15.

[6] Gutlyanskiĭ, V. Ya., O. Martio, T. Sugawa, and M. Vuorinen: On the degenerate Beltrami equation. - Trans. Amer. Math. Soc. 357:3, 2005, 875–900.

[7] Gutlyanskiĭ V. Ya, and V. I. Ryazanov: On the theory of the local behavior of quasiconformal mappings. - Izv. Ross. Akad. Nauk Ser. Mat. 59:3, 1995, 471–498.

[8] Iwaniec, T., and G. Martin: Geometric function theory and non-linear analysis. - Clarendon Press, Oxford University Press, New York, 2001.

[9] Iwaniec, T., and C. Sbordone: Quasiharmonic fields. - Ann. Inst. H. Poincaré Anal. Non Linéaire 18:5, 2001, 519–572.

[10] Morrey, C. B.: On the solutions of quasi-linear elliptic partial differential equations. - Trans. Amer. Math. Soc. 43:1, 1938, 126–166.

[11] Piccinini, L. C., and S. Spagnolo: On the Hölder continuity of solutions of second order elliptic equations in two variables. - Ann. Scuola Norm. Sup. Pisa (3) 26:2, 1972, 391–402.

[12] Reich, E., and H. R. Walczak: On the behavior of Quasiconformal Mappings at a Point. - Trans. Amer. Math. Soc. 117:2, 1965, 338–351.

[13] Ricciardi, T.: A sharp Hölder estimate for elliptic equations in two variables. - Proc. Roy. Soc. Edinburgh Sect. A 135, 2005, 165–173.

[14] Ricciardi, T.: On the best Hölder exponent for two dimensional elliptic equations in divergence form. - Proc. Amer. Math. Soc. (to appear); preprint available on arXiv:math.AP/0510606.

[15] Schatz, A.: On the local behavior of homeomorphic solutions of Beltrami's equations. - Duke Math. J. 35, 1968, 289–306.

[16] Vekua, I. N.: Generalized analytic functions. - Pergamon Press, London-Paris-Frankfurt; Addison-Wesley Publishing Co. Inc., Reading Mass., 1962.

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