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Preface

The Colloquium on Differential and Difference Equations - CDDE 2000 - was held in Brno, September 5-8, 2000. It was organized by the Faculty of Science, Masaryk University and the Mathematical Institute of the Academy of Science of the Czech Republic in cooperation with the Union of Czech Mathematicians and Physicists and the Union of Slovak Mathematicians and Physicists.

Due to the growing interest in the relationship between qualitative theory of differential and difference equations it was decided to organize a meeting in Brno which would have brought together various people working in these areas. The topic of this meeting was "Qualitative theory of differential and difference equations and their applications". It followed the tradition of the previous conferences and seminars on differential equations held in Brno, as there were Equadiff 3 (1972), Equadiff 6 (1985), Equadiff 9 (1997), Workshop on Qualitative Theory of Differential Equations (1998) and Borůvka Mathematical Symposium (1999). We hope it will be possible to continue in the tradition like this.

The Colloquium was prepared by the Organizing Committee consisting of Miroslav Bartušek (chairman), Zuzana Došlá, Ondřej Došlý, Alexander Lomtatidze and Jaromír Vosmanský. The help of the Honorary and Advisory Board was very appreciated as well. There were 92 participants at the conference from 14 countries. The scientific program consisted of 5 survey plenary lectures (O. Došlý, L. Górniewitz, I. Györi, T. Kusano and Š. Schwabik), 56 communications, 10 posters and 9 extended abstracts.

The social program which took an advantage of the nice surroundings of the Brno dam lake, where the conference site was situated, was organized, too.

The CDDE 2000 Proceedings is published as the supplementary issue of Tomus 36 (2000) of Archivum mathematicum journal and will be distributed to all subscribers of this journal as well as to the participants of CDDE 2000. Additional orders of this Proceedings ($ 35) should be sent to the Managing Editor of AM.

In this volume there are published all submitted papers which have passed an usual AM reviewing process and were accepted for publication. Volume starts with 3 survey papers by invited speakers and other 26 contributions are ordered alphabetically.

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The electronic edition of the CDDE 2000 Proceedings is available in PDF format on `http://www.emis.de/journals/`.

We would like to thank our colleagues for their help which enabled us to publish this volume in short time, the editors of Archivum mathematicum for their kind agreements to publish this Proceedings in the frame of the journal, the referees for their good work and quick responses and, last but not least, to Jiří Šremr, our PhD student, for the TEX adaptation of the great number of papers and for preparing the electronic version of this volume.

Brno, February 2001 Editors

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OSCILLATION THEORY OF LINEAR DIFFERENCE EQUATIONS

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Abstract. The survey of the basic results of oscillation theory of various linear differential equations and systems is presented. It is shown that the discrete oscillation theory is in many aspects very similar to its continuous counterpart. Some open problem are discussed.

AMS Subject Classification. 39A10

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1. Introduction

The aim of this paper is to present a brief survey of the basic results of the discrete oscillation theory, to compare these results with their continuous counterparts, and to formulate some open problems in this area.

Let us start, as a motivation for our investigation, with the very famous second order linear difference equation, namely the equation

$$x_{k+2} = x_{k+1} + x_k \tag{1}$$

which determines the Fibonacci numbers. The characteristic equation of (1) is $\lambda^2 - \lambda - 1 = 0$, hence

$$x_k^{[1]} = \left(\frac{1+\sqrt{5}}{2}\right)^k, \quad x_k^{[2]} = \left(\frac{1-\sqrt{5}}{2}\right)^k$$

is a pair of linearly independent solutions of (1). Obviously, the solution $x^{[1]}$ is a monotonically increasing sequence, whereas $x^{[2]}$ is an oscillatory sequence. From this point of view, it seems that the Sturmian separation theorem concerning the zero points of the linearly independent solutions of the Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0, \quad r(t) > 0, \tag{2}$$

has no discrete analogue.

To show that this is not the case, let us return to the motivation for the investigation of oscillatory properties of (2) (more precisely, distribution of zero points of its solutions). One of these motivations consists in the investigation of positivity of the quadratic functional

$$\mathcal{F}_c(y) := \int_a^b \left[r(t)y'^2 - c(t)y^2 \right] dt \tag{3}$$

over the class of (nontrivial, sufficiently smooth) functions y satisfying $y(a) = 0 = y(b)$. The functional $\mathcal{F}$ is (upon a certain transformation) the functional of the second variation of the fixed end points variational problem

$$\int_a^b f(t, x(t), x'(t))\, dt \to \min, \quad x(a) = A, x(b) = B. \tag{4}$$

and its positivity is a sufficient condition for an extremal to be a local minimum of (4), for a more detailed treatment of this topic see [19].

The important role in the investigation of positivity of the functional $\mathcal{F}_c$ is played by the so-called Picone identity. This identity relates the quadratic functional $\mathcal{F}_c$ to the Riccati equation

$$w' + c(t) + \frac{w^2}{r(t)} = 0 \tag{5}$$

which is related to (2) by the substitution $w := \frac{r(t)x'}{x}$. This identity reads as follows; let w be a solution of (5) which exists on the whole interval $[a, b]$, then

$$\mathcal{F}_c(y) = w(t)\, y^2\big|_a^b + \int_a^b \frac{1}{r(t)} \left(r(t)y' - w(t)y\right)^2 dt, \tag{6}$$

in particular, if $y(a) = 0 = y(b)$, this formula shows that the existence of a solution x of (2) without zero in $[a, b]$ (and hence the existence of w solving (5) on $[a, b]$) implies that $\mathcal{F}_c$ can be "completed to the square" (compare the integral term on the right-hand-side of (6)) and hence $\mathcal{F}_c$ is positive over the class of y satisfying $y(a) = 0 = y(b)$.

If we replace the integral in (4) by its partial Riemann sum, after some relabeling of variables in this extremal problem, its discrete version is

$$\sum_{k=0}^{N} f(k, x_{k+1}, \Delta x_k) \to \min, \quad x_0 = A,\ x_{N+1} = B, \tag{7}$$

for a more detailed description of this discretization process we refer to [2,22]. The investigation of sufficient conditions for a local minimum of (7) leads (using essentially the same arguments as in the continuous case) to the problem of positivity of the discrete quadratic functional

$$\mathcal{F}_d(y) := \sum_{k=0}^{N} \left[r_k (\Delta y_k)^2 - c_k y_{k+1}^2 \right], \quad \Delta y_k := y_{k+1} - y_k, \tag{8}$$

in the class of nontrivial sequences $y = \{y_k\}_{k=0}^{N+1}$ satisfying $y_0 = 0 = y_{N+1}$. This functional is connected with the Sturm-Liouville difference equation

$$\Delta(r_k \Delta x_k) + c_k x_{k+1} = 0, \quad r_k \neq 0, \tag{9}$$

in the same way as (3) and (2) in the continuous case. The discrete analogue of (5) is the equation

$$\Delta w_k + c_k + \frac{w_k^2}{r_k + w_k} = 0 \tag{10}$$

and this equation is related to (9) by the substitution $w_k = \frac{r_k \Delta x_k}{x_k}$. Here one can see already a certain difference between the discrete and continuous case, namely the presence of w in the denominator of the last expression of (10), we will return to this phenomenon later in this paper. Following the same idea as in the continuous case we reveal the discrete Picone identity

$$\mathcal{F}_d(y) = w_k y_k^2 \big|_0^{N+1} + \sum_{k=0}^{N} \frac{1}{r_k + w_k} \left(r_k \Delta y_k - w_k y_k \right)^2, \tag{11}$$

where w is a solution of (10) defined for every $k \in [0, N+1]$. In particular, the term $r + w$ plays the same role as the term r in the continuous case and hence $\mathcal{F}_d$ is positive (for nontrivial y satisfying $y_0 = 0 = y_{N+1}$) provided there exists a solution w of (10) defined for $k \in [0, N+1]$ and satisfying $w_k + r_k > 0$ for $k \in [0, N]$. Substituting for $w = \frac{r \Delta x}{x}$, the last inequality is equivalent to $r_k x_k x_{k+1} > 0$. Consequently, this leads to the following definition.

Definition 1. We say that an interval $(m, m+1]$, $m \in \mathbb{Z}$, contains a *generalized zero* of a solution x of (9) if $x_m \neq 0$ and $x_m x_{m+1} r_m \leq 0$.

The Fibonacci equation (1) can be rewritten into the (self-adjoint) form

$$\Delta \left((-1)^k \Delta x_k \right) + (-1)^k x_{k+1} = 0,$$

see [2, Chap. I]. Applying the above definition (with $r_k = (-1)^k$) to this equation we easily see that both solutions $x^{[1]}, x^{[2]}$ are actually oscillatory, they have infinitely many generalized zeros.

Finally note that the discrepancies between discrete and continuous oscillation theories are mostly caused by differences between continuous calculus (differential and integral calculus) and its discrete counterpart (the calculus of differences and sums).

2. Oscillation theory of Sturm-Liouville difference equations

Using the definition of a generalized zero from the previous section we can now formulate the main statement of the oscillation theory of Sturm-Liouville difference equations (9), the so-called Roundabout theorem, see e.g. [2].

Theorem 1. *The following statements are equivalent:*

(i) Equation (9) is disconjugate on $[0, N]$, i.e., the solution $\tilde{x}$ given by the initial condition $\tilde{x}_0 = 0$, $r_0\tilde{x}_1 = 1$ has no generalized zero in $(0, N + 1]$.
(ii) There exists a solution of (9) having no generalized zero in $[0, N + 1]$.
(iii) There exists a solution w of (10) which is defined for every $k \in [0, N + 1]$ and satisfies $r_k + w_k > 0$ for $k \in [0, N]$.
(iv) The quadratic functional $\mathcal{F}_d(y)$ is positive for every nontrivial y satisfying $y_0 = 0 = y_{N+1}$.

This theorems shows that the Sturmian separation and comparison theory does extend to (9). Indeed, the separation theorem is given by the equivalence (i) $\iff$ (ii) and the comparison theorem is "hidden" in the equivalence (i) $\iff$ (iv). Let us also remind the main ideas used in the proof of Theorem 1. The implication (i) $\implies$ (ii) follows from the continuous dependence of solutions of (9) on a parameter. More precisely, if the solution $\tilde{x}$ given in (i) has no generalized zero in $(0, N + 1]$, then the solution $x^{[\varepsilon]}$ given by the initial condition $x_0^{[\varepsilon]} = \epsilon$, $r_0 x_1^{[\varepsilon]} = 1$ has no generalized zero in $[0, N + 1]$ if $\varepsilon > 0$ is sufficiently small. The implication (ii) $\implies$ (iii) is just the Riccati substitution and the already mentioned fact that $r_k + w_k > 0$ if and only if $r_k x_k x_{k+1} > 0$. The implication (iii) $\implies$ (iv) follows immediately from Picone's identity. Finally, the implication (iv) $\implies$ (i) is proved by contradiction. If $\tilde{x}$ would have a generalized zero in $(0, N+1]$, one can construct a nontrivial $y = \{y_k\}_{k=0}^{N+1}$ with $y_0 = 0 = y_{N+1}$ such that $\mathcal{F}_d(y) \le 0$. More details concerning this proof can be found e.g. in [5].

The Roundabout theorem (observe that this name for the theorem comes from its proof) immediately suggests two main methods of the discrete oscillation theory. The first one consists in the equivalence (i) $\iff$ (iv) and is called the *variational method*, whereas the second method, leaned on the equivalence (i) $\iff$ (iii), is usually referred as the *Riccati technique*. Recall that equation (9) is said to be *nonoscillatory* if there exists $N \in \mathbb{N}$ such that (9) is disconjugate on $[N, M]$ for every $M > N$, in the opposite case (9) is said to be *oscillatory*.

To prove (via the variational method) that (9) is oscillatory, it suffices to construct for every $N \in \mathbb{N}$ a sequence $y = \{y_k\}_{k=N}^{\infty}$, such that $y_N = 0$, only finitely many y_k are nonzero (this class of sequence we will denote by $\mathcal{D}(N)$) and

$$\mathcal{F}_d(y; N, \infty) := \sum_{k=N}^{\infty} \left[r_k (\Delta y_k)^2 - c_k y_{k+1}^2 \right] < 0.$$

On the other hand, to prove nonoscillation of (9) we need to show that there exists $N \in \mathbb{N}$ such that for every nontrivial $y \in \mathcal{D}(N)$ we have $\mathcal{F}_d(y; N, \infty) > 0$.

A typical example of the oscillation criterion proved using the variational method is the discrete version of the Leighton-Wintner oscillation criterion.

Theorem 2. *Suppose that $r_k > 0$ for large k and*

$$\sum^{\infty} r_k^{-1} = \infty = \sum^{\infty} c_k. \tag{12}$$

Then equation (9) is oscillatory.

Proof. Let $N \in \mathbb{N}$ be arbitrary. Define for $N < n < m < M$ (which will be determined later) a sequence $y \in \mathcal{D}(N)$ as follows

$$y_k = \begin{cases} \left(\sum_{j=N}^{k-1} r_j^{-1}\right)\left(\sum_{j=N}^{n-1} r_j^{-1}\right)^{-1}, & N+1 \le k \le n, \\ 1, & n+1 \le k \le m-1, \\ \left(\sum_{j=k}^{M-1} r_j^{-1}\right)\left(\sum_{j=m}^{M-1} r_j^{-1}\right)^{-1}, & m \le k \le M-1, \\ 0, & k \ge M. \end{cases}$$

Then we have

$$\begin{aligned} \mathcal{F}_d(y; N, \infty) &= \sum_{k=N}^{\infty} \left[r_k(\Delta y_k)^2 - c_k y_{k+1}^2\right] = \sum_{k=N}^{M-1} \left[r_k(\Delta y_k)^2 - c_k y_{k+1}^2\right] \\ &= \left(\sum_{k=N}^{n-1} + \sum_{k=n}^{m-1} + \sum_{k=m}^{M-1}\right)\left[r_k(\Delta y_k)^2 - c_k y_{k+1}^2\right] \\ &= \left(\sum_{k=N}^{n-1} r_k^{-1}\right)^{-1} - \sum_{k=N}^{n-1} c_k y_{k+1}^2 - \sum_{k=n}^{m-2} c_k - \sum_{k=m-1}^{M-1} c_k y_{k+1}^2 + \left(\sum_{k=m}^{M-1} r_k^{-1}\right)^{-1}. \end{aligned}$$

Now, using the discrete version of the second mean value theorem of the sum calculus (see, e.g. [11]), there exists $\tilde{m} \in [m-1, M-1]$ such that

$$\sum_{k=m-1}^{M-1} c_k y_{k+1}^2 \le \sum_{k=m-1}^{\tilde{m}} c_k.$$

Let $n > N$ be fixed. Since (12) holds, for every $\varepsilon > 0$ there exist $M > m > n$ such that $\sum_{k=n}^{\tilde{m}} c_k > \mathcal{F}_d(y; N, n-1) + \varepsilon$ whenever $\tilde{m} > m$ and $\left(\sum_{k=m}^{M-1} r_k^{-1}\right)^{-1} < \varepsilon$. Consequently, we have

$$\mathcal{F}_d(y; N, \infty) = \mathcal{F}_d(y; N, n-1) - \sum_{k=n}^{\tilde{m}} c_k + \left(\sum_{k=m}^{M-1} r_k^{-1}\right)^{-1} < 0$$

what we needed to prove.

A more sophisticated application of the construction of the sequence y leads to a discrete versions of Nehari-type oscillation criteria, for more details we refer to

[11], where the variational method is used to derive oscillation criteria for $2n$-order Sturm-Liouville difference equations.

In proving nonoscillation criteria using the variational method, the following discrete version of the Wirtinger-type inequality is a very useful tool, see [23].

Theorem 3. *Let M_k be a positive sequence such that $\Delta M_k \neq 0$ for $k \geq N$. Then for every $y \in \mathcal{D}(N)$ we have*

$$\sum_{k=N}^{\infty} |\Delta M_k| y_{k+1}^2 \leq \psi_N \sum_{k=N}^{\infty} \frac{M_k M_{k+1}}{|\Delta M_k|} (\Delta y_k)^2,$$

where

$$\psi_N := \left(\sup_{k \geq N} \frac{M_k}{M_{k+1}}\right) \left\{1 + \left(\sup_{k \geq N} \frac{|\Delta M_k|}{|\Delta M_{k-1}|}\right)^{1/2}\right\}^2. \tag{13}$$

A typical example of the application of the Wirtinger inequality is the next Nehari-type nonoscillation criterion which is proved for higher order equations in [23].

Theorem 4. *Suppose that there exists a positive sequence M_k such that ΔM_k is eventually nonzero and satisfies $0 < \psi := \limsup_{N\to\infty} \psi_N < \infty$, where ψ_N is defined by (13). If*

$$\limsup_{k\to\infty} \frac{1}{M_k} \sum_{j=k}^{\infty} c_j^+ < \frac{1}{\psi}, \quad c_k^+ := \max\{0, c_k\},$$

then equation (9) is nonoscillatory.

We finish this section with a Hille-Nehari type nonoscillation criterion proved using the Riccati technique. This criterion is presented in [16] for the half-linear second order difference equation

$$\Delta(r_k \Phi(\Delta x_k)) + c_k \Phi(x_{k+1}) = 0, \quad \Phi(x) := |x|^{p-2}x, \ p > 1,$$

but for the sake of simplicity we formulate it for linear equation (9).

Observe that according to the Sturm comparison theorem for (9), to prove nonoscillation of (9), it actually suffices to find $N \in \mathbb{N}$ and a sequence w_k defined for $k \geq N$, satisfying $w_k + r_k$ and the *inequality*

$$\Delta w_k + c_k + \frac{w_k^2}{w_k + r_k} \leq 0. \tag{14}$$

Theorem 5. *Suppose that $r_k > 0$ for large k, $\sum^{\infty} c_k$ is convergent and*

$$\lim_{k\to\infty} \frac{r_k^{-1}}{\sum^{k-1} r_j^{-1}} = 0. \tag{15}$$

If

$$\limsup_{k\to\infty}\left(\sum^{k-1} r_j^{-1}\right)\left(\sum_{j=k}^{\infty} c_j\right) < \frac{1}{4}, \quad \liminf_{k\to\infty}\left(\sum^{k-1} r_j^{-1}\right)\left(\sum_{j=k}^{\infty} c_j\right) > -\frac{3}{4} \tag{16}$$

then (9) *is nonoscillatory.*

Note that assumption (15) has no analogue in the continuous version of Theorem 5 (see e.g. [12]) and necessity of this assumption in Theorem 5 is caused by the term $r_k + w_k$ in the denominator of the last term in (10). We define the sequence

$$w_k := \frac{1}{4}\left(\sum^{k-1} r_j^{-1}\right)^{-1} + \sum_{j=k}^{\infty} c_j$$

and in order to show that (16) imply that w is a solution of (14) satisfying $w_k + r_k > 0$ we need just assumption (15). In the continuous case, the denominator of the last term in the Riccati equation (5) is r, i.e. does not contain the function w and no analogue of (15) is needed in the continuous modification of this proof.

