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On the automorphism groups of convex domains in $\mathbb{C}^n$

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Abstract. We establish that every bounded convex domain in $\mathbb{C}^n$ with an automorphism orbit accumulation at a boundary point at which the domain has a sphere contact from inside admits a non-compact 1-parameter subgroup of automorphisms. Notice that this in particular implies that no Teichmüller domain of a Riemann surface of genus $g > 1$ can be holomorphically imbedded as a convex domain in $\mathbb{C}^{3g-3}$.

1 Introduction

The primary goal of this article is to give a rigorous proof of the following:

Theorem 1.1. *If a bounded convex domain in* $\mathbb{C}^n$ *possesses a non-compact automorphism orbit accumulating at a boundary point with sphere contact inside, then the automorphism group contains a non-compact* 1*-parameter subgroup.*

This statement and even more general ones were mentioned, more often than not, rather casually as a corollary to S. Frankel's widely known work in Acta Mathematica ([3]). However, several experts pointed out that Frankel's method alone does not easily imply the statement above. (In the smooth boundary case for instance, it does; even if Frankel's theorem on convergence of Frankel scaling does not require smoothness, deriving the above result from it seems to require certain extra conditions such as smoothness of the boundary at least locally. For instance, the final remark of this article may be relevant for this point.) What is proved in Frankel's paper actually is the special case in which the bounded convex domain covers holomorphically a compact complex variety. In such a case, there is an advantage of being able to choose a non-tangential automorphism orbit accumulating at a boundary point which admits a sphere contact from inside, and Frankel's method of producing a non-compact 1-parameter subgroup of automorphisms relies upon this advantage.

In general, the non-compactness of the holomorphic automorphism group with

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respect to the compact-open topology cannot ensure the existence of such a special orbit. The main point of the proof of the above stated theorem in fact lies in the fact that the automorphism orbit may *not* be non-tangential to the boundary at the accumulation point. (We use the terminology "an automorphism orbit being non-tangential to the boundary" in the sense that the automorphism orbit stays in an acute cone contained in the domain with its apex at the accumulation point.) Nonetheless, we do not preclude the possibility that a precise proof is already known and published somewhere, in which case we hope that this paper may serve as an alternative proof of this useful fact, and furthermore as yet another article which presents one of the very useful scaling methods in detail.

In this note, we also use the scaling method, but of Pinchuk. Even though it is now known that Pinchuk's scaling method on convex domains is in fact equivalent to Frankel's scaling method (e.g. [5]) in terms of convergence, Pinchuk's formulation appeals to us as more descriptive and straightforward in many applications.

Notice however that our theorem is presented here not only in order to produce merely a formal generalization of Frankel's theorem. It in particular has a relevance to the following well-known problem: *if a bounded domain in* $\mathbb{C}^n$ *has a smooth boundary and if its automorphism group is non-compact, is its automorphism group positive-dimensional*?

Furthermore, our theorem also implies the following result.

Proposition 1.2. *The Bers embedding of the Teichmüller space of a compact Riemann surface of genus* 2 *or higher cannot be a convex domain.*

Even if detailed arguments appear in the last section of this article, we briefly state the proof here. H. L. Royden's work (together with results that precede in the Teichmüller theory) implies that the holomorphic automorphism group of the Teichmüller domain (we mean Bers' embedding, in this context) is discrete and non-compact. See [9]. It is further known that the automorphism orbits of this domain can accumulate at every boundary point. As a result, if the Teichmüller domain were realized as a bounded convex domain, we may have an automorphism orbit accumulating at a point with a sphere contact from inside. Then, the domain must admit a non-compact one-parameter subgroup, according to our main theorem. This discrepancy yields the proof.

While this proposition may not be a surprising one, we point this out because this is in contrast with the recent articles by M. Abate and G. Patrizio (see [1] and the references therein), which demonstrate that the Teichmüller domains have several important special properties that seem shared only with the bounded convex domains. However, we point out here that our proposition above does not exclude the possibility that a new embedding of the Teichmüller space might turn out to be convex.

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that each automorphism orbit of a point accumulates at every boundary point of the Teichmüller domain of a compact Riemann surface of genus $g > 1$. A brief explication for this is added in the final section of this paper in a remark.

2 Line types and scaling of convex domains

Let Ω be a bounded convex domain in $\mathbb{C}^n$. Notice that we do not assume any further boundary regularity.

