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NEW EXAMPLES OF WEAKLY COMPACT APPROXIMATION IN BANACH SPACES

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Abstract. The Banach space E has the weakly compact approximation property (W.A.P.) if there is $C < \infty$ so that the identity map I_E can be uniformly approximated on any weakly compact subset $D \subset E$ by weakly compact operators V on E satisfying $\|V\| \le C$. We show that the spaces $N(\ell^p, \ell^q)$ of nuclear operators $\ell^p \to \ell^q$ have the W.A.P. for $1 < q \le p < \infty$, but that the Hardy space H^1 does not have the W.A.P.

0. Introduction

The Banach space E has the *weakly compact approximation property* (abbreviated W.A.P.) if there is $C < \infty$ so that for any weakly compact set $D \subset E$ and $\varepsilon > 0$ one finds a weakly compact operator $V \in W(E)$ satisfying

$$\sup_{x \in D} \|x - Vx\| < \varepsilon \quad \text{and} \quad \|V\| \le C. \tag{0.1}$$

Here $V \in W(E)$ if the image VB_E of the closed unit ball B_E of E is relatively weakly compact. This concept of weakly compact approximation is natural, but the resulting property differs completely from the classical bounded approximation properties defined in terms of finite rank or compact operators (for more about these properties see e.g. [C]). The W.A.P. was introduced in [AT], and some applications can be found in [AT] and [T2]. It was later more systematically studied in [OT] from the perspective of Banach space theory. The W.A.P. remains fairly rare and elusive for non-reflexive spaces (obviously any reflexive space has it). The following list reviews some of the known results.

(0.2) If E is a $\mathscr{L}^1$- or $\mathscr{L}^\infty$-space, then E has the W.A.P. if and only if E has the Schur property, see [AT, Cor. 3]. Thus ℓ^1 has the W.A.P., while c_0, $C(0,1)$ and $L^1(0,1)$ fail to have it.

(0.3) The direct sums $\ell^1(\ell^p)$ and $\ell^p(\ell^1)$ have the W.A.P. for $1 < p < \infty$, [OT, Prop. 5.3].

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(0.4) The quasi-reflexive James' space J, as well as its dual J^*, have the W.A.P., [OT, Thm. 2.2 and 3.3]. On the other hand, there is [ArT, Prop. 14.11] a quasi-reflexive hereditarily indecomposable space X that fails to have the W.A.P. Moreover, the related James' tree space JT fails to have the W.A.P., [OT, Thm. 6.5].

As the first result of this note we show that the spaces $N(\ell^p, \ell^q)$ consisting of nuclear operators have the W.A.P. for $1 < q \leq p < \infty$ (note that $N(\ell^p, \ell^q)$ is reflexive if $q > p$). This result, which was motivated by timely questions of Zacharias and Defant, includes the Schatten trace class space C_1 for $p = q = 2$. Secondly, we show that the Hardy space H^1 does not have the W.A.P., which solves a question from [AT, p. 370] in the negative.

In order to determine whether a given space E has the W.A.P. or not one is often forced to rely on very specific properties of E, and there still remains fairly concrete Banach spaces for which this property is not decided (see e.g. the Problems in Section 2 as well as [OT]).

1. The spaces $N(\ell^p, \ell^q)$ of nuclear operators have the W.A.P.

Let E and F be Banach spaces. Recall that $T\colon E \to F$ is a nuclear operator, denoted $T \in N(E, F)$, if there are sequences $(x_j^*) \subset E^*$ and $(y_j) \subset F$ so that $\sum_{j=1}^\infty \|x_j^*\|\,\|y_j\| < \infty$ and $T = \sum_{j=1}^\infty x_j^* \otimes y_j$. Here $x_j^* \otimes y_j$ denotes the rank-1 operator $x \mapsto x_j^*(x) y_j$. Then $(N(E, F), \|\cdot\|_N)$ is a Banach space, where the nuclear norm of $T \in N(E, F)$ is

$$\|T\|_N = \inf\left\{ \sum_{j=1}^\infty \|x_j^*\|\ \|y_j\| : T = \sum_{j=1}^\infty x_j^* \otimes y_j \right\}.$$

Recall that $\|ASB\|_N \leq \|A\|\,\|B\|\,\|S\|_N$ whenever $S \in N(E, F)$ and A, B are compatible bounded operators. One may isometrically identify $N(\ell^2) = C_1$, where C_1 is the Schatten trace class space, see e.g. [P, Sect. 0.b] or [Pi, Sect. 2.11].