Finally note that the oscillation theory of (9) is now deeply developed and many oscillation and nonoscillation criteria for (2) have their continuous counterparts, see e.g. [1, Chap. VI].

3. Transformation and oscillation theory of symplectic difference systems

Denote $u_k = r_k \Delta x_k$ in (9). Then we can write this equation as the 2-dimensional first order system

$$\Delta\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \begin{pmatrix} 0 & r_k^{-1} \\ -c_k & 0 \end{pmatrix}\begin{pmatrix} x_{k+1} \\ u_k \end{pmatrix} \tag{17}$$

and expanding the difference operator as recurrence system

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \mathcal{S}_k \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \quad \mathcal{S}_k := \begin{pmatrix} 1 & \frac{1}{r_k} \\ -\frac{c_k}{r_k} & 1 - \frac{c_k}{r_k} \end{pmatrix}.$$

By a direct computation it is not difficult to verify that the matrix in the last system is symplectic, i.e., it satisfies the identity $\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}$, $\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Consider now the general $2n \times 2n$ symplectic difference system

$$z_{k+1} = \mathcal{S}_k z_k, \tag{18}$$

where $z = \binom{x}{u}$, $\mathcal{S}_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}$ is a symplectic matrix, i.e., it satisfies

$$\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

$x, u \in \mathbb{R}^n$ and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathbb{R}^{n\times n}$. Symplectic difference systems cover a large variety of difference equations and systems. For example, the linear Hamiltonian difference system

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k$$

with symmetric $n \times n$ matrices B, C and the matrix $(I - A)$ invertible is a special case of (18), see [2]. Since the $2n$-order Sturm-Liouville equation

$$\sum_{\nu=0}^{n} \Delta^\nu \left(r_k^{[\nu]} \Delta^\nu y_{k+n-\nu} \right) = 0, \quad \Delta^\nu := \Delta(\Delta^{\nu-1}) \tag{19}$$

can be written as (3) with special matrices A, B, C (see, e.g. [2]), symplectic difference systems cover Sturm-Liouville equations as well.

Let $Z = \binom{X}{U}$, $\bar{Z} = \binom{\bar{X}}{\bar{U}}$ be $2n \times n$ solutions of (18), then $\Delta(Z_k^T \mathcal{J} \bar{Z}_k) = 0$, i.e., $Z_k^T \mathcal{J} \bar{Z}_k = \mathcal{M}$, where $\mathcal{M}$ is a constant $n \times n$ matrix. This identity can be regarded as the extension of the classical Casoratian identity to (18). If $\bar{Z} = Z$, $\mathcal{M} = 0$ and rank $Z_k = n$, then Z is called a *conjoined basis* of (18). Oscillatory properties of solutions of (18) are defined using the concept of a *focal point* in the same way as oscillatory properties of (9) via the concept of generalized zero.

Recall that an interval $(m, m+1]$ contains a *focal point* of a $2n \times n$ solution $Z = \binom{X}{U}$ of (18) if

$$\operatorname{Ker} X_{m+1} \subseteq \operatorname{Ker} X_m \quad \text{and} \quad D_m := X_m X_{m+1}^\dagger \mathcal{B}_m \not\geq 0$$

fail to hold. Here Ker, † and $\geq$ mean kernel, Moore-Penrose generalized inverse and nonnegative definiteness of the matrix indicated.

Let $\mathcal{R}_k = \begin{pmatrix} H_k & M_k \\ K_k & N_k \end{pmatrix}$ be symplectic $2n \times 2n$ matrices (H, K, M, N being $n \times n$ matrices) and consider the transformation

$$z_k = \mathcal{R}_k \tilde{z}_k. \tag{20}$$

This transformation transforms (18) into the system $\tilde{z}_{k+1} = \tilde{\mathcal{S}}_k \tilde{z}_k$, $\tilde{\mathcal{S}}_k = \mathcal{R}_{k+1}^{-1} \mathcal{S}_k \mathcal{R}_k$ and this new system is again symplectic as can be verified by a direct computation. Moreover, if $M_k \equiv 0$ in $\mathcal{R}_k$, then transformation (20) preserves focal points of transformed systems and hence also their oscillatory behavior as it is shown in [6]. In that paper the Roundabout theorem for (18) is presented, in particular, it is proved that the quadratic functional

$$\mathcal{F}(z) := \sum_{k=0}^{N} z_k^T \{\mathcal{S}_k^T \mathcal{K} \mathcal{S}_k - \mathcal{K}\} z_k, \quad \mathcal{K} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$$

over the class of sequences satisfying $\mathcal{K} z_{k+1} = \mathcal{K} \mathcal{S}_k z_k$, $\mathcal{K} z_0 = 0 = \mathcal{K} z_{N+1}$, and the Riccati matrix difference equation

$$Q_{k+1} = (\mathcal{C}_k + \mathcal{D} Q_k)(\mathcal{A}_k + \mathcal{B} Q_k)^{-1}$$

play the same role as (8) and (10) in the oscillation theory of (9).

In the remaining part of this section we present two particular transformations of (18) where the so-called *trigonometric difference system* appears. A trigonometric difference systems (introduced by Anderson [3]) is the symplectic difference system whose matrix satisfies the additional condition $\mathcal{J}^T \mathcal{S}_k \mathcal{J} = \mathcal{S}_k$. This means that transformation (20) with $\mathcal{R}_k = \mathcal{J}$ (the so-called *reciprocity transformation*, see [6]) transforms system (18) into itself. Hence, trigonometric system can be written in the form

$$\begin{pmatrix} s_{k+1} \\ c_{k+1} \end{pmatrix} = \begin{pmatrix} \mathcal{P}_k & \mathcal{Q}_k \\ -\mathcal{Q}_k & P_k \end{pmatrix} \begin{pmatrix} s_k \\ c_k \end{pmatrix}, \tag{21}$$

where the matrices $\mathcal{P}, \mathcal{Q}$ satisfy the identities

$$\mathcal{P}_k^T \mathcal{Q}_k = \mathcal{Q}_k^T \mathcal{P}_k, \quad \mathcal{P}_k^T \mathcal{P}_k + \mathcal{Q}_k^T \mathcal{Q}_k = I. \tag{22}$$

In particular, if $n = 1$, then (22) implies the existence of $\varphi_k \in [0, 2\pi)$ such that

$$\sin \varphi_k = \mathcal{Q}_k, \quad \cos \varphi_k = \mathcal{P}_k \tag{23}$$

and then

$$\begin{pmatrix} s_k \\ c_k \end{pmatrix} = \begin{pmatrix} \sin\left(\sum^{k-1} \varphi_j\right) \\ \cos\left(\sum^{k-1} \varphi_j\right) \end{pmatrix}, \quad \begin{pmatrix} c_k \\ -s_k \end{pmatrix} = \begin{pmatrix} \cos\left(\sum^{k-1} \varphi_j\right) \\ \sin\left(\sum^{k-1} \varphi_j\right) \end{pmatrix}$$

form the basis of the solution solution space of (21).

Theorem 6. (Trigonometric transformation, [7]) *There exist $n \times n$ matrices H and K such that H is nonsingular, $H^T K = K^T H$, and the transformation*

$$\begin{pmatrix} s \\ c \end{pmatrix} = \begin{pmatrix} H^{-1} & 0 \\ -K^T & H^T \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{24}$$

transforms the symplectic system (18) into trigonometric system (21) without changing the oscillatory behavior. Moreover, the matrices $\mathcal{P}$ and $\mathcal{Q}$ from (21) may be explicitly given by

$$\mathcal{P}_k = H_{k+1}^{-1}(\mathcal{A}_k H_k + \mathcal{B}_k K_k) \quad \text{and} \quad \mathcal{Q}_k = H_{k+1}^{-1} \mathcal{B}_k H_k^{T-1}. \tag{25}$$

The previous statement is a discrete version of the trigonometric transformation of linear Hamiltonian *differential* systems established in [10], where it is proved that any linear Hamiltonian differential system

$$x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u \tag{26}$$

with B, C symmetric, can be transformed by a transformation preserving oscillatory nature of transformed systems into the trigonometric differential system

$$s' = Q(t)c, \quad c' = -Q(t)s \tag{27}$$

with a symmetric matrix Q. The terminology trigonometric system is again justified by the scalar case $n=1$ since $\sin\left(\int^t Q(s)\,ds\right), \cos\left(\int^t Q(s)\,ds\right)$ is a solution of this system. It is known (see [26, Chap. VII] that (27) with $Q(t)\geq 0$ is oscillatory (i.e., there exists a conjoined basis $\binom{S}{C}$ and a sequence $t_n\to\infty$ such that $\det S(t_n)=0$) if and only if $\int^\infty \operatorname{Tr} Q(t)\,dt=\infty$, Tr stands for the trace of the matrix indicated. In the discrete case a necessary and sufficient condition for oscillation of (21) is known only in case when $\mathcal{Q}$ is nonsingular and reads

$$\sum^\infty \operatorname{arccotg} \lambda^{[1]}\left(\mathcal{Q}_k^{-1}\mathcal{P}_k\right)=\infty,$$

$\lambda^{[1]}(\cdot)$ denotes the least eigenvalue of the matrix indicated, see [7]. Since the matrix $\mathcal{Q}$ is given by (25), nonsingularity of $\mathcal{Q}$ is equivalent to nonsingularity of $\mathcal{B}$. However, symplectic systems with $\mathcal{B}$ nonsingular do not cover many important cases, e.g. the higher order Sturm-Liouville equation (19). For this reason it would be very useful to know a necessary and sufficient condition for oscillation of (21) also in the case when Q is allowed to be singular.

We finish this section with a discrete version of the Prüfer transformation.

Theorem 7. ([8]) *Let $Z=\binom{X}{U}$ be a $2n\times n$ matrix conjoined basis of (18). Then there exist nonsingular $n\times n$ matrix H and $n\times n$ matrices S, C such that $\binom{X}{U}$ can be expressed in the form*

$$X_k=S_k^T H_k,\quad U_k=C_k^T H_k, \tag{28}$$

where $\binom{S}{C}$ is a solution of the trigonometric system (21) satisfying $S_k^T S_k+C_k^T C_k=I$, $S_k^T C_k-C_k^T S_k=0$. The matrices $\mathcal{P},\mathcal{Q}$ are given by the formulas

$$\begin{aligned}\mathcal{P}&=(H_{k+1}^T)^{-1}\tbinom{X_k}{U_k}^T\mathcal{S}_k^T\tbinom{X_k}{U_k}H_k^{-1}-\Delta H_k,\\ \mathcal{Q}&=(H_{k+1}^T)^{-1}\tbinom{X_k}{U_k}^T\mathcal{S}_k^T\mathcal{J}\tbinom{X_k}{U_k}H_k^{-1}\end{aligned}$$

and H solves the first order system

$$\Delta H_k=(\tilde{Z}_{k+1})^T(\mathcal{S}_k\tilde{Z}_k-\Delta\tilde{Z}_k)H_k,\quad \tilde{Z}=\begin{pmatrix}S^T\\ C^T\end{pmatrix}.$$

In the continuous case, the Prüfer transformation for linear Hamiltonian differential systems (26) was established in [4] as a matrix extension of the classical Prüfer transformation for (2) proved in [25]. If $n=1$ in Theorem 7 and (18) is rewritten Sturm-Liouville equation (9) (compare (17)), then (28) reduces to

$$x_k=H_k\sin\left(\sum^{k-1}\varphi_j\right),\quad r_k\Delta x_k=H_k\cos\left(\sum^{k-1}\varphi_j\right),$$

where φ_k is given by (23), and Theorem 7 is really a discrete version of the classical Prüfer transformation.

4. Higher order linear difference equations

Consider the n-th order linear difference equation

$$L(y)_k := x_{k+n} + a_k^{[n-1]} x_{k+n-1} + \dots a_k^{[1]} x_{k+1} + a_k^{[0]} x_k = 0. \tag{29}$$

Basic facts of the qualitative theory of (29) can be found in [1,17]. One of the motivation for the investigation of oscillatory properties of linear differential and difference equations is the so-called Polya factorization. In the continuous case this problem was resolved in [24] (see also [9]) and in the discrete case it is treated in the fundamental paper of Hartman [21]. Recall now some statements of that paper. An integer $k+m$ is said to be the *generalized zero point of multiplicity* m of a sequence x_k if $x_k \neq 0$, $x_{k+1} = \dots = x_{k+m-1} = 0$ and $(-1)^{m-1} x_{k+m} x_k \leq 0$. If $m = 1$ and $n = 2$ then this definition complies with the definition of the generalized zero of (9) with $r_k \equiv 1$. Observe also that a nontrivial solution of linear equation (29) cannot have a generalized zero of multiplicity greater than $n-1$ as can be verified by a direct computation. Equation (29) is said to be *disconjugate* on the interval $[0, N]$ if every nontrivial solution has at most $n-1$ generalized zeros (counting multiplicity) in $[0, M+n]$ and the solutions satisfying $x_0 = \dots = x_j = 0$, $x_{j+1} \neq 0$, $j \in \{0, \dots, n-2\}$ have at most $n-j-2$ generalized zeros (again counting multiplicity) in $(j+1, N+n-j-1]$.

Theorem 8. *Suppose that (29) is disconjugate on $[0, N]$. Then there exists a fundamental system of solutions of this equation $x^{[1]}, \dots, x^{[n]}$ such that n Casoratians*

$$C(x^{[1]}, \dots, x^{[j]})_k := \begin{vmatrix} x_k^{[1]} & \dots & x_k^{[j]} \\ \vdots & & \vdots \\ x_{k+j-1}^{[1]} & \dots & x_{k+j-1}^{[j]} \end{vmatrix} > 0$$

for $k \in [0, N]$ and $j = 1, \dots, n$. Moreover, the operator L admits Polya's factorization

$$L(y)_k = \alpha_k^{[0]} \alpha_k^{[1]} \cdots \alpha_k^{[n-1]} \Delta \left\{ \frac{1}{\alpha_k^{[n-1]}} \Delta \left[\dots \Delta \left(\frac{y_k}{\alpha_k^{[0]}} \right) \dots \right] \right\}, \tag{30}$$

where (for $j = 2, \dots, n-1$)

(31)
$$\alpha_k^{[0]} = x_k^{[1]}, \ \alpha_k^{[1]} = \Delta \left(\frac{x_k^{[2]}}{x_k^{[1]}} \right), \ \alpha_k^{[j]} = \frac{C(x^{[1]}, \dots, x^{[j-1]})_{k+1} C(x^{[1]}, \dots, x^{[j+1]})_k}{C(x^{[1]}, \dots, x^{[j]})_{k+1} C(x^{[1]}, \dots, x^{[j]})_k}$$

Another important statement concerning Polya's factorization is the so-called Trench canonical factorization, see [20].

Theorem 9. *Suppose that (29) is eventually disconjugate, i.e., there exists $N \in \mathbb{N}$ such that this equation is disconjugate on $[N, M]$ for every $M > N$. Then* *the operator L can be expressed on $[N,\infty)$ in the form (30) with the sequences $\alpha^{[1]},\dots,\alpha^{[n-1]}$ satisfying*

$$\sum^{\infty}\alpha_k^{[j]}=\infty,\quad j=1,\dots,n-1.$$

Recall that the canonical factorization for disconjugate linear *differential operators* was established by Trench [27] and that disconjugate linear differential operators have many properties similar to those of the simple operator of the n-th derivative $\tilde{L}(y):=y^{(n)}$, see e.g. [18]. This book also represent a good motivation for discretization of continuous results.

Now let us turn out attention to the higher order, two-term, Sturm-Liouville equation

$$(-1)^n\Delta^n(r_k\Delta^n y_k)=q_k y_{k+n} \tag{32}$$

with $r_k\neq 0$. The most of the next results can be extended to the general equation (19), but to see better the similarity between the second order case (9) and higher order equations, we consider two-term equation (32) only. Since this equation can be written as a linear Hamiltonian difference system and hence also as a symplectic difference system (18), oscillatory properties of (32) are defined via those of the corresponding symplectic difference system. Denote

$$\mathcal{D}_n(N)=\{y=\{y_k\}_{k=N}^{\infty}:\, y_N=\ldots=y_{N+n-1}=0,\ \exists M>N+n-1,\ y_k=0,\ k\geq M\}$$

(observe that the class of sequences $\mathcal{D}(N)$ defined in Section 2 coincides with $\mathcal{D}_1(N)$). The quadratic functional associated with (32) is

$$\mathcal{F}(y;N,\infty)=\sum_{k=N}^{\infty}\left[r_k(\Delta^n y_k)^2-q_k y_{k+n}^2\right]$$

and equation (32) is nonoscillatory if and only if there exists $N\in\mathbb{N}$ such that $\mathcal{F}(y;N,\infty)>0$ for every nontrivial $y\in\mathcal{D}_n(N)$. This statement is a direct extension of the of the variational oscillation method for second order equations to (32). Using a modified construction from Section 2, one can prove the following higher order extension of the Leighton-Wintner criterion given in Theorem 2.

Theorem 10. ([11]) *Suppose that $r_k>0$ for large k, $\sum^{\infty} r_k^{-1}=\infty$ and there exists $j\in\{0,\dots,n-1\}$ such that $\sum^{\infty} q_k k^{(j)}=\infty$, where $k^{(j)}:=k(k-1)\cdots(k-j+1)$, $k^{(0)}=1$ is the so-called generalized j-th power. Then equation (32) is oscillatory.*

Concerning a higher order extension of the Hille-Nehari-type nonoscillation criterion given in Theorem 5, the proof of this extension is essentially the same as those of Theorem 5, only one has to apply the Wirtinger inequality n-times (instead of once as in Theorem 5). We do not formulate the result explicitly, but we refer to the recent papers [13,23].

