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LOCAL CONVEXITY PROPERTIES OF j-METRIC BALLS

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Abstract. This paper deals with local convexity properties of the j-metric. We consider convexity and starlikeness of the j-metric balls in convex, starlike and general subdomains of $\mathbf{R}^n$.

1. Introduction

The *j-distance* in a proper subdomain G of the Euclidean space $\mathbf{R}^n$, $n \geq 2$, is defined by

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x),d(y)\}}\right),$$

where $d(x)$ is the Euclidean distance between x and ∂G. If the domain G is understood from the context we use notation j instead of j_G.

The j-distance was first introduced by Gehring and Palka [GP] in 1976 in a slightly different form and in the above form, by Vuorinen [Vu2] in 1985. The j-distance is actually a metric and a proof of the triangle inequality valid for general metric spaces is given in [S]. Previously the j-metric has been studied in connection with the study of other metrics [GO, H, S, V, Vu2]. See also recent papers [HL, L]. In spite of these studies many basic questions of the j-metric remain open and some of them will be studied here.

The purpose of this paper is to study metric spaces (G, j_G) and especially local convexity properties of *j-metric balls* or in short *j-balls* defined by

$$B_j(x,M) = \{y \in G \colon j(x,y) < M\},$$

where $M > 0$ and $x \in G$. In the dimension $n = 2$ we call these *j-metric disks* or *j-disks.*

Vuorinen suggested in [Vu4] a general question about the convexity of balls of small radii in metric spaces. This paper is motivated by this question and we will provide an answer in a particular case. Our main result is the following theorem. For the definition of starlike domains see 3.3.

Theorem 1.1. *For a domain $G \subsetneq \mathbf{R}^n$ and $x \in G$ the j-balls $B_j(x,M)$ are convex if $M \in (0,\log 2]$ and strictly starlike with respect to x if $M \in \big(0,\log(1+\sqrt{2})\big]$.*

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In Section 2 we consider general properties of the j-metric and show that for any G there exists points such that there is no geodesic between them. In Section 3 we consider local convexity properties of j-balls in punctured space and in Section 4 we extend these results to an arbitrary domain $G \subsetneq \mathbf{R}^n$. We will further consider convexity of j-balls in convex domains and starlikeness of j-balls in starlike domains.

2. Properties of the j-metric

Throughout this paper $G \subsetneq \mathbf{R}^n$, $n \geq 2$, is a domain. We denote $m(x,y) = \min\{d(x), d(y)\}$ and use notation $B^n(x,M)$ for the Euclidean balls and $S^{n-1}(x,M)$ for the Euclidean spheres. We often identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$.

In 1976 Gehring and Palka [GP] also introduced the quasihyperbolic metric, which has been widely applied in geometric function theory and mathematical analysis in general, see e.g. [Vu3, V]. The *quasihyperbolic distance* between two points x and y in a proper subdomain G of the Euclidean space $\mathbf{R}^n$, $n \geq 2$, is defined by

$$k_G(x,y) = \inf_{\alpha \in \Gamma_{xy}} \int_\alpha \frac{|dx|}{d(x)},$$

where Γ_{xy} is the collection of all rectifiable curves in G joining x and y. We denote the *quasihyperbolic ball* by

$$D_G(x,M) = \{y \in G \colon k_G(x,y) < M\}.$$

The quasihyperbolic metric is closely related with the j-metric. By [GP, Lemma 2.1] j_G is always a minorant of k_G, in other words, for a proper subdomain G of $\mathbf{R}^n$ we have

$$j_G(x,y) \leq k_G(x,y)$$

for all $x, y \in G$.

The following result can be used to estimate the quasihyperbolic metric from above by the j-metric.

Proposition 2.1. [Vu3, Lemma 3.7] *Let $G \subsetneq \mathbf{R}^n$ be a domain, $x \in G$, $y \in B^n\big(x, d(x)\big)$ and $s \in (0,1)$. Then*

$$k_G(x,y) \leq \frac{1}{1-s} j_G(x,y).$$

The following lemma gives Euclidean bounds for the j-balls.

Proposition 2.2. [S, Theorem 3.8] *For a proper subdomain $G \subset \mathbf{R}^n$, $x \in G$ and $M > 0$ we have*

$$B^n\big(x, r\, d(x)\big) \subset B_j(x,M) \subset B^n\big(x, R\, d(x)\big),$$

where $r = 1 - e^{-M}$ and $R = e^M - 1$. Moreover

$$B_j(x,M) \subset \left\{ z \in G \colon e^{-M} d(x) \leq d(z) \leq e^M d(x) \right\}.$$

Remark 2.3. A similar result to Proposition 2.2 is also true for the quasihyperbolic metric see [Vu1, page 347].