Theorem 1 below is the main result of this section. Observe that in its statement the spaces $N(\ell^p, \ell^q)$ are actually reflexive for $1 < p < q < \infty$, so that only the cases $1 < q \leq p < \infty$ contain non-trivial information. In fact, $N(\ell^p, \ell^q) = K(\ell^q, \ell^p)^*$ in the trace-duality

$$\langle U, V\rangle = \mathrm{tr}(VU), \quad U \in N(\ell^p, \ell^q), \quad V \in K(\ell^q, \ell^p),$$

where the space $K(\ell^q, \ell^p)$ of compact operators $\ell^q \to \ell^p$ is reflexive once $1 < p < q < \infty$, see e.g. [R, Cor. 2.6] or [K, Sect. 2, Cor. 2]. Theorem 1 can also be rephrased in the terms of the projective tensor products $\ell^p \widehat{\otimes}_\pi \ell^q$, see the Remarks following Lemma 2.

Theorem 1. *$N(\ell^p, \ell^q)$ has the W.A.P. whenever $1 < p, q < \infty$.*

The proof of Theorem 1 is based on Lemma 2 below, which contains a basic characterization of the relatively weakly compact subsets of the non-reflexive spaces $N(\ell^p, \ell^q)$ for $1 < q \leq p < \infty$. Let (e_j) be the unit vector basis of ℓ^p for $1 < p < \infty$.

We denote the natural basis projection of ℓ^p onto $[e_1, \dots, e_n]$ by P_n, and set $Q_n = I - P_n$ for $n \in \mathbf{N}$. For $m < n$ we also put $P_{(m,n]} = P_n - P_m = P_n Q_m = Q_m P_n$, which is the natural projection of ℓ^p onto $[e_{m+1}, \dots, e_n]$. We denote the corresponding basis projections on ℓ^q by $\widetilde{P}_n, \widetilde{Q}_n$ and $\widetilde{P}_{(m,n]}$, respectively. We will frequently use the facts that $\|S - \widetilde{P}_n S P_n\|_N \to 0$ and $\|\widetilde{Q}_n S Q_n\|_N \to 0$ as $n \to \infty$ for any $S \in N(\ell^p, \ell^q)$.

Lemma 2. *Suppose that $1 < q \le p < \infty$ and let $D \subset N(\ell^p, \ell^q)$ be a bounded subset. Then D is relatively weakly compact in $N(\ell^p, \ell^q)$ if and only if*

$$\lim_{n\to\infty} \sup_{S\in D} \|\widetilde{Q}_n S Q_n\|_N = 0. \tag{1.1}$$

We first complete the proof of Theorem 1 with the help of (1.1) before establishing the more technical Lemma 2.

Proof of Theorem 1. We may assume that $1 < q \le p < \infty$ since $N(\ell^p, \ell^q)$ is reflexive for $1 < p < q < \infty$, see the comment preceding Theorem 1. Suppose that $D \subset N(\ell^p, \ell^q)$ is a weakly compact subset and let $\varepsilon > 0$ be given. Write

$$S = \big(\widetilde{P}_n S P_n + \widetilde{P}_n S Q_n + \widetilde{Q}_n S P_n\big) + \widetilde{Q}_n S Q_n \equiv \psi_n(S) + \widetilde{Q}_n S Q_n \tag{1.2}$$

for $S \in N(\ell^p, \ell^q)$ and $n \in \mathbf{N}$. Here $\psi_n \colon N(\ell^p, \ell^q) \to N(\ell^p, \ell^q)$ for $n \in \mathbf{N}$ are the bounded linear maps defined by $\psi_n(S) = \widetilde{P}_n S P_n + \widetilde{P}_n S Q_n + \widetilde{Q}_n S P_n$ for $S \in N(\ell^p, \ell^q)$. Clearly $\|\psi_n\| \le 3$ for any n. It follows from (1.1) and (1.2) that

$$\sup_{S\in D} \|S - \psi_n(S)\|_N = \sup_{S\in D} \|\widetilde{Q}_n S Q_n\|_N \to 0 \quad \text{as } n \to \infty,$$

so that $\sup_{S\in D} \|S - \psi_n(S)\|_N < \varepsilon$ once n is large enough.

Consequently it will be enough to verify that

$$\psi_n \in W(N(\ell^p, \ell^q)), \quad n \in \mathbf{N}. \tag{1.3}$$

This fact can be deduced from a suitable combination of general results, see the proofs of [LS, Prop. 2.2 and 2.3] or the survey [ST, p. 262], but we sketch a direct argument for completeness. Note first that the maps

$$\varphi_{y,y^*}(S) = (y^* \otimes y)S = S^* y^* \otimes y; \quad \phi_{x,x^*}(S) = S(x^* \otimes x) = x^* \otimes Sx$$

are weakly compact on $N(\ell^p, \ell^q)$ for any $y^* \in \ell^{q'}$, $y \in \ell^q$, $x \in \ell^p$ and $x^* \in \ell^{p'}$, where p' and q' are the respective dual exponents. In fact, $\varphi_{y,y^*}(B_{N(\ell^p,\ell^q)}) \subset \|y^*\|\,\|y\|(B_{\ell^{p'}} \otimes y)$, since $\|S^* y^* \otimes y\|_N \le \|y^*\|\,\|y\|$ for $S \in B_{N(\ell^p,\ell^q)}$. Clearly the set $B_{\ell^{p'}} \otimes y$ is relatively weakly compact in $N(\ell^p, \ell^q)$, since $z^* \mapsto z^* \otimes y$ embeds $\ell^{p'}$ isomorphically into $N(\ell^p, \ell^q)$ for $y \ne 0$. The case of ϕ_{x,x^*} is analogous.