In Theorem 10 and also in its nonoscillatory counterpart given in [13], equation (32) is viewed in a certain sense as a perturbation of the one-term (nonoscillatory) equation $(-1)^n \Delta^n (r_k \Delta^n y_k)^{(n)} = 0$ and it is shown that if the sequence q_k is "sufficiently positive", i.e., $\sum^\infty q_k k^{(j)} = \infty$, ("not too positive") then (32) becomes oscillatory (remains nonoscillatory).

To formulate an open problem connected with (32), consider the $2n$-order Sturm-Liouville *differential* equation

$$(-1)^n \left(t^\alpha y^{(n)}\right)^{(n)} = q(t)y, \tag{33}$$

where $\alpha \notin \{1, 3, \dots, 2n-1\}$ is a real constant. A typical approach when investigating oscillatory properties of (33) used e.g. in [14,15], is that this equation is not viewed as a perturbation of the one-term equation $(-1)^n (t^\alpha y^{(n)}) = 0$, but as a perturbation of the Euler-type equation

$$(-1)^n (t^\alpha y^{(n)}) + \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} y = 0 \tag{34}$$

$\gamma_{n,\alpha} = (-4)^{-n} \prod_{i=0}^{n-1} (2n - \alpha - 2i - 1)(2n + \alpha - 2i - 1)$ being the so-called critical oscillation constant. In the discrete case we also have in disposal an Euler-type equation

$$(-1)^n \Delta^{2n} x_k + \frac{\gamma}{(k + 2n - 1)^{(2n)}} x_k = 0 \tag{35}$$

whose solutions are of the form $x_k = \frac{\Gamma(\lambda+k)}{\Gamma(k)}$, $\Gamma(t) = \int_0^\infty \mathrm{e}^{-s} t^{s-1}\, ds$ being the classical Γ function, and λ is a solution of the characteristic equation $(-1)^n \lambda(\lambda - 1) \cdots (\lambda - 2n + 1) + \gamma = 0$, see [1, Chap. III]. However, equation (35) (in contrast to (34)) *is not in self-adjoint form*, since the second term on left-hand-side of this equation contains x with index k instead of $k + n$ (compare (32)). Hence the above mentioned "continuous" idea cannot be directly applied to difference equations.This suggests the following open problem; to find a two-term *self-adjoint nonoscillatory* difference equation which can be solved explicitly (like (34) in the continuous case) and to use this equation as "perturbation equation" in the oscillation theory of (32).

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TOPOLOGICAL STRUCTURE OF SOLUTION SETS: CURRENT RESULTS

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Abstract. We shall present current results concerning Browder–Gupta-type theorems, Banach principle for multivalued mappings and the inverse limit method including its applications to ordinary differential equations and differential inclusions. Consequently, Aronszajn's-type topological characterization of the set of solutions for differential equations and inclusions is considered. Note that some new results of the above mentioned type will be discussed.

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Keywords. fixed points, multivalued maps, inverse systems, acyclicity, topological structure, limit map, topological degree, admissible maps, differential inclusions, Fréchet spaces, Cauchy problem for ODE's

Table of contents

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1. Introduction

In 1923, H. Kneser proved that the Peano existence theorem can be formulated in this way that the set of all solutions is not only nonempty but also compact connected (comp. also [139], [140]). Later, in 1942 N. Aronszajn improved the Kneser theorem by showing that the set of all solutions is even R_δ-set. Evidently the characterization of the set of fixed points for some operators implies the respective result for solution sets. This paper is an attempt to give a systematic presentation of results and methods which concern the topological structure of fixed point sets and solution sets. In this subject there are three methods so called Browder–Gupta method, Banach method and inverse limit method. We survey most important results concerning the above three methods. Our considerations concentrate on simplest cases and main ideas. We included rich literature in which the reader can find further results.

Our paper is devoted for mathematicians and students interested in the topological fixed point theory or in the qualitative theory of differential equations and differential inclusions.

In what follows we shall assume that all topological spaces considered in our paper are metric.

2. Browder–Gupta type results

The famous Schauder Fixed Point Theorem or more generally the Lefschetz Fixed Point Theorem says that there exists a fixed point theorem for some classes of mappings. So, a natural question is to characterize the set of fixed points. The first result, which is still a main one, was proved in 1969 by F. Browder and C. Gupta (comp [21]). Below we shall present a slight generalization of the above mentioned result.

To do this we need some topological notions (for details see: [69]).

Definition 2.1. A space X is called *contractible* provided there exists a (continuous) homotopy $h : X \times [0,1] \to X$ such that:

$$h(x,0) = x \quad \text{for every } x \in X$$

and

$$h(x,1) = x_0 \quad \text{for every } x \in X \text{ and some fixed } x_0 \in X.$$

Definition 2.2. A space X is called an *absolute retract* (written $X \in \mathrm{AR}$) provided that for every space Y, its closed subset $B \subset Y$ and continuous map $f : B \to X$ there exists a continuous extension $\widetilde{f} : Y \to X$ of f over Y, i.e. $\widetilde{f}(x) = f(x)$ for every $x \in B$.

Definition 2.3. A space X is called an *R_δ-set* provided that there exists a sequence of compact nonempty contractible spaces $\{X_n\}$ such that:

$$X_{n+1} \subset X_n \quad \text{for every } n; \tag{2.1}$$

$$X = \bigcap_{n=1}^{\infty} X_n. \tag{2.2}$$

Let us remark (comp. [69]) that a space $X \in \mathrm{AR}$ if and only if X is a convex subset W of a normed space E or X is homeomorphic to a retract[1] of a convex subset $W \subset E$. So any absolute retract is contractible. If we restrict our considerations to compact spaces then we have:

$$\boxed{\mathrm{AR}} \subset \boxed{\mathrm{CONTRACTIBLE}} \subset \boxed{R_\delta}$$

Note that any R_δ-set is a compact nonempty connected space which is acyclic with respect to the Čech homology functor (comp. again [69]), i.e. it has the same homology as the one point space $\{x_0\}$.

Definition 2.4. Let $f : X \to Y$ be a continuous function and let $y \in Y$. We shall say that f is *proper at the point* y provided that there exists $\varepsilon > 0$ such that for any compact set $K \subset B(y, \varepsilon)$ the set $f^{-1}(K)$ is compact, where $B(y, \varepsilon)$ is the open ball in Y with the center at $y \in Y$ and radius ε.

Recall that $f : X \to Y$ is called *proper* provided that for any compact $K \subset Y$ the set $f^{-1}(K)$ is compact. Of course any proper map $f : X \to Y$ is proper at every point $y \in Y$.

Now we are able to formulate our reformulation of the Browder–Gupta theorem:

Theorem 2.1. *Let E be a Banach space and $f : X \to E$ be a continuous map such that the following conditions are satisfied:*

(2.1.1) *f is proper at $0 \in E$,*
(2.1.2) *for every $\varepsilon > 0$ there exists a continuous map $f_\varepsilon : X \to E$ for which we have:*
  (i) *$\|f(x) - f_\varepsilon(x)\| < \varepsilon$ for every $x \in X$,*
  (ii) *the map $\widetilde{f}_\varepsilon : f_\varepsilon^{-1}(B(0, \varepsilon)) \to B(0, \varepsilon)$, $\widetilde{f}_\varepsilon(x) = f_\varepsilon(x)$ for every $x \in f_\varepsilon^{-1}(B(0, \varepsilon))$, is a homeomorphism.*

Then the set $f^{-1}(\{0\})$ is an R_δ-set.

Sketch of proof. First, we have to prove that $f^{-1}(\{0\})$ is nonempty. We take for every $\varepsilon = 1/n$, $n = 1, 2 \dots$ a map $f_n : X \to E$ which satisfies (2.1.2). In view of (2.1.2)(ii) for every n we can find a point $x_n \in X$ such that $f_n(x) = 0$. It follows that:

$$\|f(x_n)\| = \|f(x_n) - f_n(x_n)\| < \frac{1}{n}.$$

So the sequence $\{f(x_n)\}$ is convergent to the point $0 \in E$. Since f is proper at $0 \in E$, we can assume without loss of generality that the sequence $\{x_n\}$ is

[1] A space A is a retract of W if there exists a continuous function $r : W \to A$ such that $r(x) = x$ for every $x \in A$ (we have assumed that $A \subset W$).

convergent to a point $x \in E$. Now from the continuity of f it follows that $f(x) = 0$ and consequently $f^{-1}(\{0\}) \neq \emptyset$.

Now let us denote by S the set $f^{-1}(\{0\})$. It follows from (2.1.1) that S is compact. Moreover, we have proved that $S \neq \emptyset$. For every $\varepsilon = 1/n$, $n = 1, 2 \ldots$ let $A_n = f_n(S)$ where f_n are chosen according to (2.1.2). Then from (2.1.2)(i) we deduce that $A_n \subset B(0, 1/n)$. Note that $\{A_n\}$ is a sequence of compact sets. We let:

$$C_n = \overline{\operatorname{conv}}(A_n).$$

It follows from the Mazur's Lemma (comp. [69] or [90]) that C_n is a compact convex subset of $B(0, 1/n)$. Now by using (2.1.2)(ii) we deduce that set $D_n = f_n^{-1}(C_n)$ is an absolute retract (because it is homeomorphic to the convex set C_n). Therefore we can proceed in the same way as in the proof of Theorem 7 ([21]) and our theorem follows from Lemma 5 in [21].

Note that assumptions in 2.1 are analogous to Theorem 7 ([21]).

Let us remark also that Theorem 2.1 has exactly the same proof if we replace the Banach space E by an arbitrary Fréchet space and open balls by convex symmetric open neighbourhoods of the zero point $0 \in E$. We shall show it in the multivalued case.

Now, we are going to explain the scope of fixed point interpretation of Theorem 2.1.

Assume that $X \subset E$ and $F: X \to E$ is a given mapping. We let $f: X \to E$, $f(x) = x - F(x)$. Then f is called the *field* associated with F. We have:

$$f^{-1}(\{0\}) = \operatorname{Fix}(F) = \{x \in X \mid F(x) = x\}.$$

Observe that if $F_\varepsilon : X \to E$ is an ε-approximation of F then f_ε $(f_\varepsilon(x) = x - F_\varepsilon(x))$ is an ε-approximation of f $(f(x) = x - F(x))$.

It is well known that if F is a compact map or k-set contraction or condensing map which has ε-approximation of the same type then all assumptions of Theorem 5.2 are satisfied for the field f $f(x) = x - F(x))$ associated with F.

We would like to conclude that Theorem 5.2 contains as a special case many results, the called generalizations of the Browder–Gupta theorem (com. [21], [38], [39], [54], [55], [56], [101], [102], [141], [147], [158], [175], [176]).

There is a natural and essential problem to formulate an appropriate multivalued version of the Browder–Gupta Theorem. In this order see: [6], [12], [19], [34], [35], [61], [62], [60], [75], [84], [88], [101], [123], [124], [54]. The most general result was obtained in 1999 by G. Gabor (see [60]). We shall present below the Gabor result.

To do this recall some notation. In what follows the symbol $\varphi : X \multimap Y$ is reserved for multivalued mappings. In this Section we shall assume that for every $x \in X$ the set $\varphi(x)$ is compact nonempty.

A map $\varphi : X \multimap Y$ is called *upper semicontinuous* (u.s.c.) provided that for every open $U \subset Y$ the set $\{x \in X \mid \varphi(x) \subset U\}$ is open; φ is called lower semicontinuous (l.s.c.) provided that for every open $U \subset Y$ the set:

$$\{x \in X \mid \varphi(x) \cap U \neq \emptyset\}$$

is open; φ is continuous, if φ is both u.s.c. and l.s.c.

A map $\varphi : X \multimap Y$ is proper provided that for every compact $K \subset Y$ the set

$$\{x \in X \mid \varphi(x) \cap K \neq \emptyset\}$$

is compact. In what follows for given $\varphi : X \multimap Y$ and $A \subset Y$ we let:

$$\varphi^{-1}(A) = \{x \in X \mid \varphi(x) \subset A\},$$
$$\varphi_+^{-1}(A) = \{x \in X \mid \varphi(x) \cap A \neq \emptyset\}.$$

Assume that $X \subset Y$ and $\varphi : X \multimap Y$ is a given multivalued map. We let

$$\mathrm{Fix}(\varphi) = \{x \in X \mid x \in \varphi(x)\}.$$

Now we are able to formulate the multivalued version of the Browder–Gupta Theorem (see: [60]).

Theorem 2.2. *Let X be a metric space, E a Fréchet space, $\{U_k\}$ a base of open convex symmetric neighbourhoods of the origin in E, and let $\varphi : X \multimap E$ be an u.s.c. proper map with compact values. Assume that there is a sequence of compact convex valued u.s.c. proper maps $\varphi_k : X \to E$ such that*

(i) $\varphi_k(x) \subset \varphi(N_{1/k}(x)) + U_k$, *for every* $x \in X$,
(ii) *if* $0 \in \varphi(x)$, *then* $\varphi_k(x) \cap \overline{U_k} \neq 0$,
(iii) *for every $k \geq 1$ and every $u \in E$ with $u \in U_k$ the inclusion $u \in \varphi_k(x)$ has an acyclic set of solutions.*

Then the set $\mathcal{S} = \varphi^{-1}(0)$ is compact and acyclic[2].

Proof. We show that $\mathcal{S}$ is nonempty. To this end, notice that for every $k \geq 1$ we can find $x_k \in X$ such that $0 \in \varphi_k(x_k)$. Assumption (i) implies that there are $z_k \in N_{1/k}(x_k)$, $y_k \in \varphi_k(z_k)$ and $u_k \in U_k$ such that $0 = y_k + u_k$. Thus $y_k \to 0$. Consider the compact set $K = \{y_k\} \cup \{0\}$. Since φ is proper, the set $\varphi_+^{-1}(K)$ is compact. Moreover, $\{z_k\} \subset \varphi_+^{-1}(K)$. Thus we can assume, without loss of generality, that $\{z_k\}$ converges to some point $x \in X$. By the upper semicontinuity of φ, we have $0 \in \varphi(x)$ and, what follows, $\mathcal{S} \neq \emptyset$.

Since φ is proper, the set $\mathcal{S}$ is compact. We show that it is acyclic. By assumption (ii), the set $A_k = \varphi_{k+}^{-1}(\overline{U_k})$ is nonempty. Consider the map $\psi : A_k \multimap \overline{U_k}$, $\psi_k(x) = \varphi_k(x) \cap \overline{U_k}$. Since $\overline{U_k}$ is contractible and ψ_k is u.s.c. convex valued surjection (see (iii)), we can apply Corollary 3.12 in [60] to obtain that A_k is acyclic.

Now we show that for every open neighbourhood U of $\mathcal{S}$ in X there exists $k \geq 1$ such that $A_k \subset U$. Indeed, assume on the contrary that there is an open neighbourhood U of $\mathcal{S}$ in X such that $A_k \not\subset U$ or every $k \geq 1$. It means that there are $x_k \in A_k$ with $x_k \notin U$ and, consequently, there are $y_k \in \varphi_k(x_k)$ such that

[2] i.e. the Čech homology of $\mathcal{S}$ are the same as a singleton $\{x_0\}$.

$y_k \in \overline{U_k}$. Assumption (i) implies that there are $z_k \in B(x_k, 1/k)$, $v_k \in \varphi(z_k)$ and $u_k \in U_k$ such that $y_k = v_k + u_k$. Therefore, $v_k = y_k - u_k \in 2U_k$ which implies that $v_k \to 0$. Consider the compact set $K_0 = \{v_k\} \cup \{0\}$. Since φ is proper, we can assume that $\{z_k\}$ and, consequently, $\{x_k\}$ converges to some point $x \in X$. Thus $x \in \mathcal{S}$. On the other hand, $x \notin U$, a contradiction and our theorem follows from Lemma 3.10 in [60].

Remark 2.1. It is easy to see that in the above result we can assume that X is a subset of a Fréchet space. Then, instead of neighbourhoods, we can consider sets $x + V_k$, where $\{V_k\}$ is the base of open convex symmetric neighbourhoods of the origin.

As a consequence of Theorem 3.6 and properties of a topological degree of u.s.c. compact convex valued maps (see e.g. [69] or [104]) one can obtain the following theorem generalizing the result of Czarnowski in [39].

Theorem 2.3. *Let $\varOmega$ be an open subset of a Fréchet space E, $\{U_k\}$ the base of open convex symmetric neighbourhoods of the origin in E, and $\varPhi : \overline{\varOmega} \multimap E$ a compact u.s.c. map with compact convex values. Suppose that $x \notin \varPhi(x)$ for every $x \in \partial\varOmega$, and the topological degree $\deg(j - \varPhi, \varOmega, 0)$ of $(j - \varPhi)$ is different from zero, where $j : \overline{\varOmega} \to E$ is an inclusion. Assume that there exists a sequence $\{\varPhi_k : \varOmega \multimap E\}$ of compact u.s.c. maps with compact convex values such that*

- (i) *$\varPhi_k(x) \subset \varPhi(x + U_k) + U_k$, for every $x \in \overline{\varOmega}$,*
- (ii) *if $x \in \varPhi(x)$, then $x \in \varPhi_k(x) + U_k$,*
- (iii) *for every $u \in \overline{U_k}$ the set $\mathcal{S}_u^k$ of all solutions to the inclusion $x - \varPhi_k(x) \ni u$ is acyclic or empty, for every $n > 0$.*

Then the fixed point set $\mathrm{Fix}(\varPhi)$ of $\varPhi$ is compact and acyclic.

Proof. Define the maps φ, $\varphi_k : \overline{\varOmega} \multimap E$, $\varphi = j - \varPhi$, $\varphi_k = j - \varPhi_k$. One can check that φ, φ_k are proper maps. To apply Theorem 2.2 it is sufficient to show that, for sufficiently big k and for every $u \in \overline{U_k}$ the set $\mathcal{S}_u^k$ is nonempty.

For each $k \geq 1$ define the map $\varPsi : \overline{\varOmega} \multimap E$, $\varPsi(x) = \varPhi_k(x) + u$, for every $x \in \overline{\varOmega}$. We prove that, for sufficiently big k, $\deg(j - \varPsi_k, \varOmega, 0) \neq 0$ which implies, by the existence property of a degree, a nonemptiness of $\mathcal{S}_u^k$.