By Proposition 2.2 the j-ball $B_j(x,M)$ shrinks towards the center $\{x\}$ as M approaches 0. The following lemma shows that the j-balls $B_j(x,M)$ exhaust the domain G.

Lemma 2.4. *Let $G \subset \mathbf{R}^n$ be a bounded domain and fix $x \in G$ and $s \in (0, d(x)]$. Then*

$$\{y \in G\colon d(y) > s\} \subset B_j\big(x, \log(1 + d/s)\big),$$

for $d = \sup_{z \in \partial G} |x - z|$.

Proof. Let us assume $d(y) > s$. Then either $m(x,y) = d(x) \geq s$ or $m(x,y) = d(y) > s$. In both cases $m(x,y) \geq s$ and since $|x-y| < d$ for all $y \in G$ we have

$$j(x,y) = \log\left(1 + \frac{|x-y|}{m(x,y)}\right) < \log\left(1 + \frac{d}{s}\right). \qquad \square$$

Let us denote the set of closest boundary points of a point x in a domain $G \subset \mathbf{R}^n$ by

$$R_x = \{z \in \partial G\colon |z - x| = d(x)\}.$$

The next result characterizes the case of equality in the triangle inequality for the j-metric. Its proof is based on the proof of the triangle inequality [S, Lemma 2.2].

Theorem 2.5. *Let $x, y, z \in G \subsetneq \mathbf{R}^n$ be distinct points and $d(x) \leq d(z)$. Then*

$$j_G(x,z) = j_G(x,y) + j_G(y,z)$$

implies that x, z and u are collinear for some $u \in R_x$ and $y \in (x,z)$ with $d(x) < d(y) < d(z)$.

Proof. By definition $j_G(x,z) < j_G(x,y) + j_G(y,z)$ is equivalent to

$$\frac{|x-z|}{m(x,z)} < \frac{|x-y|}{m(x,y)} + \frac{|y-z|}{m(y,z)} + \frac{|x-y||y-z|}{m(x,y)m(y,z)}. \tag{2.6}$$

The assumption $d(x) \leq d(z)$ implies $m(x,z) = d(x)$.

If $d(y) \leq d(x)$, then the inequality (2.6) is equivalent to

$$|x-z| < |x-y|\frac{d(x)}{d(y)} + |y-z|\frac{d(x)}{d(y)} + \frac{|x-y||y-z|}{d(y)}\frac{d(x)}{d(y)},$$

which is true, because $|x-z| \leq |x-y| + |y-z|$, $(|x-y||y-z|)/d(y) > 0$ and $d(x)/d(y) \geq 1$.

If $d(y) > d(x)$, then the inequality (2.6) is equivalent to

$$|x-z| < |x-y| + |y-z|\left(\frac{d(x) + |x-y|}{m(y,z)}\right),$$

which is false if and only if x, y and z are collinear and

$$\frac{d(x) + |x-y|}{m(y,z)} = 1.$$

If $d(x) = d(z)$, then $d(x)/m(y,z) = 1$ and

$$\frac{d(x)+|x-y|}{m(y,z)} > 1. \tag{2.7}$$

If $d(x) < d(z) < d(y)$, then the inequality (2.7) is true, because $d(x)+|x-y| \geq d(y) > d(z) = m(y,z)$. If $d(x) < d(y) \leq d(z)$, then the inequality (2.7) is true if and only if $y \notin \{k(x-u)\colon k>0\}$, where $u \in R_x$. □

The implication of Theorem 2.5 in the other direction was proved by Hästö, Ibragimov and Lindén [HIL, Corollary 3.7].

Definition 2.8. Let $G \subsetneq \mathbf{R}^n$ be a domain and γ a curve in G. If

$$j(x,y)+j(y,z)=j(x,z)$$

for all $x,z \in \gamma$ and $y \in \gamma'$, where γ' is the subcurve of γ joining x and z, then γ is a *geodesic segment* or shortly a *geodesic*. We denote a geodesic between x and y by $J[x,y]$.

By Theorem 2.5 and the result of Hästö, Ibragimov and Lindén we can easily find all geodesics $J[x,y]$ for any domain G. The geodesic needs to satisfy the triangle inequality as equality at each point and therefore the geodesic can only be a line segment l with the following property.