Finally, (1.3) follows since the individual operators defining ψ_n, such as $S \mapsto \widetilde{P}_n S Q_n$, are sums of weakly compact ones composed with bounded ones. The proof of Theorem 1 will be complete once Lemma 2 has been established. □

Proof of Lemma 2. Suppose first that (1.1) holds. According to (1.2) we get that

$$D \subset \psi_n(D) + \delta_n B_{N(\ell^p,\ell^q)}, \quad n \in \mathbf{N}, \tag{1.4}$$

where $\delta_n \equiv \sup_{S\in D} \|\widetilde{Q}_n S Q_n\|_N \to 0$ as $n \to \infty$. Here $\psi_n(D)$ is a relatively weakly compact subset of $N(\ell^p, \ell^q)$ for all n by (1.3). It is a standard fact that (1.4) then implies that D is a relatively weakly compact subset of $N(\ell^p, \ell^q)$.

Assume towards the converse implication that (1.1) fails to hold for the bounded subset $D \subset N(\ell^p, \ell^q)$. Put $\Delta = D - D$. The strategy is to exhibit a sequence $(S_k) \subset \Delta$, which is equivalent to the unit vector basis in ℓ^1. In this event (S_k) does not have any weakly convergent subsequences, so that Δ (as well as D) is not a relatively weakly compact set.

Observe first that by our assumption

$$c = \inf_{n\in\mathbf{N}} \sup_{S\in D} \|\widetilde{Q}_n S Q_n\|_N > 0, \tag{1.5}$$

since $\sup_{S\in D} \|\widetilde{Q}_n S Q_n\|_N$ is clearly non-increasing in n. We proceed to construct by induction a sequence $(S_k)_{k\geq 1} \subset \Delta$ and intertwining sequences $1 = n_1 < m_1 < n_2 < m_2 < \ldots$ of natural numbers so that the following conditions are satisfied for all $r \in \mathbf{N} = \{1, 2, \ldots\}$:

$$\|\widetilde{P}_{(n_r,m_r]} S_r P_{(n_r,m_r]}\|_N > \frac{c}{2}, \tag{1.6}$$

$$\|\widetilde{P}_{(n_j,m_j]} S_k P_{(n_j,m_j]}\|_N < \frac{c}{2^{j+k+4}} \quad \text{for } 1 \leq j, k \leq r \text{ and } j \neq k. \tag{1.7}$$

First pick $n_1 = 1$ and $S_1 \in D$ so that $\|\widetilde{Q}_{n_1} S_1 Q_{n_1}\|_N > \frac{c}{2}$, and by truncation $m_1 > n_1$ so that (1.6) holds for $r = 1$. Assume next that we have already chosen $S_1, \ldots, S_r \in \Delta$ and $1 = n_1 < m_1 < \ldots < n_r < m_r$ so that (1.6) and (1.7) holds until r. We pick by truncation $n_{r+1} > m_r$ such that

$$\|\widetilde{Q}_{n_{r+1}} S_j Q_{n_{r+1}}\|_N < \frac{c}{2^{2r+5}} \quad \text{for } j = 1, \ldots, r. \tag{1.8}$$

Note that (1.8) guarantees (1.7) for $j = r + 1$ and $1 \leq k \leq r$ regardless of our subsequent choice of $m_{r+1} > n_{r+1}$.

We next choose inductively an auxiliary sequence $(T_s)_{s\geq 1} \subset D$ and increasing indices $n_{r+1} = l_1 < l_2 < \ldots$ in such a way that

$$\|\widetilde{Q}_{l_s} T_s Q_{l_s}\|_N > \frac{2c}{3}, \quad s \in \mathbf{N}, \tag{1.9}$$

$$\|\widetilde{Q}_{l_{s+1}} T_s Q_{l_{s+1}}\|_N < \frac{c}{6}, \quad s \in \mathbf{N}. \tag{1.10}$$

This is possible by (1.5) and the fact that $Q_t T_s Q_t \to 0$ in $N(\ell^p, \ell^q)$ as $t \to \infty$. Use finite-dimensionality and the boundedness of D to find a subsequence of (T_s) such that $(\widetilde{P}_{(n_j,m_j]} T_s P_{(n_j,m_j]})_{s\geq 1}$ converges in the nuclear norm for all $j = 1, \ldots, r$ as $s \to \infty$ along this subsequence. Hence there are $s_1 < s_2$ for which the choice $S_{r+1} = T_{s_2} - T_{s_1}$ satisfies

$$\|\widetilde{P}_{(n_j,m_j]} S_{r+1} P_{(n_j,m_j]}\|_N < \frac{c}{2^{2r+5}} \quad \text{for } j = 1, \ldots, r.$$

This yields (1.7) for $k = r + 1$ and $1 \leq j \leq r$.