Since φ is a closed[3] map (see e.g. [69]), we can find, for sufficiently big k, a neighbourhood U_k of the origin such that $\varphi(\partial\varOmega) \cap \overline{U_k} = \emptyset$.

Consider the following homotopy $H_k : \overline{\varOmega} \times [0,1] \multimap E$, $H(x,t) = (1-t)\varPhi(x) + t\varPsi_k(x)$. We show that

$$Z_k = \{x \in \partial\varOmega \mid x \in H_k(x,t) \text{ for some } t \in [0,1]\} = \emptyset$$

for sufficiently big k. Suppose, on the contrary, that there are a subsequence of $\{H_k\}$ (we denote it also by $\{H_k\}$), points $x_k \in \partial\varOmega$, and numbers $t_k \in [0,1]$

[3] φ is closed provided for every closed $K \subset \overline{\varOmega}$ the set $\varPhi(K) = \bigcup_{x \in K} \varPhi(x)$ is a closed subset of E.

such that $x_k \in H_k(x_k, t_k)$, that is $x_k = (1-t_k)y_k + t_k s_k + t_k u$, for some $y_k \in \Phi(x_k)$ and $s_k \in \Phi(x_k)$. Assumption (i) implies that there are $z_k \in x_k + U_k$ and $v_k \in \Phi(z_k)$ such that $s_k \in v_k + U_k$. By the compactness of Φ, we can assume that $y_k \to y$ and $v_k \to v$. Therefore, $s_k \to v$. Moreover, we can assume that $t_k \to t \in [0,1]$. This implies that $x_k \to x_0 = (1-t)y + tv + tu$ or, equivalently, that $0 = (1-t)(x_0 - y) + t(x_0 - v) - tu$. But by the upper semicontinuity of φ, we obtain that $x_0 - y \in \varphi(x_0)$ and $x_0 - v \in \varphi(x_0)$. Since φ is convex valued, $0 \in (1-t)\varphi(x_0) + t\varphi(x_0) - tu \subset \varphi(x_0) - tu$. This implies that $\varphi(x_0) \cap \overline{U_k} \neq \emptyset$, a contradiction.

Now, by the homotopy property of a topological degree, one obtains

$$\deg(\Psi_k, \Omega, 0) = \deg(\Phi, \Omega, 0) \neq 0$$

which ends the proof of the theorem.

3. Aronszajn type results

In 1890 Peano [140] showed that the Cauchy problem

$$\left.\begin{aligned} &\dot{x}(t) = g(t, x(t)) \quad \text{for } t \in [0,a], \\ &x(0) = x_0, \end{aligned}\right\} \tag{3.1}$$

where $g : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, has local solutions although the uniqueness property does not hold in general.

This observation became a motivation for studying the structure of the set $\mathcal{S}$ of solutions to (3.1). Peano himself showed that, in the case $n = 1$, all sections $\mathcal{S}(t) = \{x(t) \mid x \in \mathcal{S}\}$ are nonempty, compact and connected (that is, a continuum) in the standard topology of the real line, for t in some neighbourhood of t_0. Kneser generalized this result in 1923 [89] into the case of arbitrary n. In 1928 Hukuhara [80] proved that $\mathcal{S}$ is a continuum in the Banach space of continuous functions with the sup norm.

A more precise characterization of $\mathcal{S}$ was found in 1942 by Aronszajn [10], who showed that $\mathcal{S}$ is an R_δ-set, i.e. it is homeomorphic to the intersection of a decreasing sequence of compact contractible spaces (or compact absolute retracts). This implies that $\mathcal{S}$ is acyclic which means that, without a lipschitzianity of the right hand side f of (3.1), the set $\mathcal{S}$ of solutions (3.1) may not be a singleton but, from the point of view of algebraic topology, it is equivalent to a point, in the sense that it has the same homology groups as one point space $\{x_0\}$.

Aronszajn's result was improved by several authors (see: [1], [3], [4]–[6], [9], [14], [15], [16], [17], [19], [23], [24], [32], [34], [13], [38]–[39], [40], [42]–[44], [46]–[47], [48]–[49], [53], [54], [60], [66], [68], [73], [75], [77]–[78], [97]–[98], [101], [121]–[138], [156]–[168], [171]–[173], [174]–[178]) but always a main tool to do it is a version of the Browder–Gupta theorem. We shall sketch it in the case of problem (3.1) first for the singlevalued case and later for the multivalued case.

The singlevalued case follows immediately from the Browder–Gupta Theorem and the Szufla's type lemma (see [164] or [68]) which we shall present below.

The following result is a slight reformulation of Lemma 1 in [164].

Theorem 3.1. *Let $E = C([0,a],\mathbb{R}^m)$ be the Banach space of continuous maps with the usual max-norm and let $X = K(0,r) = \{u \in E \mid \|u\| \le r\}$ be the closed ball in E.*

If $F : X \to E$ is a compact map and $f : X \to E$ is a compact vector field associated with F, i.e. $f(u) = u - F(u)$, such that the following conditions are satisfied:

(3.1.1) *there exists an $x_0 \in \mathbb{R}^m$ such that $F(u)(0) = x_0$, for every $u \in K(0,r)$;*
(3.1.2) *for every $\varepsilon \in]0,a]$ and for every $u, v \in X$, if $u(t) = v(t)$ for each $t \in [0,\varepsilon]$, then $F(u)(t) = F(v)(t)$ for each $t \in [0,\varepsilon]$;*

then there exists a sequence $f_n : X \to E$ of continuous proper mappings satisfying conditions (2.1.1)–(2.1.2) with respect to f.

Sketch of proof. For the proof it is sufficient to define a sequence $F_n : X \to E$ of compact maps such that:

$$\text{(i)} \qquad F(x) = \lim_{n\to\infty} F_n(x), \quad \text{uniformly in } x \in X,$$

and

$$\text{(ii)} \qquad f_n : X \to E, \quad f_n(x) = x - F_n(x), \text{ is a one-to-one map.}$$

To do this we additionally define the mappings $r_n : [0,a] \to [0,a]$ by putting:

$$r_n(t) = \begin{cases} 0, & t \in \left[0, \dfrac{a}{n}\right], \\ t - \dfrac{a}{n}, & t \in \left(\dfrac{a}{n}, a\right]. \end{cases}$$

Now we are able to define the sequence $\{F_n\}$ as follows:

$$\text{(iii)} \qquad F_n(x)(t) = F(x)(r_n(t)), \quad \text{for } x \in X,\ n = 1, 2, \ldots .$$

It is easily seen that F_n is a continuous and compact mapping, $n = 1, 2, \ldots$. Since $|r_n(t) - t| \le a/n$ we deduce from compactness of F and (iii) that

$$\lim_{n\to\infty} F_n(x) = F(x), \quad \text{uniformly in } x \in X.$$

Now we shall prove that f_n is a one-to-one map. Assume that for some $x, y \in X$ we have

$$f_n(x) = f_n(y).$$

This implies that

$$x - y = F_n(x) - F_n(y).$$

If $t \in [0, a/n]$ then we have

$$x(t) - y(t) = F(x)(r_n(t)) - F(y)(r_n(t)) = F(x)(0) - F(y)(0).$$

Thus, in view of (3.1.1), we obtain

$$x(t) = y(t), \quad \text{for every } t \in [0, a/n].$$

Finally, by successively repeating the above procedure n times we infer that

$$x(t) = y(t), \quad \text{for every } t \in [0, a].$$

Therefore f_n is a one-to-one map and the proof is complete.

Now from Theorems 2.1 and 3.1 we get:

Corollary 3.1. *Assume that f and F are as in Theorem (3.1). Then $f^{-1}(0) = \mathrm{Fix}(F)$ is an R_δ-set.*

Now we come back to problem (3.1). We shall denote by $\mathcal{S}(g, 0, x_0)$ the set of all solutions of the Cauchy problem (3.1).

Theorem 3.2 (Aronszjan). [4] *Let $g : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ be a mapping such that:*

(3.2.1) *$g(\,\cdot\,, x)$ is a measurable function for every $x \in \mathbb{R}^n$,*
(3.2.2) *$g(t, \cdot\,)$ is a continuous function for every $t \in [0, a]$,*
(3.2.3) *there exists a Lebesgue integrable function $\alpha : [0, a] \to [0, +\infty)$ such that:*

$$\|g(t, x)\| \le \alpha(t) \quad \text{for every } (t, x) \in [0, a] \times \mathbb{R}^n.$$

Then $\mathcal{S}(g, 0, x_0)$ is an R_σ-set.

Sketch of proof. We define the integral operator:

$$F : C([0, a], \mathbb{R}^n) \to C([0, a], \mathbb{R}^n)$$

by putting

$$F(u)(t) = x_0 + \int_0^t g(\tau, u(\tau))\, d\tau \quad \text{for every } u \text{ and } t. \tag{3.2}$$

Then $\mathrm{Fix}(F) = \mathcal{S}(g, 0, x_0)$. It is easy to see that F satisfies all the assumptions of Theorem 2.1. Consequently we deduce Theorem 3.2 from 3.1 and the proof is complete.

Now, let g be a Carathéodory map with linear growth. Assume further that $u \in \mathcal{S}(g, 0, x_0)$. Then we have (cf. (3.2.1))

$$u(t) = F(u)(t) = x_0 + \int_0^t g(\tau, u(\tau))\, d\tau,$$

[4] A mapping $g : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying conditions (3.2.1) and (3.2.2) will be called a Carathéodory function; if g satisfies (3.2.3) then it is called integrably bounded.

and consequently

$$\|u(t)\| \leq \|x_0\| + \int_0^a \mu(\tau)\,d\tau + \int_0^t \mu(\tau)\|u(\tau)\|\,d\tau.$$

Therefore from the well-known Gronwall inequality we get

$$\|u(t)\| \leq (\|x_0\||\gamma)\exp(\gamma) \quad \text{for every } t,$$

where $\gamma = \int_0^a \mu(\tau)\,d\tau$. We let

$$g_0 : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$$

by putting

$$g_0(t,x) = \begin{cases} g(t,x), & \text{if } \|x\| \leq M \text{ and } t \in [0,a], \\ g(t, Mx/\|x\|), & \text{if } \|x\| \geq M \text{ and } t \in [0,a], \end{cases}$$

where $M = (\|x_0\| + \gamma)\exp(\gamma)$.

Proposition 3.1. *If g is a Carathéodory map with linear growth, then*

(3.1.a) *g_0 is Carathéodory and integrably bounded; and*
(3.1.b) $\mathcal{S}(g_0, 0, x_0) = \mathcal{S}(g, 0, x_0)$.

The proof of Proposition 3.1 is straightforward (cf. [68], [69], [91]).
Now from Theorem 3.2 and Proposition 3.1 we obtain immediately:

Corollary 3.2. *If $g : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map and has linear growth, then $\mathcal{S}(g, 0, x_0)$ is an R_σ-set.*

We recall the following classical result:

Theorem 3.3. *If $g : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping which is integrably bounded and satisfies condition* (3.41) *and it is locally Lipschitz with respect to the second variable*[5]*, then $\mathcal{S}(g, 0, x_0)$ is an R_δ-set.*

In 1986 F. S. De Blasi and J. Myjak (see [47]) generalized Aronszajn's result for differential inclusions with u.s.c. convex valued right hand sides. Below we shall show the method presented in [68] (comp. also [101], [102]). For the simplicity we shall consider the following Cauchy problem:

$$\left.\begin{aligned} x'(t) &\in \varphi(t, x(t)), \\ x(0) &= x_0, \end{aligned}\right\} \tag{3.3}$$

where $\varphi : [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an u.s.c. bounded map with compact convex values.

We shall denote by $\mathcal{S}(\varphi; 0, x_0)$ the set of all solutions of (3.3). In what follows we keep all assumptions on φ contained in (3.3).

First we have:

[5] Such a mapping g is called integrably bounded measurable-locally Lipschitz.

Proposition 3.2. *If φ possesses a measurable-locally Lipschitz selector $f : [0,a]\times \mathbb{R}^n \to \mathbb{R}^n$, (written $f \subset \varphi$), i.e. $f(t,x) \in \varphi(t,x)$ for every $(t,x) \in [0,a]\times\mathbb{R}^n$, then $\mathcal{S}(\varphi; 0, x_0)$ is contractible.*

Sketch of proof. Let $f \subset \varphi$ be measurable-locally Lipschitz selector. By Theorem 3.3 the following Cauchy problem:

$$\left.\begin{aligned} x'(t) &= f(t, x(t)),\\ x(t_0) &= u_0, \end{aligned}\right\} \tag{3.4}$$

has exactly one solution for every $t_0 \in [0,a]$ and $u_0 \in \mathbb{R}^n$. For the proof it is sufficient to define a homotopy $h : \mathcal{S}(\varphi, 0, x_0)\times[0,1] \to \mathcal{S}(\varphi, 0, x_0)$ such that

$$h(x,s) = \begin{cases} x & \text{for } s = 1 \text{ and } x \in \mathcal{S}(\varphi, 0, x_0),\\ \overline{x} & \text{for } s = 0, \end{cases}$$

where $x = \mathcal{S}(\varphi, 0, x_0)$ is exactly one solution given for the Cauchy problem (3.4).

We put

$$h(x,s)(t) = \begin{cases} x(t), & 0 \le t \le sa,\\ \mathcal{S}(f, sa, x(sa))(t), & sa \le t \le a. \end{cases}$$

Then h is a continuous homotopy contracting $\mathcal{S}(\varphi, 0, x_0)$ to the point $\mathcal{S}(\varphi, 0, x_0)$.

Observe that if $\varphi : [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ is an intersection of the decreasing sequence $\varphi_k : [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ i.e. $\varphi(t,x) = \bigcap_{k=1}^{\infty}\varphi_k(t,x)$ and $\varphi_{k+1}(t,x) \subset \varphi_k(t,x)$ for almost all $t \in [0,a]$ and for all $x \in \mathbb{R}^n$, then

$$\mathcal{S}(\varphi, 0, x_0) = \bigcap_{k=1}^{\infty}\mathcal{S}(\varphi_k, 0, x_0).$$

We have (see: [102] or [69]):

Theorem 3.4. *Assume that φ is as in (3.3). Then there exists a decreasing sequence $\varphi_k : [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ of compact convex valued and bounded u.s.c. mappings such that:*

(3.4.1) $\varphi(t,x) = \bigcap_{k=1}^{\infty}\varphi_k(t,x)$ *for every $t,x) \in [0,a]\times\mathbb{R}^n$,*
(3.4.2) *every φ_k possesses a measurable locally Lipschitz selector $f_k \subset \varphi_k$.*

Now we are in the position to prove the following Aronszajn-type result:

Theorem 3.5. *Under assumptions of (3.3) the set $\mathcal{S}(\varphi; 0, x_0)$ is R_δ.*

Sketch of proof. Consider the sequence $\{\varphi_k\}$ according to (3.4). Then:

$$\mathcal{S}(\varphi; 0, x_0) = \bigcap_{k=1}^{\infty}\mathcal{S}(\varphi_k, 0, x_0).$$

In view of Proposition 3.2 the set $\mathcal{S}(\varphi_k; 0, x_0)$ is contractible. Since φ_k is u.s.c. bounded with convex compact values if follows that $\mathcal{S}(\varphi_k; 0, x_0)$ is compact nonempty (see for example [69]). Therefore $\mathcal{S}(\varphi; 0, x_0)$ is an intersection of compact nonempty and contractible spaces and hence $\mathcal{S}(\varphi; 0, x_0)$ is R_δ.

Remark 3.1. Theorem (3.5) remains true for φ a Carathéodory map with sublinear growth (see: [69] or [47]).

Above we have showed only an application of Browder-Gupta Theorem to the Cauchy problem for the first order ordinary differential equations (inclusions) in the Euclidean space $\mathbb{R}^n$. We would like to point out that another applications are possible, namely:

(A) to the Cauchy problem in Banach spaces on compact or noncompact intervals (see: [2], [4], [9], [18], [32], [34], [35], [38], [39], [40], [49], [50], [53], [55], [56], [59], [61], [62], [66]–[68], [77], [78], [88], [97], [98], [94]–[96], [121]–[138], [152], [174], [175], [171]–[173]);
(B) to higher order differential equations or inclusions (see: [14], [15], [22], [16], [29], [34], [47], [48], [107], [108], [156]–[168], [169], [170]);
(C) to more general boundary value problems both ordinary differential equations and inclusions (see: [5], [6], [12], [17], [72], [13], [93], [149]–[151], [110]–[117]);
(D) to integral equations and inclusions (see: [1], [23]–[25], [87], [171], [172]).

We shall end this section by showing you another possibility. We mean differential equations (inclusions) on compact subsets of $\mathbb{R}^n$ or more generally of Banach spaces. There are only few papers devoted this problem (see: [17], [13], [54], [72], [66], [121]–[123] [143]). For simplicity we shall restrict our considerations to subsets of $\mathbb{R}^n$ (for the Banach case see: [13], [72] and [54]).

Let K be a compact subset of $\mathbb{R}^n$. For a point $x \in K$ by T_xK we shall denote the Bouligand tangent cone to K at x.

We have (see: [66] or [69]):

$$T_xK = \left\{ y \in \mathbb{R}^n \;\middle|\; \liminf_{t \to 0^+} \frac{\operatorname{dist}(x + ty, K)}{t} = 0 \right\}.$$

A compact subset $K \subset \mathbb{R}^n$ is called a *proximate retract* provided there exists an open neighbourhood U of K in $\mathbb{R}^n$ and a retraction $r : U \to K$ such that:

$$\|x - r(x)\| = \operatorname{dist}(x, K), \quad \text{for every } x \in U.$$

It is well known that the class of all proximate retracts is quite rich, in particular it contains convex sets and C^2-manifolds.