Lemma 2.9. *Let $G \subsetneq \mathbf{R}^n$ be a domain and $J[x,y]$ be a geodesic segment with $x,y \in G$. There exists $u \in \partial G$ such that $u \in R_s$ for all $s \in J[x,y]$ and u and $J[x,y]$ are collinear.*

Proof. Let us assume, on the contrary, that there exists $z \in J[x,y]$ such that $d(z) < d(x) - |x-z|$. Now $j_G(x,z)+j_G(z,y)=j_G(x,y)$ is equivalent to

$$d(z)|x-z| + \big(d(x)+|x-z|\big)|z-y| = d(z)|x-y|.$$

We have

$$\begin{aligned} d(z)|x-y| &\leq d(z)|x-z| + d(z)|z-y| \\ &< d(z)|x-z| + \big(d(x)+|x-z|\big)|z-y| \\ &= d(z)|x-y| \end{aligned}$$

which is a contradiction. □

Theorem 2.10. *Let $G \subsetneq \mathbf{R}^n$ be a domain. Then there exist $x,y \in G$ such that there is no geodesic $J[x,y]$.*

Proof. Let us assume, on the contrary, that for all $x,y \in G$ there exists a geodesic $J[x,y]$. Since G is a domain, we can choose $x,y,z \in G$ to be three distinct noncollinear points. Now there exists a geodesic $J[x,y]$ from x to y. We may assume $d(x) < d(y)$ and then by Lemma 2.9 $B^n\big(x,d(x)\big) \subset B^n\big(y,d(y)\big) \subset G$.

On the other hand, there exists a geodesic $J[x,z]$ from x to z and therefore there has to exist a point $u \in S^{n-1}\big(x,d(x)\big) \cap \partial G$ such that x, z and u are collinear.

This is a contradiction, because x, y and u are noncollinear and therefore $u \in B^n\big(y, d(y)\big)$. □

Remark 2.11. By Theorem 2.10 a j-metric geodesic does not always exist between two points. Gehring and Osgood have proved [GO, Lemma 1] that for the quasihyperbolic metric there always exists a geodesic between two points of a domain $G \subsetneq \mathbf{R}^n$.

However, the geodesics of the j-metric are unique while the geodesics of the quasihyperbolic metric need not be unique.

3. Convexity and starlikeness of j-balls in punctured space

In this section we consider the case $G = \mathbf{R}^n \setminus \{0\}$. By definition the j-balls in punctured space $G = \mathbf{R}^n \setminus \{0\}$ are similar, which means that $B_j(x, M)$ can be mapped onto $B_j(y, M)$ for all $x, y \in G$ by rotation and stretching. We see easily that these balls are also symmetric along the line that goes through 0 and the center point.

Theorem 3.1. *Let $x \in \mathbf{R}^n \setminus \{0\}$. Then*

1) *the j-ball $B_j(x, M)$ is convex if and only if $M \in (0, \log 2]$.*
2) *the j-ball $B_j(x, M)$ is strictly convex if and only if $M \in (0, \log 2)$.*

Proof. 1) By similarity we can assume $x = e_1$ and by symmetry it is sufficient to consider only the case $n = 2$. We will consider $\partial B_j(1, M)$ for fixed M. By definition we have for $z \in \partial B_j(x, M)$

$$M = \begin{cases} \log(1 + |z-1|), & |z| \geq 1, \\ \log\left(1 + |z-1|/|z|\right), & |z| < 1, \end{cases}$$

which is equivalent to

$$e^M - 1 = \begin{cases} |z-1|, & |z| \geq 1, \\ |1 - 1/z|, & |z| < 1. \end{cases}$$

For $|z| \geq 1$ the $\partial B_j(1, M)$ is an arc of a circle with center 1 and radius $e^M - 1$. For $|z| < 1$ the $\partial B_j(1, M)$ is a circle that goes through points $1/(e^M)$ and $1/(2 - e^M)$ and has center on the real axis. This means that the center of the circle is $c = 1/\big(e^M(2 - e^M)\big)$ and the radius of the circle is $|e^M - 1|/|e^M(2 - e^M)|$. Now $c > 1$, if $M \leq \log 2$, and $c < 0$, if $M > \log 2$. Therefore $\partial B_j(1, M)$ is convex for $M \leq \log 2$ and not convex for $M > \log 2$.

2) We have $c \in (1, \infty)$, where c is as above. Therefore $B_j(x, M)$ is strictly convex. In the case $M = \log 2$ we have $c = \infty$ and $B_j(x, M)$ is not strictly convex. □

Remark 3.2. For fixed $x \in G$ the quasihyperbolic ball $D_G(x, M)$ is strictly convex in $G = \mathbf{R}^n \setminus \{0\}$ if and only if $M \in (0, 1]$ [K].

Clearly $B_j(x, M)$ is never smooth. We will next define starlikeness of a domain.

Definition 3.3. Let $G \subset \mathbf{R}^n$ be a bounded domain and $x \in G$. We say that G is *starlike with respect to* x if each line segment from x to $y \in G$ is contained in G. The domain G is *strictly starlike with respect to* x for $x \in G$ if each ray from x meets ∂G at exactly one point.

The next theorem determines the values of M for which the j-ball $B_j(x, M)$ is strictly starlike with respect to x.