It remains to find $m_{r+1} > n_{r+1}$ and to verify (1.6) for $r+1$. For this observe that by (1.9), (1.10) and the fact $l_{s_2} > n_{r+1}$ one has

$$\begin{aligned}\|\widetilde{Q}_{n_{r+1}} S_{r+1} Q_{n_{r+1}}\|_N &\ge \|\widetilde{Q}_{l_{s_2}} S_{r+1} Q_{l_{s_2}}\|_N \ge \|\widetilde{Q}_{l_{s_2}} T_{s_2} Q_{l_{s_2}}\|_N - \|\widetilde{Q}_{l_{s_2}} T_{s_1} Q_{l_{s_2}}\|_N \\ &> \frac{2c}{3} - \|\widetilde{Q}_{l_{s_1+1}} T_{s_1} Q_{l_{s_1+1}}\|_N > \frac{c}{2}.\end{aligned}$$

By truncation we may then pick $m_{r+1} > n_{r+1}$ so that (1.6) holds for $r+1$. This completes the induction step.

Let $(c_k) \in \ell^1$, $(c_k) \neq 0$, be an arbitrary sequence. Define $E_k = [e_{n_k+1}, \ldots, e_{m_k}] \subset \ell^p$ and $F_k = [f_{n_k+1}, \ldots, f_{m_k}] \subset \ell^q$ for $k \in \mathbf{N}$. By finite-dimensional trace-duality and the fact that E_k, F_k are 1-complemented subspaces there is $U_k \in L(F_k, E_k) = N(E_k, F_k)^*$ so that $\|U_k\| = 1$ and

$$\langle U_k, \widetilde{P}_{(n_k,m_k]} S_k P_{(n_k,m_k]}\rangle = \frac{|c_k|}{c_k} \|\widetilde{P}_{(n_k,m_k]} S_k P_{(n_k,m_k]}\|_N, \quad k \in \mathbf{N}.$$

We may choose $U_k = 0$ in case $c_k = 0$. Then $U = \sum_{k=1}^{\infty} U_k$ defines a bounded operator $\ell^q \to \ell^p$ satisfying $\|U\| = \sup_k \|U_k\| = 1$ since $q \le p$ by assumption.

Clearly $\|\sum_{k=1}^{\infty} c_k S_k\|_N \le M \sum_{k=1}^{\infty} |c_k|$, where $M = \sup_{S\in\Delta} \|S\|_N$. Towards the converse estimate we first observe that

$$\langle U_k, \widetilde{P}_{(n_r,m_r]} S_r P_{(n_r,m_r]}\rangle = \mathrm{tr}(\widetilde{P}_{(n_r,m_r]} S_r P_{(n_r,m_r]} U_k) = 0$$

for $k \neq r$ in the trace-duality $L(\ell^q, \ell^p) = N(\ell^p, \ell^q)^*$. Hence it follows from (1.6) that

$$\begin{aligned}\Big\|\sum_{k=1}^{\infty} c_k \widetilde{P}_{(n_k,m_k]} S_k P_{(n_k,m_k]}\Big\|_N &\ge \langle U, \sum_{k=1}^{\infty} c_k \widetilde{P}_{(n_k,m_k]} S_k P_{(n_k,m_k]}\rangle \\ &= \sum_{k=1}^{\infty} |c_k| \cdot \|\widetilde{P}_{(n_k,m_k]} S_k P_{(n_k,m_k]}\|_N \ge \frac{c}{2} \sum_{k=1}^{\infty} |c_k|.\end{aligned} \tag{1.11}$$

We also need the general fact that

$$\Big\|\sum_{r=1}^{\infty} \widetilde{P}_{(n_r,m_r]} S P_{(n_r,m_r]}\Big\|_N \le \|S\|_N, \quad S \in N(\ell^p, \ell^q). \tag{1.12}$$