Now, let $\varphi : [0, a] \times K \multimap \mathbb{R}^n$ be an u.s.c. map which is bounded and compact convex valued. We shall assume also the following:

$$\varphi(t, x) \cap T_xK \neq \emptyset, \quad \text{for every } (t, x) \in [0, a] \times K. \tag{3.5}$$

For such a map φ we consider the following Cauchy problem:

$$\left.\begin{aligned} &x'(t) \in \varphi(t, x(t)), \\ &x(0) = x_0, \qquad x_0 \in K, \end{aligned}\right\} \tag{3.6}$$

where solutions are considered as absolutely continuous functions $x : [0, a] \to \mathbb{R}^n$ such that $x(t) \in K$ for every $t \in [0, a]$.

Let $\mathcal{S}_K(\varphi; 0, x_0)$ denote the set of all solutions of (3.6).

In 1992 S. Plaskacz proved (see: [143])

Theorem 3.6. *Under all of the above assumptions the set $\mathcal{S}_K(\varphi; 0, x_0)$ is R_δ.*

For the proof of Theorem 3.6 we recommend [143] or [66] or [69].

Remark 3.2. There exists a recent result of R. Bader and W. Kryszewski ([13]) where Theorem 3.6 is taken up for regular sets in Hilbert spaces and Carathéodory-type mappings.

4. Fixed points of multivalued contractions and applications

The Banach contraction principle is one of few fixed point theorems, where, besides the existence, some further information is included, namely how the unique fixed point can be successively approximated with arbitrary accuracy. In the case of a multivalued contraction we have the set of fixed points. So a natural question of its topological characterization arises. In this section we shall review most important results of this type. For more details we recommend: [20], [7], [37], [54], [61], [70], [71], [105], [144], [145].

For a metric space (X, d), by $C(X)$ we shall denote the family of all closed nonempty subsets of X For $A \in C(X)$ and $\varepsilon > 0$, we let

$$0_\varepsilon(A) = \{x \in X \mid \exists y \in A, d(x, y) < \varepsilon\}.$$

Let $A, B \in C(X)$. We define the Hausdorff distance $d_H(A, B)$ between A and B as follows:

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B) \text{ and } B \subset O_\varepsilon(A)\}.$$

It is well known that $d_H(A, B)$ can be equal to infinity. If we restrict our considerations to the family $BC(X)$ of all bounded closed and nonempty subsets of X, then d_H is a metric $BC(X)$, the so called Hausdorff metric.

Let E be a Banach space and A, B, C, $D \in BC(E)$. It is easy to see that:

$$\begin{array}{lll} d_H(A+B, C+D) \leq & d_H(A, C) + d_H(B, D), & (i) \\ d_H(\{x+A\}, \{y\}) = & d_H(\{x\}, \{y-A\}), & (ii) \\ d_H(tA, tB) & \leq d_H(A, B), \quad \text{for } t \in [0,1], & (iii) \end{array}$$

where $A + B = \{x + y \mid x \in A \text{ and } y \in B\}$ is the algebraic sum of A and B and $tA = \{tx \mid x \in A\}$.

Recall that a mapping $F : Y \to BC(X)$ is called Hausdorff-continuous if it is continuous w.r.t. the metric d in Y and d_H in $BC(X)$.

F is called *measurable* if, for every closed $U \subset X$, the set $F_+^{-1}(U)$ is measurable.

Proposition 4.1 ([69]). *A map $F : Y \to BC(X)$ is Hausdorff-continuous with compact values if and only if F is both u.s.c. and l.s.c.*

Note that, for $F : Y \to BC(X)$, the Hausdorff continuity implies only l.s.c. (see again [69]). It is easy to see that the following proposition is true.

Proposition 4.2. *If $F : Y \to BC(X)$ is l.s.c. with connected values and $F(Y) = \bigcup_{y\in Y} F(y) = X$, then X is connected, provided Y is connected.*

If what follows we need some additional topological notions. A metric space (X, d) is C^n (i.e. n-connected) if, for every $k \leq n$, every continuous map from the k-sphere S^k into X is null homotopic (i.e. homotopic to a constant map). Namely,

it means that every continuous map $f : S^k \to X$ has a continuous extension over the closed ball K^{n+1}, where S^n and K^{n+1} stand for the unit sphere and the unit closed ball in the Euclidean $(n + 1)$-space $\mathbb{R}^n$, respectively.

A space X is C^∞ (i.e. infinitely connected), if it is C^n, for every n. A collection $\mathcal{E} \subset 2^X$ is equi-LC^n if, for every $y \in \bigcup\{B \mid B \in \mathcal{E}\}$, every neighbourhood V of y in X contains a neighbourhood W of y in X such that, for all $B \in \mathcal{E}$ and $k \leq n$, every map from S^k into $W \cap B$ is null-homotopic over $V \cap E$ (i.e. a homotopy taking values in $V \cap E$). We shall also make use of the following (comp. [69]).

Theorem 4.1 (Michael's Selection Theorem). *Let X be a metric space and Y be a complete metric space. Let $F : X \to BC(Y)$ be a l.s.c. map such that the topological dimension* $\dim X \leq n + 1$ *and $F(x)$ is C^n and for all $x \in X$ with the collection $\{F(x) \mid x \in X\}$ equi-LC^n. Then F has a continuous selection.*

A mapping $F : X \to C(X)$ is called a *multivalued contraction* if there exists $\alpha < 1$ such that:

$$d_H(F(x), F(y)) < \alpha d(x, y), \quad \text{for every } x, y \in X.$$

In 1970, H. Covitz and S. B. Nadler proved:

Theorem 4.2 ([37]). *If X is a complete metric space and $F : X \to C(X)$ is a contraction, then* $\mathrm{Fix}(F) = \{x \in X \mid x \in F(x)\} \neq \emptyset$.

Let $F : X \to C(X)$ be a contraction. Obviously, the set $\mathrm{Fix}(F)$ is not a singleton, in general. For example, let $F(x) = A$, for every $x \in A$, be a constant map. Evidently, F is a contraction and $\mathrm{Fix}(F) = A$.

The following theorem is due to B. Ricceri ([145]).

Theorem 4.3. *Let E be a Banach space and let $F : E \to C(E)$ be a contraction such that $F(x)$ is convex, for every $x \in E$. Then* $\mathrm{Fix}(F)$ *is a retract of E.*

In 1991, A. Bressan, A. Cellina and A. Fryszkowski proved:

Theorem 4.4 ([20]). *If $E = L^1(T)$ is the space of integrable functions on a measure space T and $F : E \to BC(E)$ is a contraction with decomposable*[6] *values, then* $\mathrm{Fix}(F)$ *is a compact* AR*-space.*

[6] $A \subset L^1(T)$ is decomposable if, for every $\gamma, \mu \in A$ and a measurable subset $J \subset T$, we have:

$$(\gamma \cdot \chi_J + \mu\chi_{T\setminus J}) \in A,$$

where χ_S is the characteristic function of the subset $S \subset T$.

In view of [70] and [71], we would like to generalize both 4.4 and 4.5.

A simple argument shows that the following proposition [71], Proposition 1.1 is true.

Proposition 4.3. *Let X be a separable metric space and let X_0 be a nonempty closed subset of X. If $X \in \mathrm{AR}$ and, for any separable space Y and any nonempty closed set $Y_0 \subset Y$, every continuous function $f_0 : Y_0 \to X_0$ admits a continuous extension over Y, then $X_0 \in \mathrm{AR}$.*

Let (T, F, μ) be a finite, positive, nonatomic measure space and let $(E, \|\cdot\|)$ be a Banach space. We denote by $L^1(T, E)$ the Banach space of all (equivalent classes of) μ-measurable functions $u : T \to E$ such that the function $t \to \|u(t)\|$ is μ-integrable, equipped with the norm

$$\|u\|_{L^1(T,E)} = \int_T \|u(t)\| \, d\mu.$$

We always assume that the space $L^1(T, E)$ is separable. The multifunction $F : X \to C(X)$ is called *Lipschitzean* if there exists a real number $L \geq 0$ such that $d_H(F(x'), F(x'')) \leq Ld(x', x'')$, for all $x', x'' \in X$. If $L < 1$, we say that F is a multivalued contraction. It can be easily checked that any Lipschitzean multifunction is l.s.c. The following property of Lipschitz multifunctions will play an important role in proving the main result of this section.

Proposition 4.4. *Let (X, d) be a metric space and let $F : X \to C(X)$ be a Lipschitzean multifunction. Set, for every $x \in X$, $\varphi(x) = d(x, F(x))$. Then the function $\varphi : X \to [0, +\infty)$ is Lipschitzean.*

Proof. Let $L \leq 0$ be such that $d_H(F(x'), F(x'')) \leq Ld(x', x'')$, for all $x', x'' \in X$. Pick $x', x'' \in X$ and choose $\varepsilon > 0$. Owing to the definition of φ, there exists $z' \in F(x')$ fulfilling

$$-\varphi < -d(x', z') + \varepsilon.$$

Using the inequality $d(z', F(x'')) \leq Ld(x', x'')$, we can find $z'' \in F(x'')$ such that,

$$d(z', z'') < Ld(x', x'') + \varepsilon.$$

Therefore,

$$\begin{aligned} e(x'') - e(x') &< d(x'', F(x'')) - d(x', z') + \varepsilon \\ &\leq d(x'', z'') - d(x', z') + \varepsilon < (L+1)d(x', x'') + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, we actually have

$$\varphi(x'') - \varphi(x') \leq (L+1)d(x', x'')$$

and, interchanging x' with x'',

$$\varphi(x') - \varphi(x'') \leq (L+1)d(x', x'').$$

This completes the proof.

We now recall the notion of the Michael family of subsets of a metric space [71], Definition 1.4.

Definition 4.1. Let X be a metric space and let $M(X)$ be a family of a closed subsets of X, satisfying the following conditions:

(4.1.1) $X \in M(X)$, $\{x\} \in M(X)$, for all $x \in X$, and, if $\{A_i\}_{i\in I}$ is any sub-class of $M(X)$, then $\bigcap_{i\in I} A_i \in M(X)$;
(4.1.2) for every $k \in \mathbb{N}$ and every $x_1, x_2, \ldots, x_k \in X$, the set

$$A(x_1, x_2, \ldots, x_k) = \bigcap \{A \mid A \in M(X), x_1, x_2, \ldots, x_k \in A\}$$

is infinitely connected;
(4.1.3) to each $\varepsilon > 0$, there corresponds $\delta > 0$ such that, for any $A \in M(X)$, any $k \in \mathbb{N}$, and any $x_1, x_2, \ldots, x_k \in O_\delta(A)$, one has $A(x_1, x_2, \ldots, x_k) \subseteq O_\varepsilon(A)$;
(4.1.4) $\overline{A \cap B(x, r)} \in M(X)$, for all $A \in M(X)$, $x \in X$, and $r > 0$;

then we say that $M(X)$ is the *Michael family* of subsets of X.

This concept is closely related to the existence of continuous selections. Indeed, we have the following (see [69] or GMS):

Proposition 4.5. *Let X, Y be two metric spaces and let $F: X \to C(Y)$ be a l.s.c. multifunction. If Y is complete and there exists a Michael family $M(Y)$ of subsets of Y such that $F(x) \in M(Y)$, for each $x \in X$, then, for any nonempty closed set $X_0 \subseteq X$, every continuous selection f_0 from $F|_{X_0}$ admits a continuous extension f over X such that $f(x) \in F(x)$, for all $x \in X$.*

The proceeding result gains interest if we realize that significant classes of sets are the examples of the Michael families.

Example 4.1. Let X be a convex subset of a normed space and let $M(X)$ be the class of all sets $A \subseteq X$ such that $A = \emptyset$ or A is convex and closed in X. Then $M(X)$ is a Michael family of subsets of X.

Example 4.2 (comp. [70]). Let X be a metric space and let $M(X)$ be a simplicial convexity on X, whose elements are closed in X. Then $M(X)$ is a Michael family of subsets of X.

Definition 4.2. Let X be a metric space, let $F: X \to C(X)$ be l.s.c., and let $\mathcal{D}$ be a family of metric spaces. We say that F has the *selection property* w.r.t. $\mathcal{D}$ if, for any $Y \in \mathcal{D}$, any pair of continuous functions $f: Y \to X$ and $h: Y \to (0, +\infty)$ such that

$$G(y) = \overline{F((y)) \cap B(f(y), h(y))} \neq \emptyset, \quad y \in Y,$$

and any nonempty closed set $Y_0 \subseteq Y$, every continuous selection g_0 from $G|_{Y_0}$ admits a continuous extension g over Y fulfilling $g(y) \in G(y)$, for all $y \in Y$. If $\mathcal{D}$ is a family of all metric spaces, then we say that F has *selection property* (in symbols, $F \in SP(X)$).

Such notion has some meaningful features, as the remarks below point out.

Remark 4.1. Let X be a metric space and let $F : X \to C(X)$ be a l.s.c. m ultifunction. If X is complete and there exists a Michael family $M(X)$ of subsets of X such that $F(x) \in M(X)$, for all $x \in X$, then $F \in SP(X)$. This is an immediate consequence of Proposition 3.6.

Remark 4.2. Let X be a nonempty closed subset of $L^1(T,E)$ and $F : X \to C(X)$ be a l.s.c. multifunction with decomposable values. Then, arguing as in [71], it is possible to see that F has the selection property w.r.t. the family of all separable metric spaces.

We are now in a position to prove the main result of this section (see: [70] or [71]).

Theorem 4.5. *Let X be a complete absolute retract and let $F : X \to C(X)$ be a multivalued contraction. Suppose $F \in SP(X)$. Then the set* $\mathrm{Fix}(F)$ *is a complete absolute retract.*

Proof. Since $\mathrm{Fix}(F)$ is nonempty and closed in X, we only have to show that if Y is a metric space, $Y^\star$ is a nonempty closed subset of Y, and $f^\star : Y^\star \to \mathrm{Fix}(F)$ is a continuous function, then there exists a continuous extension $f : Y \to \mathrm{Fix}(F)$ of $f^\star$ over Y. Let d be the metric of Y, let $L \in (0,1)$ be such that $d_H(F(x'),F(x'')) \le Ld(x',x'')$, for all $x',x'' \in X$, and let $M \in (1,L^{-1})$. The assumption $X \in \mathrm{AR}$ yields a continuous function $f_0 : Y \to X$ fulfilling $f_0(y) = f^\star(y)$ in Y. We claim that there is a sequence $\{f_n\}$ of continuous functions from Y into X with the following properties:

(i) $f_n|Y^\star = f^\star$, for every $n \in \mathbb{N}$,
(ii) $f_n(y) \in F(f_{n-1}(y))$, for all $y \in Y$, $n \in \mathbb{N}$,
(iii) $d(f_n(y), f_{n-1}(y)) \le L^{n-1} d(f_1(y), f_0(y) + M^{1-n}$, for every $y \in Y$, $n \in \mathbb{N}$.

To see this, we proceed by induction on n. It follows from Proposition 3.4 that the function $h_0 : Y \to (0,+\infty)$, defined by

$$h_0(y) = d(f_0(y), F(f_0(y))) + 1, \quad y \in Y,$$

is continuous; moreover, one clearly has $F(f_0(y)) \cap B(f_0(y), h_0(y)) \ne \emptyset$, for all $y \in Y$. Having in mind that $F \in SP(X)$, we obtain a continuous function $f_1 : Y \to X$ satisfying $f_1(y) = f^\star(y)$ in $Y^\star$ and $f_1(y) \in F(f_0(y))$ in Y. Hence, conditions (i), (ii), and (iii) are true for f_1. Now, suppose that we have constructed p continuous functions $f_1, f_2, \dots, f_p$ from Y into X in such way that (i), (ii), and (iii) hold, whenever $n = 1, 2, \dots, p$. Since F is Lipschitzean with the constant L, (ii) and (iii) apply for $n = p$, and $LM < 1$, for every $y \in Y$, we achieve

$$\begin{aligned} d(f_p(y), F(f_p(y))) &\le d_H(F(f_{p-1}(y)), F(f_p(y))) \le Ld(f_{p-1}(y), f_p(y)) \\ &\le L^p d(f_1(y), f_0(y)) + LM^{1-p} \end{aligned}$$

$$\leq L^p d(f_1(y), f_0(y)) + M^{-p},$$

and subsequently

$$F(f_p(y)) \cap B(f_p(y), L^p d(f_1(y), f_0(y)) + M^{-p}) \neq \emptyset.$$

Because of the assumption $F \in SP(X)$, this produces a continuous function $f_{p+1} : Y \to X$ with the properties:

$$\begin{array}{ll} f_{p+1}|Y^\star \qquad\qquad = f^\star; \quad f_{p+1}(y) \in F(f_p(y)), & \text{for every } y \in Y; \\ d(f_{p+1}(y), f_p(y)) \leq L^p d(f_1(y), f_0(y)) + M^{-p}, & \text{for all } y \in Y. \end{array}$$

Thus, the existence of the sequence $\{f_n\}$ is established. We next define, for any $a > 0$, $Y_a = \{y \in Y \mid d(f_1(y), f_0(y)) < a\}$. Obviously, the family of sets $\{Y_a \mid a > 0\}$ is an open convering of Y. Moreover, due to (iii) and the completeness of X, the sequence $\{f_n\}$ converges uniformly on each Y_a. Let $f : Y \to X$ be the point-wise limit of $\{f_n\}$. It can be easily seen that the function f is continuous. Furthermore, owing to (i), one has $f|_{Y^\star} = f^\star$. Finally, the range of f is a subset of $\text{Fix}(F)$, because, by (ii), $f(y) \in F(f(y))$, for all $y \in Y$. This completes the proof.

The same arguments as in the proof of Theorem 4.5 actually lead to the following more general result.

Theorem 4.6. *Let $\mathcal{D}$ be a family of metric spaces, let X be a complete absolute retract, and let $F : X \to C(X)$ be a multivalued contraction having the selection property w.r.t. $\mathcal{D}$. Then, for any $Y \in \mathcal{D}$ and any nonempty closed set $Y_0 \subseteq Y$, every continuous function $f_0 : Y_0 \to \text{Fix}(F)$ admits a continuous extension over Y.*

Theorem 4.6 has a variety of special cases of a particular interest. As an example, Remark 4.10 combined with Theorem 4.6 lead to

Theorem 4.7. *Let X be a complete absolute retract and let $F : X \to C(X)$ be a multivalued contraction. If there exists a Michael family $M(X)$ of subsets of X such that $F(x) \in M(X)$, for all $x \in X$, then the set $\text{Fix}(F)$ is an absolute retract.*

Evidently, Theorem 4.6 generalizes earlier results formulated in (4.5) and (4.6). For details concerning 4.6 see: [70] and [71]. Now, we would like to study the topological dimension of the set $\text{Fix}(F)$ for some multivalued contractions. Note that the above mentioned problem was initiated by J. Saint Raymond [144]. At first, we recall the following result (see: [144] or [7]).