Theorem 3.4. *For $x \in \mathbf{R}^n \setminus \{0\}$ the j-ball $B_j(x, M)$ is strictly starlike with respect to x if and only if $M \in \big(0, \log(1+\sqrt{2})\big]$.*

Proof. Because the j-balls are similar it is sufficient to consider $x = e_1$. By symmetry it is sufficient to consider the case $n = 2$ and the part of $\partial B_j(1, M)$ that is above the real axis. If $M \geq \log 3$, then $B_j(1, M) = B^2(1, r) \setminus B^2(c, s)$, where c, r and s are given in the proof of Theorem 3.1 and $B^2(c, s) \subset B^2(1, r)$. Therefore $B_j(1, M)$ can be starlike with respect to 1 only for $M < \log 3$.

Let us assume $M < \log 3$. By the proof of Theorem 3.1 $B_j(1, M) = B^2(1, r) \setminus B^2(c, s)$. Let us denote the point of intersection of $S^1(1, r)$ and $S^1(c, s)$ above the real axis by z. Now z is also the point of intersection of the unit circle and the boundary $\partial B_j(1, M)$. Let us denote by l the line that goes through points 1 and z. Now $B_j(1, M)$ is strictly starlike with respect to 1 if and only if $l \cap B^2(1, r) \cap B^2(c, s) = \emptyset$. If z is a tangent of $S^1(c, s)$, then the circles $S^1(1, r)$ and $S^1(c, s)$ are perpendicular and M has the largest value such that $B_j(1, M)$ is starlike with respect to 1.

By the proof of Theorem 3.1 we have $c = -1/e^M(e^M - 2)$, $r = |1 - z| = e^M - 1$, $|1 - c| = (e^M - 1)^2/e^M(e^M - 2)$ and $s = |z - c| = (e^M - 1)/e^M(e^M - 2)$. Let us assume that z is a tangent of $S^1(c, s)$. Now by the Pythagorean Theorem

$$\frac{(e^M - 1)^4}{e^{2M}(e^M - 2)^2} = (e^M - 1)^2 + \frac{(e^M - 1)^2}{e^{2M}(e^M - 2)^2},$$

which is equivalent to $e^{2M} - 2e^M - 1 = 0$ and therefore

$$M = \log(1 + \sqrt{2}). \qquad \square$$

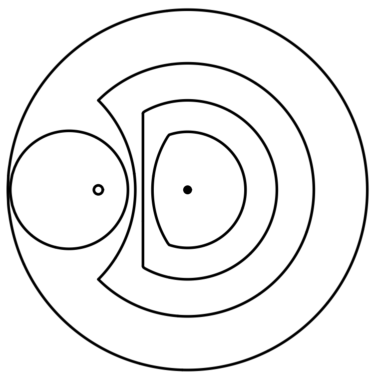

Figure 1. The boundaries of j-disks $j(1, M)$ in punctured plane $G = \mathbf{R}^2 \setminus \{0\}$ with $M = 0.5$, $M = \log 2$, $M = \log(1 + \sqrt{2})$ and $M = 1.1 \approx \log 3$.

Example 3.5. Let us consider the starlikeness of j-balls $B_j(x,M)$ with respect to $z \in B_j(x,M)$ for $M > \log 2$. By choosing $z = (e^{-M}+\varepsilon)x/|x|$ for $\varepsilon > 0$ and letting ε approach to zero we see that $B_j(x,M)$ is not starlike with respect to z.

On the other hand, if we choose $z = (e^M - \varepsilon)x/|x|$ for $\varepsilon > 0$ and $M < \log\big((3+\sqrt{5}/2\big)$, we see that $B_j(x,M)$ is strictly starlike with respect to z for small enough ε.

Remark 3.6. For fixed $x \in G$ the quasihyperbolic ball $D_G(x,M)$ is strictly starlike with respect to x in $G = \mathbf{R}^n \setminus \{0\}$ if and only if $M \in (0,\kappa]$ [K], where $\kappa \approx 2.83297$.

4. Convexity and starlikeness of j-balls

We will consider convexity and starlikeness of j-balls $B_j(x,M)$ for $M > 0$ in convex, starlike and general domains.