The block diagonalization estimate (1.12) is proved for the nuclear norm $\|\cdot\|_N$ exactly as in the case of the operator norm in [LT, pp. 20–21]. By combining (1.12), (1.11) and (1.7) we get that

$$\begin{aligned}&\Big\|\sum_{k=1}^{\infty} c_k S_k\Big\|_N \ge \Big\|\sum_{r=1}^{\infty} \widetilde{P}_{(n_r,m_r]} \Big(\sum_{k=1}^{\infty} c_k S_k\Big) P_{(n_r,m_r]}\Big\|_N \\ &\ge \frac{c}{2}\sum_{k=1}^{\infty} |c_k| - \sum_{r=1}^{\infty} \Big(\sum_{k<r} |c_k| \cdot \|\widetilde{P}_{(n_r,m_r]} S_k P_{(n_r,m_r]}\|_N + \sum_{k>r} |c_k| \cdot \|\widetilde{P}_{(n_r,m_r]} S_k P_{(n_r,m_r]}\|_N\Big)\end{aligned}$$

$$\geq \left(\frac{c}{2} - c \cdot \sum_{r,k=1}^{\infty} 2^{-(r+k+4)} \right) \sum_{s=1}^{\infty} |c_s| \geq \frac{3c}{8} \sum_{k=1}^{\infty} |c_k|.$$

Hence the sequence $(S_k) \subset \Delta$ is equivalent to the unit vector basis in ℓ^1. This completes the proof of Lemma 2 as noted above. $\square$

Actually, there is a somewhat simpler proof for Lemma 2. The alternative argument constructs a sequence $(S_k) \subset D$ and a related block-diagonal operator $U \in L(\ell^q, \ell^p)$ so that $|\langle S_k, U \rangle| > \frac{c}{2}$ for $k \in \mathbf{N}$, whence one may deduce that (S_k) has no weakly convergent subsequences in $N(\ell^p, \ell^q)$. However, the argument in Lemma 2 establishes a stronger fact, which is an analogue of a result of Kadec and Pełczyński for non-weakly compact subsets of $L^1(0,1)$, see [W, III.C.12].

Corollary 3. *If $1 < q \leq p < \infty$ and $D \subset N(\ell^p, \ell^q)$ is a bounded subset which is not relatively weakly compact, then the difference set $D - D$ contains a sequence (S_k) equivalent to the unit vector basis in ℓ^1.*

Remarks. (1) Clearly Lemma 2 does not hold for $1 < p < q < \infty$, since in this case $N(\ell^p, \ell^q)$ is reflexive, but $D = \{e_n^* \otimes f_n : n \in \mathbf{N}\}$ does not satisfy (1.1). Here $(e_n^*) \subset \ell^{p'}$ is the biorthogonal basis.

(2) Theorem 1 can be restated as follows by using the known (partial) correspondence between spaces of nuclear operators and projective tensor products: *$\ell^p \widehat{\otimes}_\pi \ell^q$ has the W.A.P. whenever $1 < p, q < \infty$.* This follows from the isometric identification $\ell^p \widehat{\otimes}_\pi \ell^q = N(\ell^{p'}, \ell^q)$, but one may also translate the argument of Theorem 1 into the setting of tensor products. We refer to [DF] for the requisite background.

The scope of Theorem 1 within the class of spaces $N(E,F)$ of nuclear operators (or the related projective tensor products) remains unclear. For instance, it follows from Theorem 1 that $N(\ell^p \oplus \ell^q)$ has the W.A.P. for $1 < p < q < \infty$, since $N(\ell^p \oplus \ell^q)$ is linearly isomorphic to $N(\ell^p) \oplus N(\ell^q) \oplus N(\ell^p, \ell^q) \oplus N(\ell^q, \ell^p)$. The following questions appear natural.

Problems. (1) If $N(E,F)$ has the W.A.P., then E^* and F must also have this property, since $E^* \subset N(E,F)$ and $F \subset N(E,F)$ as complemented subspaces. *Are there E and F so that E^* and F have the W.A.P., but $N(E,F)$ fails to have the W.A.P.?*

(2) Let E and F be reflexive Banach spaces having unconditional Schauder bases. *Does $N(E,F)$ always have the W.A.P.? As an important special case, does $N(L^p(0,1))$ have the W.A.P. for $1 < p < \infty$ and $p \neq 2$?*

(3) Recall that Y has the Schur property if $\|y_n\| \to 0$ as $n \to \infty$ for any weak-null sequence $(y_n) \subset Y$. By applying the construction in [BP] to ℓ^1 one obtains a separable $\mathscr{L}^\infty$-space X so that $\ell^1 \subset X$ isometrically and X/ℓ^1 has the Schur property. It is then easy to check that X has the Schur property, so that X has the W.A.P. by (0.2). *Does $X \widehat{\otimes}_\pi X$ have the W.A.P.?*

The Banach space E has the *inner weakly compact approximation property* (inner W.A.P.) if there is $C < \infty$ so that for any weakly compact operator $U \in W(E, Z)$, where Z is an arbitrary Banach space, and $\varepsilon > 0$ there is $V \in W(E)$ satisfying

$$\|U - UV\| < \varepsilon \text{ and } \|V\| \le C. \tag{1.13}$$

This property, first considered in [T1] and [T2], is less intuitive than the W.A.P. It is a (pre)dual property to W.A.P. in the following sense: If X has the inner W.A.P., then X^* has the W.A.P., see [T1, Prop. 3.4]. The converse does not hold: the Johnson–Lindenstrauss space JL fails to have the inner W.A.P., but JL^* has the W.A.P., see [T2, Thm. 1.4].