Proposition 4.6. *If $F : X \to BC(X)$ is a contraction with compact values, then $\text{Fix}(F)$ is compact.*

The following result due to Z. Dzedzej and B. Gelman ([58]) is a generalization of the result obtained by J. Saint Raymond ([144]).

Theorem 4.8. *Let E be a Banach space and $F : E \to BC(E)$ be a contraction with convex values and a constant $\alpha < 1/2$. Assume, furthermore, that the topological dimension $\dim F(x)$ of $F(x)$ is greater or equal to n, for some n and every $x \in E$. If $\mathrm{Fix}(F)$ is compact, then $\dim \mathrm{Fix}(F) > n$.*

Problem 1. Is it possible to prove 4.8, for $E = X$, to be a complete AR-space and $F : X \to CB(X)$ with values belonging to a Michael family $M(X)$?

Following D. Miklaszewski, we would like to discuss some generalizations of 4.8.

Theorem 4.9. *Let X be a retract of a Banach space E, and $F : X \to BC(X)$ be a compact continuous multivalued map with values being such elements of the Michael family $M(X)$ that $F(x) \setminus \{x\} \in C^{k-2}$, for every $x \in \mathrm{Fix}(F)$. Then the set $\mathrm{Fix}(F)$ has the dimension greater or equal to k.*

Proof. Suppose on the contrary that $\dim(\mathrm{Fix}(F)) < k$. Let us consider the maps $\psi : \mathrm{Fix}(F) \to BC(E)$ and $\varphi : \mathrm{Fix}(F) \to E \setminus \{0\}$ defined by the formulae: $\psi(x) = F(x) - x = \{y - x \mid y \in F(x)\}$ and $\varphi = \psi(x) \setminus \{0\} = (F(x) \setminus \{x\}) - x$. We are going to prove that the family $\{\varphi(x) \mid x \in \mathrm{Fix}(F)\}$ is equi-LC^{∞}. Let $y \in \varphi(x_0)$ and r be a positive number such that $0 \notin B_E(y, 3r)$. Suppose that the set $B_E(y, r) \cap \varphi(x)$ is non-empty, for a fixed point x of F. Then $B_E(y, r) \cap \varphi(x) = [(B_E(y + x, r) \cap F(x)) - x]$. Let $z \in B_E(y + x, r) \cap F(x)$. It is easy to show that $B_E(y + x, r) \cap F(x) \subset B_E(y + x, 3r) \cap F(x)$. But the second set of these three sets being in the Michael family $M(X)$ is C^{∞} as well as its translation, so the inclusion of $B_E(y, r) \cap \varphi(x)$ into the set $B_E(y, 3r) \cap \varphi(x)$ is homotopically trivial, and the family $\{\varphi(x) \mid x \in \mathrm{Fix}(F)\}$ is equi-LC^{∞}. It follows from Theorem 1.8 that φ has a selection f. Then the map $g : \mathrm{Fix}(F) \to X$ defined by the formula: $g(x) = f(x) + x$ is a selection of F. We conclude that, in view of Theorem 4.10, there exists a selection h of F being an extension of g. But h has a fixed point $x' \in \mathrm{Fix}(F)$, $h(x') = g(x') = f(x') + x' = x'$, $f(x') = 0 \in \varphi(x)$, which is a contradiction.

In the case when $\dim X < +\infty$, by analogous considerations as in the proof of 4.9 we obtain:

Theorem 4.10. *Let X be a retract of a Banach space E and $F : X \to BC(X)$ be a continuous (i.e. both l.s.c. and u.s.c.) map such that $\overline{F(X)} = \overline{\bigcup\{F(x) \mid x \in X\}}$ is a compact set. Assume that the values of F satisfy the following conditions:*

(i) *$F(x) \setminus \{x\}$ is C^{k-2}, for every $x \in \mathrm{Fix}(F)$,*
(ii) *$F(x)$ is C^k for every $x \in X$,*
(iii) *$\{F(x) \mid x \in \mathrm{Fix}(F)\}$ is equi-LC^{k-2} in E,*
(iv) *$\{F(x) \mid x \in X\}$ is equi-LC^k in X.*

Them $\dim(\mathrm{Fix}(F)) \geq k$.

The proof of 4.10 is quite analogous to that of 4.9. Finally, note that one can show an example of a continuous (i.e. both l.s.c. and u.s.c.) map with contractible values of the local dimension 2 such that (iii) and (iv) are satisfied, but the dimension of the set of fixed points equals 1.

It is evident that the above results can be applied directly to differential inclusions where the right hand side is a measurable-Lipschitz multivalued map $f : [0,a] \times \mathbb{R}^n \to CB(\mathbb{R})$. A very general application to the so called almost-periodicity problem for differential inclusions in Banach spaces is presented in Section 5 of [7].

Namely, we shall give a topological characterization of the set of solutions of some boundary value problems for differential inclusions of order k.

Let E be a separable Banach space and let $\phi : [0,a] \times E^k \multimap E$ be a multivalued mapping, where $E^k = E \times \ldots \times E$ (k-times).

We shall consider the following problem

$$\left.\begin{aligned} x^{(k)}(t) &\in \phi(t, x(t), x'(t), \ldots, x^{(k-1)}(t)) \\ x(0) &= x_0 \\ x'(0) &= x_1 \\ &\vdots \\ x^{(k-1)}(0) &= x_{k-1}, \end{aligned}\right\} \tag{4.1}$$

where the solution $x : [0,a] \to E$ is understood in the sense of t almost everywhere (a.e., $t \in [0,a]$) and $x_0, \ldots, x_{k-1} \in E$.

Observe that for $k = 1$ problem (4.1) reduces to the well-known Cauchy problem for differential inclusions. In what follows we shall denote by $\mathcal{S}(\phi, x_0, \ldots, x_{k-1})$ the set of all solutions of (4.1).

Our first application of Theorem 4.6 is the following:

Theorem 4.11. *Assume that φ is a mapping with compact values. Assume further that the following conditions hold:*

(4.11.1) *φ is bounded, i.e. there is an $M > 0$ such that $\|y\| \le M$ for every $t \in [0,a]$, $x \in E^k$ and $y \in \varphi(t,x)$,*
(4.11.2) *the map $\varphi(\cdot\,, x)$ is measurable for each $x \in E^k$,*
(4.11.3) *φ is a Lipschitz map with respect to the second variable, i.e. there exists an $L > 0$ such that for every $t \in [0,a]$ and for every $z = (z_1, \ldots, z_k)$, $y = (y_1, \ldots, y_k) \in E^k$ we have:*

$$d_H(\varphi(t,z), \varphi(t,y)) \le L \sum_{i=1}^{k} \|z_i - y_i\|.$$

Then the set $\mathcal{S}(\varphi, x_0, \ldots, x_{k-1})$ of all solutions of the problem (4.1) is an AR-*space.*

Sketch of proof. For the proof we define (single-valued) mappings[7]:
$h_j : M([0,a],E) \to AC^j$, $j = 0,\dots,k-1$, by putting

$$(h_j(z))(t) = x_0 + tx_1 + \dots + (t^j/j!)x_j + \int_0^t \int_0^{s_1} \dots \int_0^{s_j} z(s)\,ds\,ds_j \dots ds_1,$$

where $AC^j = \{u \in C^j([0,a],E) : u^{(j)}$ is absolutely continuous$\}$ and for $u \in AC^j$ we put:

$$\|u\| = \|u\|_{C^j} + \sup\operatorname{ess}_{t\in[0,a]}\{\|u^{(j+1)}(t)\|\}.$$

Now consider a multivalued mapping $\psi : M([0,a],E) \to M([0,a],E)$ defined as follows:

$$\psi(x) = \{z \in M([0,a],E) \mid z(t) \in \varphi(t, h_{k-1}(x)(t),\dots,h_0(x)(t)),\ \text{for a.e. } t \in [0,a]\}.$$

It follows from the Kuratowski–Ryll–Nardzewski Selection Theorem and (4.11.1) that ψ is well defined (with closed decomposable values in $M([0,a],E)$. Moreover, it is easy to see that $h_{k-1}(\mathrm{Fix}(\psi)) = \mathcal{S}(\varphi, x_0,\dots,x_{k-1})$. Consequently, since h_{k-1} is a homeomorphism onto its image, in view of Theorem 4.6, it is sufficient to show that ψ is a contractive mapping. We shall do this by using the $M([0,a],E)$-version of Bielecki's method and the Kuratowski–Ryll–Nardzewski Theorem. In fact it is enough to see that for every $u, z \in M([0,a],E)$ and for every $y \in \psi(u)$ there is a $v \in \psi(z)$ such that

$$(\star) \qquad \|y - v\|_1 \le \alpha\|u - z\|_1,$$

where $\alpha \in [0,1)$ and $\|w\|_1 = \sup\operatorname{ess}_{t\in[0,a]}\{e^{-Lakt}\|w(t)\|\}$ is the Bielecki norm in $M([0,a],E)$. Observe that using Theorem 4.2 (in [66]) for ψ and z, we get a mapping $v \in \psi(z)$ and now ($\star$) follows directly from 4.11.3. The proof of Theorem 4.11 is complete.

Remark 4.3. Note that if we impose more assumptions on φ then we are able to get better information on $= \mathcal{S}(\varphi, x_0,\dots,x_{k-1})$ (for details see [7], [66], [69]).

Now following [12], [61], [62] we would like to add that if we consider problem (4.1) for $k = 1$ and in Theorem 4.11 we assume moreover that $\varphi(t,x)$ is convex and $\dim\varphi(t,x) \ge n$ for some n and every $(t,x) \in [0,a] \times E$, then, in view of Theorem 4.9 we get that:

$$\dim \mathcal{S}(\varphi, x_0) \ge n.$$

Finally, let us remark that if we reject the assumption that φ has compact values, then still a characterization of $\mathcal{S}(\varphi, x_0)$ is possible (see Theorem 3.1 in [105]).

[7] here $M([0,a],E)$ is the Banach space of continuous essentialy bounded mappings.

5. The inverse limit method

The inverse limit method in differential equations and inclusions is quite new and it was indicated in 1999 by J. Andres, G. Gabor and L. Górniewicz (see: [5], [6] and [60]).

We shall start from the topological preparation. By an inverse system of topological spaces we mean a family $\mathcal{IS} = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$, where Σ is a set directed by the relation $\leq$, X_α is a topological (Hausdorff) space for every $\alpha \in \Sigma$ and $\pi_\alpha^\beta : X_\alpha \to X_\beta$ is a continuous mapping for every two elements $\alpha, \beta \in \Sigma$ such that $\alpha \leq \beta$. Moreover, for each $\alpha \leq \beta \leq \gamma$ the following conditions should hold: $\pi_\alpha^\alpha = \mathrm{id}_{X_\alpha}$ and $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$.

A subspace of the product $\Pi_{\alpha\in\Sigma} X_\alpha$ is called a *limit of the inverse system* $\mathcal{IS}$ and it is denoted by $\lim_{\leftarrow} \mathcal{IS}$ or $\lim_{\leftarrow}\{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ if

$$\lim_{\leftarrow} \mathcal{IS} = \{(x_\alpha) \in \Pi_{\alpha\in\Sigma} X_\alpha \mid \pi_\alpha^\beta(x_\beta) = x_\alpha \text{ for all } \alpha \leq \beta\}.$$

An element of $\lim_{\leftarrow} \mathcal{IS}$ is called a *thread* or a *fibre* of the system $\mathcal{IS}$. One can see that if we denote by $\pi_\alpha : \lim_{\leftarrow} \mathcal{IS} \to X_\alpha$ a restriction of the projection $p_\alpha : \Pi_{\alpha\in\Sigma} X_\alpha \to X_\alpha$ onto the α-th axis, then we obtain $\pi_\alpha = \pi_\alpha^\beta \pi_\beta$ for each $\alpha \leq \beta$.

Now we summarize some useful properties of limits of inverse systems which are well known (comp. [60]):

Proposition 5.1. *Let $\mathcal{IS} = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ be an inverse system.*

(5.1.1) *The limit* $\lim_{\leftarrow} \mathcal{IS}$ *is a closed subset of* $\Pi_{\alpha\in\Sigma} X_\alpha$.
(5.1.2) *If, for every $\alpha \in \Sigma$, X_α is*
- (i) *compact, then* $\lim_{\leftarrow} \mathcal{IS}$ *is compact;*
- (ii) *compact and nonempty, then* $\lim_{\leftarrow} \mathcal{IS}$ *is compact and nonempty;*
- (iii) *a continuum, then* $\lim_{\leftarrow} \mathcal{IS}$ *is a continuum;*
- (iv) *acyclic, and* $\lim_{\leftarrow} \mathcal{IS}$ *is nonempty,* $\lim_{\leftarrow} \mathcal{IS}$ *is acyclic;*
- (v) *metrizable, Σ is countable, and* $\lim_{\leftarrow} \mathcal{IS}$ *is nonempty, then* $\lim_{\leftarrow} \mathcal{IS}$ *is metrizable.*

The following further information is useful for applications.

Proposition 5.2 ([60]). *Let $\mathcal{IS} = \{X_n, \pi_n^p, \mathbb{N}\}$ be an inverse system. If each X_n is an R_δ-set, then so is* $\lim_{\leftarrow} \mathcal{S}$.

The following example shows that a limit of an inverse system of compact absolute retracts does not have to be an absolute retract.

Example 5.1. Consider a family $\{X_n\}_{n=1}^\infty$ of subsets of $\mathbb{R}^2$ defined as follows:

$$X_n = \left(\left[0, \frac{1}{n\pi}\right] \times [-1,1]\right) \cup \left\{(x,y) \,\middle|\, y = \sin\frac{1}{x} \text{ and } \frac{1}{n\pi} < x \leq 1\right\}.$$

One can see that for each $m, n \geq 1$ such that $m \geq n$ we have $X_m \subset X_n$.

Define the maps $\pi_n^m : X_m \to X_n$, $\pi_n^m(x) = x$. Therefore $\mathcal{IS} = \{X_n, \pi_n^m, \mathbb{N}\}$ is an inverse system of compact absolute retracts. It is evident that $\lim_{\leftarrow} \mathcal{IS}$ is homeomorphic to the intersection of all X_n. On the other hand

$$X = \bigcap_{n=1}^{\infty} X_n = \{(0,y) \mid y \in [-1,1]\} \cup \left\{(x,y) \,\middle|\, y = \sin\frac{1}{x} \text{ and } 0 < x \leq 1\right\}$$

and X is not an absolute retract since, for instance, X is not locally connected.

Note that in [60] the following information on a limit of an inverse system of absolute retracts has been formulated.

Proposition 5.3. *Let* $\mathcal{IS} = \{X_n, \pi_n^p, \mathbb{N}\}$ *be an inverse system of compact absolute retracts such that* $X_n \subset X_p$ *and* π_n^p *is a retraction for all* $n \leq p$. *Then* $\lim_{\leftarrow} \mathcal{IS}$ *has the fixed point property, i.e. every continuous map* $f : \lim_{\leftarrow} \mathcal{IS} \to \lim_{\leftarrow} \mathcal{IS}$ *has a fixed point.*

Example 5.2. Consider the inverse system $\mathcal{S} = \{X_n, \pi_n^p, \mathbb{N}\}$ such that $X_n = [n, \infty)$ and $\pi_n^p : X_p \hookrightarrow X_n$ are inclusion maps for $n \leq p$. It is obvious that $\lim_{\leftarrow} \mathcal{S}$ is homeomorphic to the intersection of all X_n which is an empty set.

Let us give important examples of inverse systems.

Example 5.3. Let, for every $m \in \mathbb{N}$, $C_m = C([0,m], \mathbb{R}^n)$ be a Banach space of all continuous functions of the closed interval $[0,m]$ into $\mathbb{R}$, and $C = C([0,\infty), \mathbb{R}^n)$ be an analogous Fréchet space of continuous functions.

Consider the maps $\pi_m^p : C_p \to C_m$, $\pi_m^p(x) = x|_{[0,m]}$. It is easy to see that C is isometrically homeomorphic to a limit of the inverse system $\{C_m, \pi_m^p, \mathbb{N}\}$. The maps $\pi_m : C \to C_m$, $\pi_m(x) = x|_{[0,m]}$ correspond to suitable projections.

Remark 5.1. In the same manner as above we can show that Fréchet spaces $C(J, \mathbb{R}^n)$, where J is an arbitrary interval, $L^1_{loc}(J, \mathbb{R}^n)$ of all locally integrable functions, $AC_{loc}(J, \mathbb{R}^n)$ of all locally absolutely continuous functions and $C^k(J, \mathbb{R}^n)$ of all continuously differentiable functions up to the order k can be considered as limits of suitable inverse systems.

More generally, every Fréchet space is a limit of some inverse system of Banach spaces.

Now we introduce the notion of multivalued maps of inverse systems. Suppose that two systems $\mathcal{IS} = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ and $\mathcal{IS}' = \{Y_{\alpha'}, \pi_{\alpha'}^{\beta'}, \Sigma'\}$ are given.

Definition 5.1. By a *multivalued map of the system* $\mathcal{IS}$ *into the system* $\mathcal{IS}'$ we mean a family $\{\sigma, \varphi_{\sigma(\alpha')}\}$ consisting of a monotone function $\sigma : \Sigma' \to \Sigma$, that is $\sigma(\alpha') \leq \sigma(\beta')$, and of multivalued maps $\varphi_{\sigma(\alpha')} : X_{\sigma(\alpha')} \multimap Y_{\alpha'}$ with nonempty values, defined for every $\alpha' \in \Sigma'$ and such that

$$\pi_{\alpha'}^{\beta'} \varphi_{\sigma(\beta')} = \varphi_{\sigma(\alpha')} \pi_{\sigma(\alpha')}^{\sigma(\beta')}, \tag{5.1}$$

for each $\alpha' \leq \beta'$.