Let us consider j-balls in a domain G with a finite number of boundary points. The case $\operatorname{card}\partial G = 1$ is identical to $G = \mathbf{R}^n \setminus \{0\}$. If $\partial G = \{y_1, y_2\}$, then $B_{j_G}(x,M) = B_{j_{\mathbf{R}^n\setminus\{y_1\}}}(x,M) \cap B_{j_{\mathbf{R}^n\setminus\{y_2\}}}(x,M)$. This is clear, because the j-distance between a and b depends only on the closest boundary point of the end points a and b. Similarly for $\partial G = \{y_1, y_2, \dots, y_m\}$ we have

$$B_{j_G}(x,M) = \bigcap_{i=1}^{m} B_{j_{\mathbf{R}^n\setminus\{y_i\}}}(x,M).$$

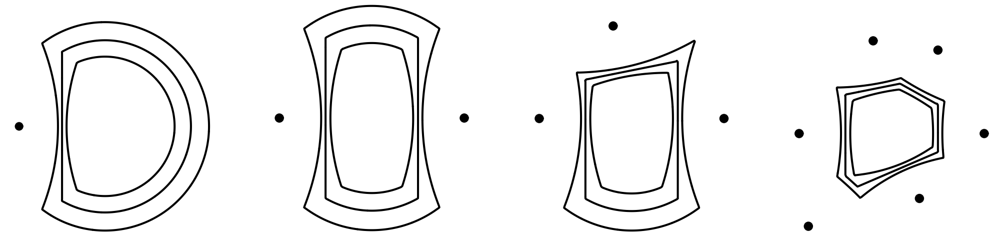

Figure 2. The boundaries of j-disks in a domain with 1, 2, 3 and 6 boundary points.

This gives an idea to prove Theorem 1.1, which shows that j-balls are convex in any domain G for small radius M.

Proof of Theorem 1.1. Let $x \in G$ be arbitrary. We claim that

$$A = B_{j_G}(x,M) = \bigcap_{z\in\partial G} B_{j_{\mathbf{R}^n\setminus\{z\}}}(x,M) = B. \tag{4.1}$$

Let $y \in B$. We can choose $z' \in \partial G$ with

$$j_{\mathbf{R}^n\setminus\{z'\}}(x,y) = \min_{z\in\partial G} j_{\mathbf{R}^n\setminus\{z\}}(x,y).$$

Because $z' \in \partial G$ we have $j_G(x,y) \le j_{\mathbf{R}^n\setminus\{z'\}}(x,y)$ and therefore $y \in A$.

On the other hand, let $y \in A$. By definition there is a point $z' \in \partial G$ with $\min\{|x-z'|, |y-z'|\} = \min_{z \in \partial G}\{|x-z|, |y-z|\}$. Now $j_{\mathbf{R}^n \setminus \{z'\}}(x,y) \le j_G(x,y)$ and $y \in B$.

By Theorem 3.1 each $B_{j_{\mathbf{R}^n \setminus \{z\}}}(x,M)$ is convex for $0 < M \le \log 2$ and (4.1) $B_{j_G}(x,M)$ is an intersection of convex domains and therefore it is convex.

If $M \in (0, \log 2]$, then $B_{j_G}(x,M)$ is convex and therefore also starlike with respect to x. If $M \in (\log 2, \log(1+\sqrt{2})]$, then

$$B_j(x,M) = B \setminus \left(\bigcup_{z \in \partial G} A_z \right),$$

where $B = B^n\big(x, (e^M-1)d(x)\big)$ and $A_z = B^n(c_z z, r_z)$ for $c_z = |z|/(e^M(2-e^M))$ and $r_z = |z||1-e^{-M}|/|e^M-2|$. Let us assume that $B_j(x,M)$ is not strictly starlike with respect to x. Now there exists $a, b \in B$ such that $b \in (x,a)$, $a \in B_j(x,M)$ and $b \notin B_j(x,M)$. Now $b \in B^n(c_z z, r_z)$ for some $z \in \partial G$. By the proof of Theorem 3.4 $a \in B^n(c_z z, r_z)$, which is a contradiction. □

Corollary 4.2. *For a domain $G \subsetneq \mathbf{R}^n$ and $x \in G$ the j-balls $B_j(x,M)$ are simply connected if $M \in \big(0, \log(1+\sqrt{2})\big]$.*

Proof. By Theorem 1.1 $B_{j_G}(x,M)$ is starlike with respect to x and therefore also simply connected. □

Corollary 4.3. *For a domain $G \subsetneq \mathbf{R}^n$ and $x \in G$ the j-balls $B_j(x,M)$ are strictly convex if $M \in (0, \log 2)$.*

Proof. By the proof of Theorem 1.1 and Theorem 3.1

$$B_j(x,M) = \bigcap_{z \in \partial G} (B_{z,1} \cap B_{z,2}),$$

where $B_{z,i}$ is a Euclidean ball and $x \in B_{z,i}$. Therefore $B_j(x,M)$ is strictly convex. □