The argument of Theorem 1 yields that the spaces $K(\ell^p, \ell^q)$ of compact operators (alternatively, the ε-tensor products $\ell^p \widehat{\otimes}_\varepsilon \ell^q = K(\ell^{p'}, \ell^q)$) have the inner W.A.P. for $1 < p, q < \infty$.

Corollary 4. *The spaces $K(\ell^p, \ell^q)$ have the inner W.A.P. whenever $1 < p, q < \infty$.*

Proof. It is again enough to consider the case $1 < p \le q < \infty$. To check (1.13) suppose that $U \colon K(\ell^p, \ell^q) \to Z$ is a weakly compact operator, where Z is a Banach space. Consider the operators ϕ_n defined on $K(\ell^p, \ell^q)$ by

$$\phi_n(S) = \widetilde{P}_n S P_n + \widetilde{P}_n S Q_n + \widetilde{Q}_n S P_n, \quad S \in K(\ell^p, \ell^q),$$

for $n \in \mathbf{N}$. It is not difficult to verify that $\phi_n^* = \psi_n \in L(N(\ell^q, \ell^p))$ in trace duality, where $\psi_n(S) = P_n S \widetilde{P}_n + P_n S \widetilde{Q}_n + Q_n S \widetilde{P}_n$ for $S \in N(\ell^q, \ell^p)$. Here $\psi_n \in W(N(\ell^q, \ell^p))$ for $n \in \mathbf{N}$ by (1.3). Moreover, the argument of Theorem 1 applied to the relatively weakly compact subset $U^*(B_{Z^*}) \subset N(\ell^q, \ell^p)$ yields that

$$\|U - U\phi_n\| = \|U^* - \psi_n U^*\| \to 0 \quad \text{as } n \to \infty.$$

Hence $K(\ell^p, \ell^q)$ has the inner W.A.P. □

2. H^1 does not have the W.A.P.

Let $\mathbf{D}$ be the unit disk in the complex plane. The Hardy space H^1 consists of the analytic maps $f \colon \mathbf{D} \to \mathbf{C}$ for which

$$\|f\| = \sup_{0<r<1} \int_0^{2\pi} |f(re^{it})| \, dm(t) < \infty,$$

where m is normalized Lebesgue measure on $[0, 2\pi]$ (identified with $\mathbf{T} = \partial\mathbf{D}$). It is a classical fact that H^1 is isometrically isomorphic via a.e. radial limits to the closed subspace

$$H^1(\mathbf{T}) = \Big\{ f \in L^1(\mathbf{T}) : \hat{f}(n) = \int_0^{2\pi} e^{-int} f(e^{it}) \, dm(t) = 0, \ n < 0 \Big\}$$

of $L^1(\mathbf{T})$. Recall that $L^1(\mathbf{T})$ does not have the W.A.P. by (0.2). This observation uses the fact that $L^1(\mathbf{T})$ has the Dunford–Pettis property (DPP), that is, any weakly compact $U \in W(L^1(\mathbf{T}))$ maps weak-null sequences to norm-null ones. By contrast

$H^1 = H^1(\mathbf{T})$ does not have the DPP, and $W(H^1)$ is a larger class (e.g. as it contains the Paley projections onto the Hilbertian subspaces spanned by lacunary sequences). Thus the known results about the W.A.P. do not resolve the natural question [AT, p. 370] whether H^1 has the W.A.P. In this section we settle this problem in the negative.

Theorem 5. *H^1 does not have the W.A.P.*

Proof. Let $g_n(z) = z^n$ for $z \in \mathbf{C}$ and $n = 0, 1, 2, \dots$. Consider

$$H = \{g_n : n \in \mathbf{N}\} \subset H^1.$$

Then H is relatively weakly compact in H^1, since (g_n) is a weak-null sequence. It will be enough to establish the following claim.