Given a sequence $Q := \{q_j \in \Omega \mid j = 1, 2, \dots\}$ that converges to a boundary point $p \in \partial\Omega$, we will now describe the associated Pinchuk scaling sequence ([8]) which is a divergent sequence of $\mathbb{C}$-affine linear mappings of $\mathbb{C}^n$.

2.1 Centering of the sequence Q. Fix a point $q_j \in Q$ and its index j. Then choose a boundary point $p_j \in \partial\Omega$ satisfying

$$\|q_j - p_j\| = \min_{x \in \partial\Omega} \|q_j - x\|.$$

Such p_j is not uniquely determined in general, and hence we simply choose one. On the other hand, the choice of p_j implies the uniqueness of the supporting real hyperplane to Ω at p_j. Now, we may choose a unitary transformation $T_j : \mathbb{C}^n \to \mathbb{C}^n$ such that the complex affine transformation $\psi_j : \mathbb{C}^n \to \mathbb{C}^n$ defined by $\psi_j(z) := T_j(z - p_j)$, for each $z \in \mathbb{C}^n$, satisfies the relation

$$\psi_j(\Omega) \subset \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \operatorname{Re} z_n > 0\},$$

where $\operatorname{Re} z$ denotes the real part of the complex number z as usual. Needless to say, the supporting hyperplane to $\psi_j(\Omega)$ is defined by the equation $\operatorname{Re} z_n = 0$.

We call the sequence $\{\psi_j\}_j$ "*the centering maps*" throughout the rest of this note.

2.2 Line types and stretching factors. We now introduce correct coordinate changes and scaling factors to build a version of Pinchuk's scaling process.

We again fix an index j and the point $q_j \in Q$. Then we consider the complex orthogonal complement $V_{n-1}^{(j)}$ in $\mathbb{C}^n$ of the line joining the origin and the point $\psi_j(q_j)$, and the "projected slice"

$$\Omega_{n-1}^{(j)} = \{z \in V_{n-1} \mid z + \psi_j(q_j) \in \psi_j(\Omega)\}.$$

Equip $V_{n-1}^{(j)}$ with the Hermitian inner product inherited from $\mathbb{C}^n$. Then, $\Omega_{n-1}^{(j)}$ is a domain in $V_{n-1}^{(j)}$ containing the origin. Choose a point in $\partial\Omega_{n-1}^{(j)}$ that is closest to the origin. It is not unique in general, and hence we select one. Let us denote it by $x_{n-1}^{(j)}$.

We continue this process as long as it is possible to proceed. $V_{n-2}^{(j)}$ will denote the complex orthogonal complement in $V_{n-1}^{(j)}$ of the vector $x_{n-1}^{(j)}$. We consider

$$\Omega_{n-2}^{(j)} = \Omega_{n-1}^{(j)} \cap V_{n-1}^{(j)}.$$

Again, $x_{n-2}^{(j)}$ will be a point in $\partial\Omega_{n-2}^{(j)}$ that is one of the closest points to the origin.

By an induction on this process, we obtain mutually orthogonal vectors $x_1^{(j)}, \ldots, x_{n-1}^{(j)}$. Let $x_n^{(j)} = \psi_j(q_j)$. Then the vectors

$$e_\ell^{(j)} := \frac{x_\ell^{(j)}}{\|x_\ell^{(j)}\|} \quad (\ell = 1, \ldots, n)$$

form an orthonormal basis for $\mathbb{C}^n$. We now denote by

$$\lambda_\ell^{(j)} = \frac{1}{\|x_\ell^{(j)}\|}$$

for every $\ell = 1, \ldots, n$. Here the superscript (j) emphasizes the dependence of each $\lambda_\ell^{(j)}$ on j.

We now consider, for each j, the complex linear mapping $\Lambda_j : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$\Lambda_j(e_\ell^{(j)}) = \lambda_\ell^{(j)} e_\ell^{(j)} \quad (\ell = 1, \ldots, n).$$

Then the *Pinchuk stretching sequence* we are going to use throughout this paper is the sequence of complex affine linear mappings given by

$$\sigma_j := \Lambda_j \circ \psi_j : \mathbb{C}^n \to \mathbb{C}^n \quad (j = 1, 2, \ldots),$$

which is the "stretching" followed by the "centering" of the sequence Q.

2.3 Scaling with the automorphism orbits. Now we consider the case *when the sequence Q above is in particular given as $q_j = \varphi_j(q_0)$, where $q_0 \in \Omega$ and $\varphi_j \in$* $\mathrm{Aut}(\Omega)$. Then we exploit the following convergence theorems for the *scaling sequence*

$$\omega_j(z) = \sigma_j \circ \varphi_j(z).$$

(Notice that this is the "normalization" of Pinchuk's stretching sequence by the non-compact automorphism sequence φ_j. This was again first introduced by S. Pinchuk.)