Bounds of Theorem 1.1 are sharp as $G = \mathbf{R}^n \setminus \{0\}$ shows. Also the bound $\log(1+\sqrt{2})$ of Corollary 4.2 is sharp. This can be seen by choosing $G = \mathbf{R}^2 \setminus \{0, z\}$ for a certain z and considering $B_j(e_1, M)$ for $M > \log(1+\sqrt{2})$. By the proof of Theorem 3.1 we know that

$$B_j(e_1, M) = B^2(e_1, r_1) \setminus B^2(c, r_2)$$

for $r_1 = e^M - 1$, $c = e_1/\big(e^M(2-e^M)\big)$ and $r_2 = (e^M-1)/\big(e^M(e^M-2)\big)$. Let l be the tangent line of $B^2(c, r_2)$ that goes through e_1. Denote $\{y\} = S^1(c, r_2) \cap l$. Choose z to be the reflection of 0 in the line l. By a simple computation we have

$$|y - e_1| = \frac{e^M - 1}{\sqrt{e^M(e^M-2)}} < r_1.$$

Let us denote by c' the reflection of c in the line l. Now $B_{j_{\mathbf{R}^2\setminus\{0,z\}}}(e_1,M) = B^2(e_1,r_1)\setminus\big(B^2(c,r_2)\cup B^2(c',r_2)\big)$ and therefore $B_j(e_1,M)$ is disconnected for $M > \log(1+\sqrt{2})$.

Similar counterexamples can be constructed for $n>2$. Let us assume $n\ge 2$ and $M>\log(1+\sqrt{2})$. Now we choose

$$G = \mathbf{R}^n \setminus \big(S^{n-1}(z,|z|)\setminus B^n(e_1,1)\big),$$

where $z\in S^{n-1}(e_1,e^M-1)$ and the line $[z,e_1]$ is a tangent of $S^{n-1}(c,r)$ for $c = e_1/\big(e^M(2-e^M)\big)$ and $r = |1-e^M|/|e^M(2-e^M)|$. Let $y\in[z,e_1]\cap S^{n-1}(e_1,e^M-1)$. Now $j_G(e_1,y)=M$ and $j_G\big(e_1,\frac{1}{2}(z+y)\big)<M$. Therefore $B_j(e_1,M)$ is disconnected.

Remark 4.4. The idea of the proof of Theorem 1.1 cannot be used for the quasihyperbolic metric. We always have

$$D_G(x,M)\subset \bigcap_{z\in\partial G} D_{\mathbf{R}^n\setminus\{z\}}(x,M)$$

but inclusion in the other direction is not always true. For example $G=\mathbf{R}^n\setminus\{0,e_1\}$, $x=e_1/4$ and $M=1$ gives an counterexample. Now $y=e_1(1-1/e)$ is on the boundary $\partial D_G(x,M)$ because

$$k_G(x,y) = k_{\mathbf{R}^n\setminus\{0\}}(x,e_1/2) + k_{\mathbf{R}^n\setminus\{e_1\}}(e_1/2,y) = \log 2 + \log(e/2) = 1.$$

On the other hand, $z = e_1\big(1-3/(4e)\big)$ belongs to the boundary $\partial D_{\mathbf{R}^n\setminus\{e_1\}}(x,M)$. Now $0.632\approx|y|<|z|\approx 0.724$ and therefore $D_{\mathbf{R}^n\setminus\{0\}}(x,M)\cap D_{\mathbf{R}^n\setminus\{e_1\}}(x,M)\not\subset D_G(x,M)$.

The next theorem states convexity of j-balls in convex domains.

Theorem 4.5. *Let $M>0$, $G\subsetneq\mathbf{R}^n$ be a convex domain and $x\in G$. Then j-balls $B_j(x,M)$ are convex.*

Proof. By Theorem 1.1 we need to consider only the case $M>\log 2$. Let us divide G into two parts $D_1=\{z\in G\colon d(z)\ge d(x)\}$ and $D_2=G\setminus D_1$. We will first show that convexity of G implies convexity of D_1. Let us assume that D_1 is not convex. There exists $a,b\in D_1$ such that $c=(a+b)/2\notin D_1$ and $d(a)=d(x)=d(b)$. Now $B^n\big(a,d(x)\big)$ and $B^n\big(b,d(x)\big)$ does not contain any points of ∂G, but $B^n(c,r)$ for some $r<d(x)$ contains at least one point of ∂G. Therefore G is not convex, which is a contradiction.

Let us consider $B_j(x,M)\cap D_1$. By definition of the j-metric we have for $y\in\partial B_j(x,M)\cap D_1$

$$|x-y| = d(x)\big(e^M-1\big)$$

and therefore $\partial B_j(x,M)\cap D_1$ is a subset of $S^{n-1}(x,r)$, where $r=d(x)\big(e^M-1\big)$. By convexity of D_1 the domain $B_j(x,M)\cap D_1$ is convex.