A map of systems $\{\sigma, \varphi_{\sigma(\alpha')}\}$ induces a *limit map* $\varphi : \lim_{\leftarrow} \mathcal{IS} \multimap \lim_{\leftarrow} \mathcal{IS}'$ defined as follows:

$$\varphi(x) = \Pi_{\alpha' \in \Sigma} \varphi_{\sigma(\alpha')}(x_{\sigma(\alpha')}) \cap \lim_{\leftarrow} \mathcal{IS}.$$

In other words, a limit map is a map such that

$$\pi_{\alpha'}\varphi = \varphi_{\sigma(\alpha')}\pi_{\sigma(\alpha')} \tag{5.2}$$

for every $\alpha' \in \Sigma'$.

Since a topology of a limit of an inverse system is the one generated by the base consisting of all sets of the form $\pi_\alpha(U_\alpha)$, where α runs over an arbitrary set cofinal in Σ and U_α are open subsets of the space X_α, it is easy to prove the following continuity property for limit maps:

Proposition 5.4 (see [5], Proposition 2.7). *Let $\mathcal{IS} = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ and $\mathcal{IS}' = \{Y_{\alpha'}, \pi_{\alpha'}^{\beta'}, \Sigma'\}$ be two inverse systems, and $\varphi : \lim_{\leftarrow} \mathcal{IS} \multimap \lim_{\leftarrow} \mathcal{IS}'$ be a limit map induced by the map $\{\sigma, \varphi_{\sigma(\alpha')}\}$.*

If, for every $\alpha' \in \Sigma$, $\varphi_{\sigma(\alpha')}$ is

(i) *u.s.c., then φ is u.s.c.;*
(ii) *l.s.c., then φ is l.s.c.;*
(iii) *continuous, then φ is continuous* (*continuous means both u.s.c. and l.s.c.*).

The following crucial result allows us to study a topological structure of fixed point sets of limit maps.

Theorem 5.1 ([60]). *Let $\mathcal{IS} = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ be an inverse system, and $\varphi : \lim_{\leftarrow} \mathcal{IS} \multimap \lim_{\leftarrow} \mathcal{IS}$ be a limit map induced by a map $\{\mathrm{id}, \varphi_\alpha\}$, where $\varphi_\alpha : X_\alpha \multimap X_\alpha$. If fixed point sets of φ_α are acyclic, and the fixed point set of φ is nonempty, then it is acyclic, too.*

Theorem 5.2. *Let $\mathcal{IS} = \{X_n, \pi_n^p, \mathbb{N}\}$ be an inverse system, and $\varphi : \lim_{\leftarrow} \mathcal{IS} \multimap \lim_{\leftarrow} \mathcal{IS}$ be a limit map induced by a map $\{\mathrm{id}, \varphi_n\}$, where $\varphi_n : X_n \multimap X_n$. If fixed point sets φ_n are compact R_δ, then the fixed point set of φ is R_δ, too.*

Corollary 5.1. *Let $\mathcal{IS} = \{X_n, \pi_n^p, \mathbb{N}\}$ be an inverse system, and $\varphi : \lim_{\leftarrow} \mathcal{IS} \multimap \lim_{\leftarrow} \mathcal{IS}$ be a limit map induced by a map $\{\mathrm{id}, \varphi_n\}$, where $\varphi_n : X_n \multimap X_n$. If all X_n are Fréchet spaces and all φ_n are contractions.*

Remark 5.2. Note that, following [70], we can prove Corollary 5.1 for a little larger class of multivalued maps (see [5], Corollary 2.9).

The inverse system approach described above gives us an easy way to study a topological structure of solution sets of differential problems on noncompact intervals. To illustrate it consider the following example:

Example 5.4. Let $F : [0,\infty) \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a Carathéodory map, i.e.

(i) values of F are nonempty, compact and convex for all $(t,x) \in [0,\infty) \times \mathbb{R}^n$,
(ii) $F(t,\,\cdot\,)$ is u.s.c. for a.a. $t \in [0,\infty)$,
(iii) $F(\,\cdot\,,x)$ is measurable for all $x \in \mathbb{R}^n$,

with at most linear growth, i.e. there exists a locally integrable function $\alpha : [0,\infty) \to [0,\infty)$ such that, for every $x \in \mathbb{R}^n$ and for a.a. $t \in [0,\infty)$,

$$|F(t,x)| \leq \alpha(t)(1 + \|x\|),$$

where $|F(t,x)| = \sup\{|y| \mid y \in F(t,x)\}$.

Consider the Cauchy problem

$$\left.\begin{aligned} &\dot{x}(t) \in F(t,x(t)) \quad \text{for a.a. } t \in [0,\infty), \\ &x(0) = x_0. \end{aligned}\right\} \tag{5.3}$$

We shall show, using the inverse systems technique, that the set of solutions $\mathcal{S}$ of problem (5.3) is R_δ. To do it, consider the family of Cauchy problems

$$\left.\begin{aligned} &\dot{x}(t) \in F(t,x(t)) \quad \text{for a.a. } t \in [0,m], \\ &x(0) = x_0, \end{aligned}\right\} \tag{5.4}$$

where $m \geq 1$. It is well known (see [46]) that, for every $m \geq 1$, the set $\mathcal{S}_m$ of the above problem is compact R_δ.

On the other hand, $\mathcal{S}_m$ is a fixed point set of the following map $\Psi_m : C_m = C([0,m],\mathbb{R}^n) \multimap C_m$,

$$\Psi_m(x) = \left\{ x_0 + \int_0^t u(s)\,ds \;\middle|\; u \in L^1([0,m],\mathbb{R}^n) \text{ and } u(t) \in G(t,x(t)) \text{ for a.a. } t \in [0,m] \right\}.$$

One can check that $\{\Psi_m\}$ is a map of the inverse system $\{C([0,m],\mathbb{R}^n), \pi^p_m, \mathbb{N}\}$, where $\pi^p_m(x) = x|_{[0,m]}$ for every $x \in C([0,p],\mathbb{R}^n)$. Moreover, it induces the limit map on $C([0,\infty),\mathbb{R}^n)$

$$\Psi(x) = \left\{ x_0 + \int_0^t u(s)\,ds \;\middle|\; u \in L^1_{loc}([0,\infty),\mathbb{R}^n) \text{ and } u(t) \in G(t,x(t)) \text{ for a.a. } t \in [0,\infty) \right\}$$

with the fixed point set $\mathcal{S}$. By Theorem 5.2 it follows that $\mathcal{S}$ is compact R_δ, as required.

Note that the above result on a topological structure of the solution set of the Cauchy problem on a halfline can be obtained by using different techniques (see e.g. [4] for the proof by using the Scorza–Dragoni type result).

Further applications of the inverse systems approach can be found in [5] and [60].

6. Concluding remarks and comments

Above we have presented different techniques of characterization of the set of fixed points and consequently solution sets for differential equations and inclusions. Now, we would like to show some consequences, which can be obtained by using the topological structure of solution sets.

We would like to show only results connected with multivalued Poincaré operator indicated by G. Dylawerski and L. Górniewicz in 1983 ([57]). We recommend also the following papers: [3], [8], [66], [69], [72], [91], [143], [45].

We shall formulate the simplest version. Most general results of this type are contained in [45].

Let $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and bounded map. We shall consider both the Cauchy problem:

$$\left.\begin{aligned} x'(t) &= f(t, x(t)), \\ x(t_0) &= x_0, \end{aligned}\right\} \tag{6.1}$$

and the following periodic problem:

$$\left.\begin{aligned} x'(t) &= f(t, x(t)), \\ x(0) &= x(a). \end{aligned}\right\} \tag{6.2}$$

We shall associate with (6.2) the multivalued Poincaré operator:

$$P_a : \mathbb{R}^n \multimap \mathbb{R}^n$$

defined as a composition of the following two maps:

$$\mathbb{R} \overset{P}{\mapsto} C([0, a], \mathbb{R}^n) \overset{e_a}{\to} \mathbb{R}^n,$$

where $P(x) = S(f; 0, x)$ and $e_a(x) = x(a)$. It follows from the Aronszajn Theorem that P has R_δ-values. On the other hand it is well known that P is u.s.c. (comp. [69] or [11]). Hence $P_a = e_a \circ P$ as a composition of u.s.c. R_δ-valued map P with a continuous map e_a is admissible in the sense of [67]. Therefore the topological degree of the field $(\mathrm{id}_{\mathbb{R}^n} - P)$ on any ball $B(0, r) \subset \mathbb{R}^n$ such that[8] $\mathrm{Fix}(P) \cap \partial B(0, r) = \emptyset$ is well defined (see: [67], [45] or [92]). In what follows P is called the Poincaré operator associated with (6.2).

The following proposition is straightforward.

Proposition 6.1. *If* $\mathrm{Fix}(P_a) \neq \emptyset$*, then problem* (6.2) *has a solution.*

In the terms of topological degree theory Proposition 6.3 can be expressed as follows:

Proposition 6.2. *Assume that for some* $r > 0$ *we have* $\mathrm{Fix}(P) \cap \partial B(0, r) = \emptyset$*. If the topological degree* $\deg(\mathrm{id}_{\mathbb{R}^n} - P_a, B(0, r))$ *of* $\mathrm{id}_{\mathbb{R}^n} - P_a$ *with respect to* $B(0, r)$ *is different from zero, then problem* (6.2) *has a solution.*

[8] $\partial B(0, r)$ denotes the boundary of $B(0, r)$ in $\mathbb{R}^n$.

In order to calculate the topological degree of Poincaré field $\mathrm{id}_{\mathbb{R}^n} - P_a$ we shall use the guiding function (or potential function) method (see: [91] or [45]).

A C^1-map $V : \mathbb{R}^n \to \mathbb{R}$ is called a guiding function (potential) for f provided that there exists $r_0 > 0$ such that:

$$\langle \operatorname{grad} V(x), f(t,x)\rangle > 0 \tag{6.3}$$

for every $t \in [0,a]$ and $x \in \mathbb{R}^n$ such that $\|x\| \geq r_0$, where $\operatorname{grad} V(x) = \Big(\frac{\partial f}{\partial x_1}(x), \dots , \frac{\partial f}{\partial x_n}(x)\Big)$ is the gradient of V at the point x and $\langle\ ,\ \rangle$ stands for the inner product in $\mathbb{R}^n$.

It follows from (6.3) that for every $r \geq r_0$ and $x \in \mathbb{R}^n$ such that $\|x\| \geq r_0$ we have $\operatorname{grad} V(x) \neq 0$ so from the localization property of the topological degree it follows that for every $r \geq r_0$ we have $\deg(\operatorname{grad} V, B(0,r)) = \deg(\operatorname{grad} V(x), B(0, r_0))$. We let:

$$\operatorname{Ind}(V) = \deg(\operatorname{grad} V(x), B(0,r)). \tag{6.4}$$

Finally we obtain:

Theorem 6.1. *If f has a potential V such that* $\operatorname{Ind}(V) \neq 0$*, then* $\deg(P_a, B(0, r)) \neq 0$ *for some $r \geq r_0$.*

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GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS AND DISCRETE SYSTEMS

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Abstract. This note is an attempt to show the possibility to deal with discrete equations in the frame of generalized ordinary differential equations defined by Jaroslav Kurzweil in 1957.
Generalized ordinary differential equations form a tool which makes it possible to use a unified approach to classical ordinary differential equations as well as discrete systems.

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1. Integration

The concept of a generalized ordinary differential equation is based on a special integration process wich is interesting by itself and plays a nice and important role in integration theory and in real analysis in general.

Assume that a bounded interval $[a, b] \subset \mathbb{R}$ is given, $-\infty < a < b < \infty$.

A finite set of points

$$D := a = \alpha_0 \le \tau_1 \le \alpha_1 \le \cdots \le \alpha_{k-1} \le \tau_k \le \alpha_k = b$$

with $\alpha_0 < \alpha_1 < \cdots < \alpha_k$ is called a *partition* of the interval $[a, b]$.

A positive function $\delta : [a, b] \to (0, \infty)$ will be called a *gauge* on the interval $[a, b]$.

The partition D is called δ- *fine* (with respect to the gauge δ) if

$$[\alpha_{j-1}, \alpha_j] \subset [\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)].$$

Assume that a function $U(\tau,t):[a,b]\times[a,b]\to\mathbb{R}^n$ is given.

For a partition D we denote by

$$S(U,D)=\sum_{j=1}^{K}(U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1}))$$

the integral sum corresponding to the function U and the partition D. The fundamental definition reads as follows.

Definition 1. The function $U:[a,b]\times[a,b]\to\mathbb{R}^n$ is called *Kurzweil integrable* over $[a,b]$ (shortly $U\in K([a,b])$) if there is a $J\in\mathbb{R}^n$ such that for every $\varepsilon>0$ there is a gauge δ on $[a,b]$ and

$$\|S(U,D)-J\|<\varepsilon$$

if D is a δ-fine partition of $[a,b]$.

We use the formal notation $J=\int_a^b DU(\tau,t)$ for the *generalized Kurzweil integral* of U over $[a,b]$.

Remark 1. Typical situations are for example $U(\tau,t)=f(\tau)\cdot t$ or $U(\tau,t)=f(\tau)\cdot g(t)$ where $f,g:[a,b]\to\mathbb{R}$ or $f:[a,b]\to\mathbb{R}^n$, $g:[a,b]\to\mathbb{R}$ or $f:[a,b]\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}^n$.

Looking for example at the sum $S(U,D)$ if $U(\tau,t)=f(\tau)\cdot t$, we can see easily that

$$S(U,D)=\sum_{j=1}^{K}f(\tau_j)(\alpha_j-\alpha_{j-1})$$

is the usual Riemann integral sum corresponding to the function $f:[a,b]\to\mathbb{R}$.

Reading the definition of the integral $\int_a^b DU(\tau,t)=\int_a^b f(s)d(s)$ we can see that it differs from the classical Darboux type definition of the Riemann integral in only one point, namely our δ is a gauge, i.e. a function which need not be a constant, instead of the positive constant gauge used for defining the Riemann integral.

Nevertheless, this slight change in the definition has dramatic consequences for the concept of the integral and integrability of functions.

It is well known that a function $f:[a,b]\to\mathbb{R}$ is integrable in the sense of our definition for $U(\tau,t)=f(\tau)\cdot t$ if and only if it is integrable in the sense of Perron (the narrow Denjoy integral) and this is a nonabsolutely convergent integral including the Lebesgue integral.

The definition of the integral is based strongly on the following statement which goes back to a paper of P. Cousin from 1895.

Lemma 1. *If δ is an arbitrary gauge on $[a, b]$, then there is a partition D of $[a, b]$ which is δ-fine.*

(See e.g. [2], [3].)

The generalized Kurzweil integral given by Definition 1 has all the good properties usual in reasonable integration theories. Among others we have the following

Theorem 1. *If $U, V \in K([a, b])$ and $c_1, c_2 \in \mathbb{R}$, then $c_1 U + c_2 V \in K([a, b])$ and*

$$\int_a^b D[c_1 U(\tau, t) + c_2 V(\tau, t)] = c_1 \int_a^b DU(\tau, t) + c_2 \int_a^b DV(\tau, t).$$

If $U \in K([a, b])$, then $U \in K([c, d])$ for every $[c, d] \subset [a, b]$.

If $c \in [a, b]$ and $U \in K([a, c])$ and $U \in K([c, b])$, then $U \in K([a, b])$ and

$$\int_a^b DU(\tau, t) = \int_a^c DU(\tau, t) + \int_c^b DU(\tau, t).$$

(See Theorems 1.9, 1.10 and 1.11 in [3].)

Also a less usual result holds for the integral.

Theorem 2. *If $U \in K([a, c])$ for every $c \in [a, b)$ and*

$$\lim_{c \to b-} \left(\int_a^c DU(\tau, t) - [U(b, c) - U(b, b)] \right) = J \in \mathbb{R}, \tag{1}$$

then $U \in K([a, b])$ and

$$\int_a^b DU(\tau, t) = J.$$

If $U \in K([c, b])$ for every $c \in (a, b]$ and

$$\lim_{c \to a+} \left(\int_c^b DU(\tau, t) + U(a, c) - U(a, a) \right) = J \in \mathbb{R}, \tag{2}$$

then $U \in K([a, b])$ and

$$\int_a^b DU(\tau, t) = J.$$

(See Theorem 1.14 in [3].)

Remark 2. The property of the integral presented in the previous Theorem 2 is sometimes called Hake's Theorem and it is essential when considering generalized ordinary differential equations.

Let us mention that e.g. in the special case of $U(\tau,t) = f(\tau)\cdot t$ the relation (1) represents the existence of the limit

$$\lim_{c\to b-}\int_a^c f(s)ds = J \in \mathbb{R}$$

and by Theorem 2 we have the existence $\int_a^b f(s)ds$ as well as the equality

$$\lim_{c\to b-}\int_a^c f(s)ds = \int_a^b f(s)ds.$$

This property is not possessed by the Riemann or Lebesgue integral. This is a typical property of the Denjoy-Perron integral.

2. Generalized ordinary differential equations

Let us have a function $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and assume that $[\alpha,\beta] \subset \mathbb{R}$ is a compact interval.

A function $x : [\alpha,\beta] \to \mathbb{R}^n$ is called a *solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau} = DF(x,t) \tag{3}$$

on the interval $[\alpha,\beta]$ if

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau),t)$$

for every $s_1, s_2 \in [\alpha,\beta]$. (The integral on the right hand side of this relation is the integral presented in Definition 1 in the previous section.)

It can be shown easily that $x : [\alpha,\beta] \to \mathbb{R}^n$ is a solution of (3) if and only if

$$x(s) = x(\gamma) + \int_\gamma^s DF(x(\tau),t)$$

for every $s \in [\alpha,\beta]$ where $\gamma \in [\alpha,\beta]$ is fixed.

Theorem 2 yields the following

Proposition 1. *If $x : [\alpha,\beta] \to \mathbb{R}^n$ is a solution of (3) then*

$$\lim_{s\to\sigma}[x(s) - [F(x(\sigma),s) - F(x(\sigma),\sigma)]] = x(\sigma)$$

for every $\sigma \in [\alpha,\beta]$.