Claim. *There is no weakly compact operator $U \colon H^1 \to H^1$ so that*

$$\sup_{h \in H} \|h - Uh\| < 1/2. \tag{2.1}$$

Proof of the Claim. Suppose to the contrary that there is an operator $U \in W(H^1)$ satisfying (2.1). We next modify U by applying the averaging technique of Rudin [Ru1]. Let τ_s be the isometric translation operator on $H^1(\mathbf{T})$ defined by $\tau_s f(e^{iu}) = f(e^{i(u+s)})$ for $s, u \in [0, 2\pi]$. Then the H^1-valued average

$$\widetilde{U} f = \int_0^{2\pi} (\tau_{-s} U \tau_s) f \, dm(s), \qquad f \in H^1,$$

yields a bounded linear operator $H^1 \to H^1$. Moreover, $\widetilde{U} \in W(H^1)$ according to the Dunford–Pettis characterization of the relatively weakly subsets of $L^1(\mathbf{T})$ as the uniformly integrable ones. Note that

$$\begin{aligned} \|g_n - \widetilde{U} g_n\| &= \left\| \int_0^{2\pi} (\tau_{-s}\tau_s g_n - \tau_{-s} U \tau_s g_n) \, dm(s) \right\| \\ &\le \int_0^{2\pi} \|g_n - U(g_n)\| \, dm(s) < 1/2 \end{aligned} \tag{2.2}$$

for all $n \in \mathbf{N}$ by (2.1) and the identity $\tau_s g_n = e^{ins} g_n$.

The construction in [Ru1] (alternatively, see [Ru2, 5.19]) guarantees that there is a bounded complex sequence $(\lambda_n)_{n \ge 0}$ so that

$$\widetilde{U} g_n = \lambda_n g_n, \quad n \in \mathbf{N} \cup \{0\}. \tag{2.3}$$

In other words, $\widetilde{U}$ is a weakly compact Fourier multiplier operator on H^1 which is determined by $(\lambda_n)_{n \ge 0}$.

Consequently $\|g_n - \widetilde{U} g_n\| = |1 - \lambda_n| < 1/2$ for $n \in \mathbf{N}$ by (2.2) and (2.3). However, this estimate contradicts the fact, isolated below in Lemma 6, that $\inf_{n \ge 1} \frac{1}{n} |\sum_{k=1}^n \lambda_k| = 0$ holds for any such weakly compact Fourier multiplier on H^1. This yields the Claim, and the proof of Theorem 5 will be complete once Lemma 6 has been established below. $\square$

Let $\Lambda = (\lambda_k)_{k\geq 0}$ be a bounded sequence of complex numbers, and define the corresponding formal Fourier multiplier T_Λ by $T_\Lambda(g_k) = \lambda_k g_k$ for $k \geq 0$.

Lemma 6. *Let $\Lambda = (\lambda_k)_{k\geq 0} \in \ell^\infty$ be a complex sequence for which the corresponding Fourier multiplier operator $T_\Lambda \in W(H^1)$. Then*

$$\inf_{n\geq 1} \frac{1}{n}\Big|\sum_{k=1}^{n} \lambda_k\Big| = 0. \tag{2.4}$$

Proof. Let A be the closure of $T_\Lambda(B_{H^1})$ in H^1, and put

$$G := \overline{\mathrm{abco}}\{gf : f \in A\,, \|g\|_{L^\infty} \leq 1\},$$

where the absolutely convex closure is taken in $L^1 \equiv L^1(\mathbf{T})$. The uniform integrability criterion implies that G is a weakly compact subset of L^1.

Assume contrary to (2.4) that there is $c > 0$ so that $|a_n| \geq c$ for all $n \geq 1$, where $a_n := \frac{1}{n}\sum_{k=1}^{n} \lambda_k$ for $n \geq 1$. Note that $|a_n| \leq \|\Lambda\|_\infty$ for $n \geq 1$. Consider for each fixed $j \geq 1$ the shifted sequence $\Lambda_j := (\lambda_{k+j})_{k\geq 0} \in \ell^\infty$ as well as the averages

$$\widetilde{\Lambda}_n := \frac{1}{na_n}\sum_{j=1}^{n} \Lambda_j \in \ell^\infty$$

for $n \geq 1$. Observe that the sequence $\widetilde{\Lambda}_n$ converges coordinatewise to $(1, 1, \ldots)$ as $n \to \infty$. In fact, by our counterassumption the k:th coordinate $b_k^{(n)} = \frac{1}{na_n}\cdot\sum_{j=1}^{n} \lambda_{j+k}$ of $\widetilde{\Lambda}_n$ satisfies

$$|b_k^{(n)} - 1| = \frac{1}{n|a_n|}\Big|\sum_{j=1}^{n} \lambda_{j+k} - \sum_{j=1}^{n} \lambda_j\Big| \leq \frac{2k}{cn}\|\Lambda\|_\infty \to 0, \quad n \to \infty.$$