Let us then show that each chord with end points in $B_j(x,M)\cap D_2$ is contained in $B_j(x,M)$. By definition for $y\in\partial B_j(x,M)\cap D_2$ we have

$$d(y)=\frac{|x-y|}{e^M-1}. \tag{4.6}$$

Let us assume $y_1,y_2\in B_j(x,M)\cap D_2$ and $z=(y_1+y_2)/2\notin B_j(x,M)$. If $z\in D_1$, then $z\in B_j(x,M)$ because $B_j(x,M)\subset B^n(x,r)$. Therefore we may assume $z\in D_2\setminus B_j(x,M)$. By (4.6) we have $d(y_i)>|x-y_i|/(e^M-1)$ for $i\in\{1,2\}$ and $d(z)<|x-z|/(e^M-1)$. Since $M>\log 2$ we have $c=1/(e^M-1)<1$. Now the boundary ∂G is outside $B^n(y_1,c|x-y_1|)\cup B^n(y_2,c|x-y_2|)$ and has a point in $B^n(z,c|x-z|)$, see Figure 3.

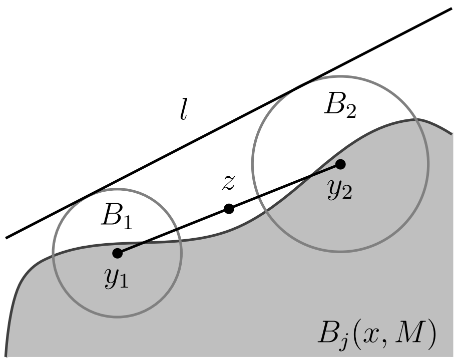

Figure 3. Line l, Euclidean balls $B_1=B^n(y_1,c|x-y_1|)$ and $B_2=B^n(y_2,c|x-y_2|)$ and points y_1, y_2 and z.

We will show that for $c<1$ the domain G is not convex. Let us denote by l a line that is a tangent to balls $B^n(y_1,c|x-y_1|)$ and $B^n(y_2,c|x-y_2|)$. Because $d(y_i,l)=c|x-y_i|$ for $i\in\{0,1\}$ we have

$$d(z,l)=\frac{c|x-y_1|+c|x-y_2|}{2}. \tag{4.7}$$

By the triangle inequality

$$|x-z|=\left|\frac{x-y_1}{2}+\frac{x-y_2}{2}\right|\le\frac{|x-y_1|}{2}+\frac{|x-y_2|}{2}$$

and by (4.7)

$$d(z,l)=\frac{c}{2}(|x-y_1|+|x-y_2|)\ge c|x-z|.$$

Now the domain G is not convex, which is a contradiction, and each chord with end points in $B_j(x,M)\cap D_2$ is contained in $B_j(x,M)$.

Since each chord with end points in $B_j(x,M)\cap D_2$ is contained in $B_j(x,M)$, $B_j(x,M)\cap D_2\subset B^n(x,r)$, D_1 is convex and $\partial B_j(x,M)\cap D_1\subset S^{n-1}(x,r)$ the j-ball $B_j(x,M)$ is convex. □

Theorem 4.8. *Let $M > 0$ and $G \subsetneq \mathbf{R}^n$ be a starlike domain with respect to $x \in G$. Then the j-balls $B_j(x, M)$ are starlike with respect to x.*

Proof. By Theorem 1.1 we need to consider $M > \log(\sqrt{2}+1)$ which is equivalent to $e^M - 1 > \sqrt{2}$. Let us divide G into two parts $D_1 = \{z \in G\colon d(z) \geq d(x)\}$ and $D_2 = G \setminus D_1$.

Similarly as in the proof of Theorem 4.5 the boundary $\partial B_j(x, M) \cap D_1$ is a subset of a sphere $S^{n-1}(x, r)$ and $B_j(x, M) \subset S^{n-1}(x, r)$. Therefore it is sufficient to show that for each $y \in B_j(x, M) \cap D_2$ the line segment $[x, y]$ is in $B_j(x, M)$.

We will show that all chords $[x, y]$ for $y \in B_j(x, M) \cap D_2$ are contained in $B_j(x, M)$. Let us assume, on the contrary, that there exists $y_1, y_2 \in \big(\partial B_j(x, M)\big) \cap D_2$ with $y_1 \in (x, y_2)$ and $z = (y_1 + y_2)/2 \notin \overline{B_j}(x, M)$. Let us first assume $z \in D_1$. Now $j_G(x, z) > j_G(x, y_2)$ is equivalent to $|x - z|/d(x) > |x - y_2|/d(y_2)$. By the selection of y_1 and y_2 we have $|x - z| < |x - y_2|$ and $d(x) > d(y_2)$ implying $|x - z|/d(x) < |x - y_2|/d(y_2)$, which is a contradiction.