Proposition 2.1. *The scaling sequence $\omega_j : \Omega \to \mathbb{C}^n$ $(j = 1, 2, \ldots)$ introduced above has the following convergence property: every subsequence of $(\omega_j)_j$ admits a subsequence that converges uniformly on compact subsets to a biholomorphic embedding, say $\hat{\omega}$ of Ω into $\mathbb{C}^n$.*

Then we also have information on the set convergence of $\omega_j(\Omega)$ as j tends to ∞. In order to explain this as plainly as possible, we remark that

$$\hat{\omega}_j(q_0) = (1, 0, \dots, 0) \quad \text{for } j = 1, 2, \dots$$

Let us introduce the notation

$$\tau(z_1, \dots, z_n) = (z_1 - 1, 0, \dots, 0) : \mathbb{C}^n \to \mathbb{C}^n$$

and let $B(0; R)$ represent the open ball in $\mathbb{C}^n$ with radius R centered at the origin $(0, \dots, 0)$. Then, we have the following result which we usually call *the local Hausdorff set-convergence of* $\omega_j(\Omega)$ *to* $\hat{\omega}(\Omega)$.

Proposition 2.2. *Let* $R > 0$ *be arbitrarily given. Then, for every* $\varepsilon > 0$ *there exists* $N > 0$ *such that for every* $j > N$ *we have*

$$\begin{aligned}(1 - \varepsilon)[(\tau \circ \omega_j(\Omega)) \cap B(0; R)] &\subset (\tau \circ \hat{\omega}(\Omega)) \cap B(0; R) \\ &\subset (1 + \varepsilon)[(\tau \circ \omega_j(\Omega)) \cap B(0; R)].\end{aligned}$$

For the proof of this precise version of statements, we would like to refer to Kim–Krantz ([5]). We remark however that estimates for the Kobayashi metric on convex domains also prove the same result, which was observed by E. Bedford and S. Pinchuk presumably before the writing of [5].

2.4 An analysis of scaled limit domains. In this section, we focus on the geometric shape of the local Hausdorff set limit of the sequence $\sigma_j(\Omega)$ introduced above. First of all, the Banach selection theorem implies that one can always extract a subsequence of $\sigma_j(\Omega)$ that converges to a convex (in general unbounded) domain in $\mathbb{C}^n$, in the sense of local Hausdorff set convergence. (See the last proposition of the preceding subsection for this terminology.)

Let us momentarily forget the automorphism sequences here, and simply take a point sequence $Q = \{q_j \mid j = 1, 2, \dots\}$ in the domain Ω that converges to a boundary point $p \in \partial\Omega$ as j tends to ∞. Then, for each $j = 1, 2, \dots$ we choose a boundary point $p_j \in \partial\Omega$ that is one of the closest points to q_j among the boundary points of Ω, repeating the same process as we constructed the Pinchuk stretching sequences earlier in this article.

Then we consider the sets

$$\Sigma_j = \{z \in \Omega \mid z - p_j = \lambda(q_j - p_j) \text{ for some } \lambda \in \mathbb{C}\}$$

which we call the *j-th principal slice* of Ω. We are now interested in the sequence $\sigma_j(\Sigma_j)$. Again, for an arbitrary $R > 0$, we restrict ourselves to the closed ball $\bar{B}(0; R)$, and consider the usual Hausdorff limit of the sequence $\sigma_j(\Sigma_j) \cap \bar{B}(0; R)$ there. *In the case when p is a smooth point in the sense that there is a sphere contact from inside* Ω, *the Hausdorff limit in this coincides with the set*

$$\{(z_1, 0, \dots, 0) \in \mathbb{C}^n \mid \operatorname{Re} z_1 \geqslant 0\} \cap \bar{B}(0; R).$$

Since $R > 0$ is arbitrary, we may now conclude: if $p \in \partial\Omega$ is a smooth boundary point in the sense that it admits a sphere contact from inside Ω, the local Hausdorff limit domain, say $\hat{\Omega}$, of the sequence $\sigma_j(\Omega)$ has a real one-dimensional straight line in its boundary.