On the other hand, the inclusion

$$T_{\widetilde{\Lambda}_n}(B_{H^1}) \subset c^{-1}G, \quad n \geq 0, \tag{2.5}$$

follows from the identity $T_{\widetilde{\Lambda}_n} = \frac{1}{na_n}\sum_{j=1}^{n} T_{\Lambda_j}$, where it is not difficult to check that $T_{\Lambda_j}(f) \in \overline{g_j}\cdot T_\Lambda(B_{H^1}) \subset G$ for $f \in B_{H^1}$. The coordinatewise convergence of $\widetilde{\Lambda}_n$ combined with (2.5) imply by approximation that $B_{H^1} \subset c^{-1}G$, which is impossible. □

Remarks. (1) By removing the uniform bound $C < \infty$ in (0.1) one obtains a strictly weaker approximation property, see [OT, Example 6.8]. The argument in Theorem 5 shows that H^1 even fails to have this weaker property.

(2) Note that the related quotient space L^1/H^1_0, where $H^1_0 = \{f \in H^1 : f(0) = 0\}$, also fails to have the W.A.P. This observation can be deduced from the facts that L^1/H^1_0 has the DPP (see e.g. [Pe, Cor. 8.1.(b)]), but not the Schur property.

Let $VMOA$ be the closed subspace of $BMOA$ consisting of the analytic functions $f\colon \mathbf{D} \to \mathbf{C}$ having vanishing mean oscillation on the boundary $\mathbf{T}$. Fefferman's duality theorem implies that $VMOA^* \approx H^1$ (up to linear isomorphism). We refer e.g. to the survey [G, Sect. 7] for an exposition and for more information about the space $BMOA$. Theorem 5 and the duality result [T1, Prop. 3.4] has the following consequence.

Corollary 7. *$VMOA$ does not have the inner W.A.P.*

References

[ArT] Argyros, S., and A. Tolias: Methods in the theory of hereditarily indecomposable Banach spaces. - Mem. Amer. Math. Soc. 170:806, 2004.

[AT] Astala, K., and H.-O. Tylli: Seminorms related to weak compactness and to Tauberian operators. - Math. Proc. Cambridge Phil. Soc. 107, 1990, 367–375.

[BP] Bourgain, J., and G. Pisier: A construction of $\mathscr{L}^\infty$-spaces and related Banach spaces. - Bol. Soc. Brasil. Mat. 14, 1983, 109–123.

[C] Casazza, P. G.: Approximation properties. - In: Handbook of Banach spaces, vol. I, North-Holland, 2001, 271–316.

[DF] Defant, A. and K. Floret: Tensor norms and operator ideals. - North-Holland, 1993.

[G] Girela, D.: Analytic functions of bounded mean oscillation. - In: Complex function spaces (Mekrijärvi, 1999), Univ. Joensuu Dept. Math. Rep. Ser. 4, 2001, 61–170.

[K] Kalton, N. J.: Spaces of compact operators. - Math. Ann. 208, 1973, 267–278.

[LT] Lindenstrauss, J., and L. Tzafriri: Classical Banach spaces I. Sequence spaces. - Ergebnisse der Mathematik 92, Springer-Verlag, 1977.

[LS] Lindström, M., and G. Schlüchtermann: Composition of operator ideals. - Math. Scand. 84, 1999, 284–296.

[OT] Odell, E., and H.-O. Tylli: Weakly compact approximation in Banach spaces. - Trans. Amer. Math. Soc. 357, 2005, 1125–1159.

[Pe] Pełczyński, A.: Banach spaces of analytic functions and absolutely summing operators. - CBMS Regional Conference Series 30, Amer. Math. Soc., 1977.

[Pi] Pietsch, A.: Eigenvalues and s-numbers. - Cambridge University Press, 1987.

[P] Pisier, G.: Factorization of linear operators and geometry of Banach spaces. - CBMS Regional Conference Series 60, Amer. Math. Soc., 1986.

[Ru1] Rudin, W.: Projections onto invariant subspaces. - Proc. Amer. Math. Soc. 13, 1962, 429–432.

[Ru2] Rudin, W.: Functional analysis. - Tata McGraw-Hill, 1973.

[R] Ruess, W.: Geometry of operator spaces. - In: Functional analysis: Surveys and recent results III, North-Holland, 1984, 59–78.

[ST] Saksman, E., and H.-O. Tylli: Multiplications and elementary operators in the Banach space setting. - In: Methods in Banach space theory (J. M. F. Castillo and W. B. Johnson, eds.) London Mathematical Society Lecture Notes Series 337, Cambridge University Press, 2006, 253–292.

[T1] Tylli, H.-O.: The essential norm of an operator is not self-dual. - Israel J. Math. 91, 1995, 93–110.

[T2] Tylli, H.-O.: Duality of the weak essential norm. - Proc. Amer. Math. Soc. 129, 2001, 1437–1442.

[W] Wojtaszczyk, P.: Banach spaces for analysts. - Cambridge University Press, 1991.

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