Let us then assume $z \in D_2$. Now

$$\frac{|x - y_1|}{d(y_1)} = \frac{|x - y_2|}{d(y_2)} = e^M - 1 < \frac{|x - z|}{d(z)}$$

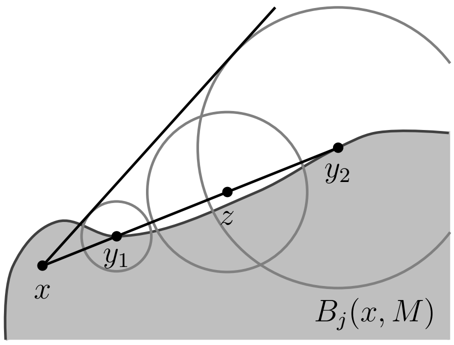

Figure 4. Selection of points y_1 and y_2. Gray circles are $B^n\big(y_1, d(y_1)\big)$, $B^n\big(z, d(z)\big)$ and $B^n\big(y_2, d(y_2)\big)$.

and therefore the boundary ∂G does not intersect $B^n\big(y_1, d(y_1)\big)$ or $B^n\big(y_2, d(y_2)\big)$ and contains a point in $B^n\big(z, d(z)\big)$, see Figure 4. This means that G is not starlike with respect to x, which is a contradiction. $\square$

Remark 4.9. (1) Let us consider the domain $G = B^n(0, 1) \cup B^n(e_1, 1/4) \cup B^n(2e_1, 1)$ and show that the j-ball $B = B_j(0, \log 3)$ is connected but the j-sphere $S = \{z \in G\colon j_G(0, z) = \log 3\}$ is disconnected. We have

$$j_G(0, e_1) = \log\left(1 + \frac{1}{1/4}\right) = \log 5$$

and therefore all points $x \in G$ with $x_1 = 1$ are neither in B nor on the boundary ∂B. We also have $B, \partial B \subset B^n(0, 1) \cup B^n(2e_1, 1)$. For all $y \in B^n(2e_1, 1) \setminus \{u \in$

$G\colon \angle 0\,2e_1\,u < \operatorname{atan}(1/4)\}$ we have

$$j_G(0,y) = \log\left(1 + \frac{|y|}{1-|2-y|}\right) \geq \log(1+2) = \log 3,$$

because $|y| + 2|2-y| \geq 2$. For all $y \in B^n(2e_1,1) \cap \{u \in G\colon \angle 0\,2e_1\,u < \operatorname{atan}(1/4)\}$ we have

$$j_G(0,y) = \log\left(1 + \frac{|y|}{d(y)}\right) \geq \log\left(1 + \frac{|y_1|}{d(y_1)}\right) \geq \log(1+2) = \log 3$$

and therefore $B \subset B^n(0,1)$ and it is connected.

Let us now consider S and denote $z \in S$. If $z \in B^n(2e_1,1)$, then $z = 2e_1$. If $z \in B^n(0,1)$, then $z \in \partial B$. Now $S = \partial B \cup \{2e_1\}$ and it is disconnected. In particular, we see that

$$\overline{\{z \in G\colon j_G(0,z) < \log 3\}} \neq \{z \in G\colon j_G(0,z) \leq \log 3\}.$$

(2) We have seen that in convex domains the j-balls are convex and in starlike domains the j-balls are starlike. However in simply connected domains the j-balls need not be simply connected. Let us consider $G = B^n(0,1) \cup B^n(e_1,h) \cup B^n(2e_1,1)$ for $h \in (0,1)$. Clearly G is simply connected. Let us consider $B = B_j(0,\log 4)$. We have

$$j_G(0,2e_1) = \log\left(1 + \frac{2}{1}\right) = \log 3$$

and therefore $2e_1 \in B$. Let $x = (x_1,\ldots,x_n) \in G$ with $x_1 = 1$. Now

$$j_G(0,x) \geq j_G(0,e_1) = \log\left(1 + \frac{1}{h}\right)$$

and $x \notin B$ for $h < 1/3$. For $h = 1/4$ the j-ball B is not even connected. Instead of the radius $\log 4$ we could choose any $r > \log 3$.

Questions 4.10. We pose some open questions concerning the quasihyperbolic metric and quasihyperbolic balls.

(1) Is it true that for any domain $G \subsetneq \mathbf{R}^n$ and $x \in G$ the quasihyperbolic ball $D_G(x,M)$ is strictly convex if $M \in (0,1]$?
(2) Is it true that for any domain $G \subsetneq \mathbf{R}^n$ and $x \in G$ the quasihyperbolic ball $D_G(x,M)$ is strictly starlike with respect to x if $M \in (0,\kappa]$ for $\kappa \approx 2.83297$?
(3) Are the quasihyperbolic geodesics unique in every simply connected domain $G \subsetneq \mathbf{R}^2$?

For the case $\mathbf{R}^n \setminus \{0\}$ see Remarks 3.2 and 3.6.

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