2.5 Proof of Theorem 1. We use the arguments of the preceding section with the sequence of automorphisms φ_j of Ω and a point $q_0 \in \Omega$ such that $\varphi_j(q_0)$ converges to $p \in \partial\Omega$, where p admits a sphere contact from inside Ω. Now, we consider $Q = \{\varphi_j(q_0) \mid j = 1, 2, \ldots\}$. We use the notation of stretching sequences and scaling sequences of the preceding section. Then, we see that

$$\omega_j(\Omega) = \sigma_j \circ \varphi_j(\Omega) = \sigma_j(\Omega)$$

for every j, because $\varphi_j(\Omega) = \Omega$. Therefore, the scaled limit domain $\hat{\omega}(\Omega) = \hat{\Omega}$. In particular, $\hat{\omega}(\Omega)$ has a straight line, say ℓ, in its boundary.

Now, recall that the convex hull of a straight line and a point away from this line is a parallel strip. Due to the convexity and this fact, we see immediately that every point of the domain $\hat{\Omega}$ admits a line contained in Ω through that point, which is in fact a parallel translation of ℓ. Let $v \in \mathbb{C}^n$ be a direction vector of ℓ. Then, it is now evident that the map

$$g_t(z) = z + tv$$

defines an automorphism of $\hat{\Omega} = \hat{\omega}(\Omega)$ for every $t \in \mathbb{R}$. Since Ω is biholomorphic to $\hat{\omega}(\Omega)$, this shows that $\mathrm{Aut}(\Omega)$ now admits a non-compact one parameter subgroup.

3 Remarks

3.1 Orbit accumulating boundary points of Teichmüller domains. Now, we would like to explain briefly the following well known fact in the Teichmüller theory: *let R be a Riemann surface of genus $g > 1$, and let $\mathcal{T}_g(R)$ be the Bers imbedding of the Teichmüller space in $\mathbb{C}^{3g-3}$. Then, for every boundary point of $\mathcal{T}_g(R)$, there exists an automorphism orbit accumulating at it*. Now, we explain how this is obtained. In this setting, one first takes $3g - 3$ simple closed geodesics which are maximal and not homotopically trivial, which in turn give rise to a Dehn twist, say φ. Then consider the sequence $\{\varphi^j \mid j = 1, 2, \ldots\}$ consisting of iterates of φ, which is a holomorphic automorphism of $\mathcal{T}_g(R)$. For each point $p \in \mathcal{T}_g(R)$, the point sequence $\{\varphi^j(p)\}$ converges to a boundary point q of $\mathcal{T}_g(R)$. It is known also that q is independent of the choice of p. Moreover, it is also known that q is a "maximal cusp". Conversely, to every maximal cusp on the boundary of $\mathcal{T}_g(R)$, there correspond $3g - 3$ geodesics which have the properties just described. Therefore, the automorphism orbit accumulating boundary points of the Teichmüller domain include cusp boundary points. Finally, a theorem of McMullen states that maximal cusps are dense in the boundary of Teichmüller space. This establishes the claim above.

3.2 The case that orbit accumulating boundary points are singular. We discuss the case when a bounded convex domain Ω admits a sequence $\{\varphi_j\}_j$ of automorphisms and whose automorphism orbit $\{\varphi_j(q) \mid j = 1, 2, \ldots\}$ (for every $q \in \Omega$) accumulates at a boundary point p at which $\partial\Omega$ is not smooth. Non-smoothness again means that the tangent cone to Ω at p is not a half space bounded by a hyperplane. (Or equivalently, the supporting hyperplane is not unique at p.) In this case, what matters most is the sequence of *principal slices* which we defined in Section 2.4. Recall that the scaled limit domain contains the real 2-dimensional *tangent cone, say* Γ, *of the limit of the principal slices* in its closure. Then, by the aforementioned convex hull arguments, the scaled limit domain must satisfy the property that for everyone of its interior point, say x, the domain contains $\{z + x \mid z \in \Gamma\}$ sharing the boundary with it. This seems very similar to the case of smooth accumulation points. However, notice here that it is possible that Γ is not a half plane, and that in such a case, there seem to be no obvious ways to conclude that the scaled limit domain contains a non-compact one-parameter family of automorphisms. (The set Γ does have homothety automorphisms for instance; yet it does not seem likely that they extend immediately to automorphisms of the scaled limit domain.) At this point, it may be fair to say that this is a limitation of this version of scaling, but it may be possible to show that this case also dispenses with the existence of one-parameter families of automorphisms. We do have some progress in a different direction which has relevance to this problem. Nevertheless, we choose not to include it here lest the coherence of this article may be affected.

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