Moreover, if the limit

$$\lim_{s\to\sigma+}[F(x(\sigma),s)-F(x(\sigma),\sigma)]=J^{+}(\sigma)\in\mathbb{R}^{n}$$

or

$$\lim_{s\to\sigma-}[F(x(\sigma),s)-F(x(\sigma),\sigma)]=J^{-}(\sigma)\in\mathbb{R}^{n}$$

exists, then

$$\lim_{s\to\sigma+}x(s)=x(\sigma+)=x(\sigma)+J^{+}(\sigma)$$

or

$$\lim_{s\to\sigma-}x(s)=x(\sigma-)=x(\sigma)+J^{-}(\sigma),$$

respectively.

This proposition shows that in the solution of the generalized ordinary differential equation (3) discontinuities can occur if $J^{+}(\sigma)$ or $J^{-}(\sigma)$ is different from zero. Consequently, a solution of (3) can be a discontinuous function in general.

Details on these concepts and properties of a solution of a generalized ordinary differential equation (3) can be found in [3].

Let us now turn our attention to a class of functions $F:\mathbb{R}^{n}\times\mathbb{R}\to\mathbb{R}^{n}$ which leads to a reasonable theory for equations of the form (3).

Assume that $h:\mathbb{R}\to\mathbb{R}$ is a nondecreasing function and that $\omega:[0,\infty)\to\mathbb{R}$ is continuous, increasing with $\omega(0)=0$ (a modulus of continuity).

Let us define the class $\mathcal{F}(h,\omega)$ of functions $F:\mathbb{R}^{n}\times\mathbb{R}\to\mathbb{R}^{n}$ satisfying

$$\|F(x,t_2)-F(x,t_1)\|\le|h(t_2)-h(t_1)| \tag{4}$$

and

$$\|F(x,t_2)-F(x,t_1)-[F(y,t_2)-F(y,t_1)]\|\le\omega(\|x-y\|)\cdot|h(t_2)-h(t_1)| \tag{5}$$

for $x,y\in\mathbb{R}^{n}$, $t_1,t_2\in\mathbb{R}$. (See 3.8 Definition in [3].)

The main statement concerning the class $\mathcal{F}(h,\omega)$ is a local existence result for a solution of (3) which has to satisfy a given initial condition.

Theorem 3. *If $\widetilde{x}\in\mathbb{R}^{n}$, $t_0\in\mathbb{R}$ and $F\in\mathcal{F}(h,\omega)$, then there exist $\Delta^{-},\Delta^{+}>0$ such that on $[t_0-\Delta^{-},t_0+\Delta^{+}]$ there exists a solution $x:[t_0-\Delta^{-},t_0+\Delta^{+}]\to\mathbb{R}^{n}$ of the generalized ordinary differential equation (3) for which $x(t_0)=\widetilde{x}$.*

(See 4.2 Theorem in [3].)

3.10 Lemma in [3] states the following:

If $F\in\mathcal{F}(h,\omega)$ and $x:[\alpha,\beta]\to\mathbb{R}^{n}$ is a solution of (3), then for every $s_1,s_2\in[\alpha,\beta]$ we have

$$\|x(s_2)-x(s_1)\|\le|h(s_2)-h(s_1)|. \tag{6}$$

This implies immediately that if $F \in \mathcal{F}(h,\omega)$ and $x : [\alpha,\beta] \to \mathbb{R}^n$ is a solution of (3) on $[\alpha,\beta]$ then $x \in BV([\alpha,\beta])$ (x is a function of bounded variation on $[\alpha,\beta]$) and

$$\mathrm{var}_\alpha^\beta x \le h(\beta) - h(\alpha) < \infty$$

if $-\infty < \alpha < \beta < \infty$.

Moreover, if h is continuous from the left (i.e. $\lim_{s\to t-} h(s) = h(t-) = h(t)$) then $x(t-) = x(t)$ and the solution of (3) is continuous from the left. This is an easy consequence of the inequality (6).

Concerning the uniqueness of solutions of (3) we have the following general result.

Theorem 4. *If $F \in \mathcal{F}(h,\omega)$, $h(t-) = h(t)$ and*

$$\lim_{v\to 0+} \int_v^u \frac{1}{\omega(r)} dr = \infty$$

for some $u > 0$, then every solution of (3) with $x(t_0) = \widetilde{x}$ is locally unique for $t > t_0$.

(See 4.8 Theorem in [3].)

Remark 3. If $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $g(x,\cdot)$ is Lebesgue measurable for $x \in \mathbb{R}$ and

$$\|g(x,s)\| \le m(s),$$

$$\|g(x,s) - g(y,s)\| \le l(s)\omega(\|x-y\|),$$

where $m, l \in L^1_{\mathrm{loc}}(\mathbb{R})$, then for

$$G(x,t) = \int_0^t g(x,s)ds : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

we have $G \in \mathcal{F}(h,\omega)$ with

$$h(t) = \int_0^t m(s)ds + \int_0^t l(s)ds.$$

The following result connects generalized ordinary differential equations with the classical ordinary differential equations in the Carathéodory sense.

A function $x : [\alpha,\beta] \to \mathbb{R}^n$ is a solution of

$$\dot{x} = g(x,t)$$

if and only if x is a solution of the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DG(x,t)$$

on $[\alpha, \beta]$.

If $J_i : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\|J_i(x) - J_i(y)\| \leq \omega(\|x - y\|)$$

for $i \in \mathbb{N}$, $x, y \in \mathbb{R}^n$ and if $H_d : \mathbb{R} \to \mathbb{R}$ is given for $d \in \mathbb{R}$ by the relations

$$H_d(t) = 0, \quad t \leq d, \quad H_d(t) = 1, \quad t > d,$$

then define

$$F(x,t) = G(x,t) + \sum_{j=1}^{\infty} J_j(x) H_j(t).$$

The function $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is well defined, it belongs to a certain class $\mathcal{F}(h, \omega)$ and the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DF(x,t)$$

is equivalent to the so called system with impulses given by the ordinary differential equation

$$\dot{x} = g(x,t)$$

and the conditions

$$x(i+) = x(i) + J_i(x(i)), \quad i \in \mathbb{N}$$

describing the jumps of a solution at the instants $i \in \mathbb{N}$.

Let us now consider the function

$$F(x,t) = \sum_{i=1}^{\infty} J_i(x) H_i(t).$$

with J_i, H_i, $i \in \mathbb{N}$ given above and assume for simplicity that

$$\|J_i(x)\| < K = \text{const.}, \quad x \in \mathbb{R}^n. \tag{7}$$

Then $F \in \mathcal{F}(h, \omega)$ with $h(t) = K \sum_{i=1}^{\infty} H_i(t)$.

Note that the assumption (7) of the uniform boundedness of the functions J_i is very strong and restrictive. We use it for simplicity only, in fact for a reasonable theory it is sufficient to require (7) on compact subsets of $\mathbb{R}^n$ only and moreover the constant K need not be the same for all $i \in \mathbb{N}$.

Consider the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DF(x,t) = D[\sum_{i=1}^{\infty} J_i(x) H_i(t)],$$

i.e. the integral equation

$$x(s) = x(\gamma) + \int_\gamma^s D[\sum_{j=1}^\infty J_j(x(\tau))H_j(t)],\ s \in [0, \infty),$$

or more conveniently

$$x(s) = x(\gamma) + \int_\gamma^s \sum_{j=1}^\infty J_j(x(t))dH_j(t),\ s \in [0, \infty), \tag{8}$$

where $\gamma \in \mathbb{R}$ is fixed.

Since $F(x, s_2) - F(x, s_1) = 0$ for $s_1, s_2 \in (j, j+1]$, $j \in \mathbb{N}$ and for $s_1, s_2 \in [0, 1]$, we get for a solution x of (8) on $[0, \infty)$ the relation

$$x(s_2) = x(s_1)$$

if $s_1, s_2 \in (j, j+1]$, $j \in \mathbb{N}$ or $s_1, s_2 \in [0, 1]$, i.e the solution x is constant on $[0, 1]$ and on intervals $(j, j+1]$, $j \in \mathbb{N}$.

Moreover, we have

$$x(j+) = x(j) + J_j(x(j)), \quad j \in \mathbb{N}.$$

If we assume that $\gamma = 0$ and $x(\gamma) = x(0) = \widetilde{x} \in \mathbb{R}^n$, then for a solution x of (8) on $[0, \infty)$ we have

$$x(s) = \widetilde{x}, \quad s \in [0, 1],$$

$$x(s) = x(1) + J_1(x(1)), \quad s \in (1, 2],$$

$$x(s) = x(k) + J_k(x(k)), \quad s \in (k, k+1], \quad k \in \mathbb{N}.$$

The solution of (8) is evidently a piecewise constant function defined on $[0, \infty)$ which is constant on the intervals $[0, 1]$, $(k, k+1]$, $k \in \mathbb{N}$

3. Discrete equations

Let us consider equations of the form

$$x_{k+1} = S_k(x_k), \quad k \in \mathbb{N} \tag{9}$$

where $S_k : \mathbb{R}^n \to \mathbb{R}^n$ with

$$\|S_k(x) - S_k(y)\| \le \omega_1(\|x - y\|) \tag{10}$$

and $\omega_1 : [0, \infty) \to [0, \infty)$ has the character of a modulus of continuity.

Given $x_1 = \widetilde{x} \in \mathbb{R}^n$, by (9) a sequence (x_k), $k \in \mathbb{N}$ in $\mathbb{R}^n$ is uniquely determined.

Also, if $x_{k^*} \in \mathbb{R}^n$ is given for some $k^* \in \mathbb{N}$, the values x_k for $k \geq k^*$, $k \in \mathbb{N}$ can be computed according to (9).

In this situation it is sometimes useful to know the "ancestors" of x_{k^*}, i.e. the values x_k for $k \in \mathbb{N}$, $k < k^*$ for which (9) is satisfied and of course it is nice to have these values determined uniquely. For this reason we require that

$$\text{the inverse } S_k^{-1} : \mathbb{R}^n \to \mathbb{R}^n \text{ to } S_k \text{ exists for } k \in \mathbb{N}$$

and S_k^{-1} is defined on the whole $\mathbb{R}^n$, i.e. the range $\mathcal{R}(S_k)$ of S_k equals $\mathbb{R}^n$ for every $k \in \mathbb{N}$.

Let us set

$$J_k(x) = S_k(x) - x \tag{11}$$

for $x \in \mathbb{R}^n$, $k \in \mathbb{N}$. Then by (10) we have

$$\|J_k(x) - J_k(y)\| \leq \omega_1(\|x - y\|) + \|x - y\| = \omega(\|x - y\|)$$

and $\omega : [0, \infty) \to [0, \infty)$ has again the shape of a modulus of continuity.

Let us now consider the generalized ordinary differential equation of the form (8) with J_k given by (11).

It can be seen immediately that given $\widetilde{x} \in \mathbb{R}^n$ the sequence (x_k), $k \in \mathbb{N}$ defined by the discrete system (9) with $x_1 = \widetilde{x}$ is such that the piecewise constant function defined by $x(s) = x_1 = \widetilde{x}$ for $s \in [0, 1]$, $x(s) = x_k$ for $s \in (k, k+1]$, $k \in \mathbb{N}$ is a solution of the generalized ordinary differential equation (8) and vice versa: if x is a solution of the generalized ordinary differential equation (8) on $[0, \infty)$ with $x(0) = \widetilde{x}$ then $x_{k+1} = x(s)$, $s \in (k, k+1]$, $k \in \mathbb{N}$ gives the sequence in $\mathbb{R}^n$ defined by (9) with $x_1 = \widetilde{x}$.

We conclude this section by stating that

there is a one-to-one correspondence between sequences (x_k), $k \in \mathbb{N}$ given by (9) and solutions of the generalized ordinary differential equation in the special form (8), where $J_k(x) = S_k(x) - x$ for $x \in \mathbb{R}^n$, $k \in \mathbb{N}$.

4. Some possible applications

Results known for generalized ordinary differential equations can be used for the investigation of discrete systems of the form (9).

For example, there are many stability concepts known for discrete systems (9) (see e.g. the book [1]). They are mostly motivated by analogous concepts for classical ordinary differential equations.

Let us define a new stability concept for discrete equations (9)

$$x_{k+1} = S_k(x_k), \quad k \in \mathbb{N},$$

where we assume that $S_k(0) = 0$ for every $k \in \mathbb{N}$.

The sequence $x_k \equiv 0$ evidently satisfies (9) and we will consider stability of this sequence.

Definition 2. $x_k \equiv 0$ is called *variationally stable* if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $y_{k_0}, \dots, y_{k_0+l}$, $l \in \mathbb{N}$ satisfies

$$\|y_{k_0}\| < \delta \text{ and } \sum_{j=k_0}^{k_0+l} \|S_j(y_j)\| < \delta,$$

then $\|y_j\| < \varepsilon$ for $j = k_0, \dots, k_0 + l$.

$x_k \equiv 0$ is called *variationally attractive* if there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$ there exist $K(\varepsilon) \in \mathbb{N}$, $\gamma(\varepsilon) > 0$ such that if $y_{k_0}, \dots, y_{k_0+l}$, $l \in \mathbb{N}$ satisfy

$$\|y_{k_0}\| < \delta_0 \text{ and } \sum_{j=k_0}^{k_0+l} \|S_j(y_j)\| < \gamma(\varepsilon),$$

then $\|y_j\| < \varepsilon$ provided $j = k_0 + K(\varepsilon), \dots, k_0 + l$.

$x_k \equiv 0$ is called *asymptotically variationally stable* if it is variationally stable and variationally attractive.

Another concept is given by the following definition.

Definition 3. $x_k \equiv 0$ is called *stable with respect to perturbations* if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $p_{k_0}, \dots, p_{k_0+l}$, $l \in \mathbb{N}$ satisfies

$$\sum_{j=k_0}^{k_0+l} \|p_j\| < \delta,\ y_{k_0} \in \mathbb{R}^n,\ \|y_{k_0}\| < \delta$$

and

$$y_{k+1} = S_k(y_k) + p_k, \quad k = k_0, \dots, k_0 + l,$$

then $\|y_j\| < \varepsilon$ for $j = k_0, \dots, k_0 + l$.

$x_k \equiv 0$ is called *attractive with respect to perturbations* if there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$ there exist $K(\varepsilon) \in \mathbb{N}$, $\gamma(\varepsilon) > 0$ such that if

$$\|y_{k_0}\| < \delta_0 \text{ and } \sum_{j=k_0}^{k_0+l} \|p_j\| < \gamma,$$

then for

$$y_{k+1} = S_k(y_k) + p_k, \quad k = k_0, \dots, k_0 + l,$$

we have $\|y_j\| < \varepsilon$ if $j = k_0 + K(\varepsilon), \dots, k_0 + l$.

$x_k \equiv 0$ is called *asymptotically stable with respect to perturbations* if it is stable and attractive with respect to perturbations.

Similar concepts have been presented for generalized differential equations in Chapter 10 of [3]. Presenting the results from [3] in terms of discrete systems we can state e.g. the following

Theorem 5. $x_k \equiv 0$ *is variationally stable if and only if it is stable with respect to perturbations.*

$x_k \equiv 0$ *is variationally attractive if and only if it is attractive with respect to perturbations.*

For characterizing e.g. the concept of variational stability of $x_k \equiv 0$ for (9) the following Ljapunov-type result can be derived using the theory of generalized ordinary differential equations (see Theorems 10.13 and 10.23 in [3]).

Theorem 6. $x_k \equiv 0$ *is variationally stable if and only if there is a sequence of functions* $V_k : \overline{B_d} \subset \mathbb{R}^n \to \mathbb{R}$, $d > 0$ *(*$\overline{B_d} = \{x \in \mathbb{R}^n; \|x\| \le d\}$ *is the closed ball in* $\mathbb{R}^n$ *centered at* 0 *with radius* d*) such that*

$$a(\|x\|) \le V_k(x), \quad V_k(0) = 0,$$

$$|V_k(x) - V_k(y)| \le K\|x - y\|$$

for $x, y \in \overline{B_d}$*,* K *is a constant and* $a : [0, \infty) \to \mathbb{R}$ *is a continuous increasing function such that* $a(r) = 0$ *if and only if* $r = 0$*.*

There is a fairly complete theory for linear generalized ordinary differential equations (see Chapter VI in [3]) which can be used in the above described way for investigating linear discrete systems of the form

$$x_{k+1} = S_k x_k + b_k, \quad k \in \mathbb{N},$$

where $S_k \in L(\mathbb{R}^n)$ are $n \times n$-matrices, $b_k \in \mathbb{R}^n$, $k \in \mathbb{N}$. With the assumption of existence of the inverse S_k^{-1}, $k \in \mathbb{N}$ we get a theory of linear systems with nice properties and the results known for linear generalized ordinary differential equations presented in [3] lead to results for linear discrete systems like variation-of-constant formula, periodic systems, Floquet theory, multipliers, etc.

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ON EXISTENCE OF SINGULAR SOLUTIONS OF *N*-TH ORDER DIFFERENTIAL EQUATIONS

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Abstract. In the paper sufficient conditions are given under which the equation $y^{(n)} = f(t, y, \ldots, y^{(n-2)})g(y^{(n-1)})$ has a singular solution $y : [T, \tau) \to \mathbb{R}, \tau < \infty$ fulfilling $\lim_{t\to\tau_-} y^{(i)}(t) = c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-2$ and $\lim_{t\to\tau_-} |y^{(n-1)}(t)| = \infty$.

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Consider the n-th order differential equation

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-2)})g(y^{(n-1)}) \tag{1}$$

where $n \geq 2, f \in C^o(\mathbb{R}_+ \times \mathbb{R}^{n-1}), g \in C^o(\mathbb{R}), \mathbb{R}_+ = [0, \infty), \mathbb{R} = (-\infty, \infty)$, there exists $\alpha \in \{-1, 1\}$ such that

$$\alpha f(t, x_1, \ldots, x_{n-1})x_1 > 0 \quad \text{for} \quad x_1 \neq 0 \quad \text{and} \quad g(x) \geq 0 \quad \text{for} \quad x \in \mathbb{R}. \tag{2}$$

Hence, (1) fulfills the sign condition.

A solution y defined on $[T, \tau) \subset \mathbb{R}_+$ is called singular if $\tau < \infty$ and y cannot be defined for $t = \tau$. A singular solution y is called nonoscillatory if $y \neq 0$ in a left neighbourhood of τ, otherwise it is called oscillatory.

The problem of the existence of a nonoscillatory singular solution y of (1) fulfilling

$$y^{(i)}(t)y(t) > 0, i = 0, 1, \ldots, n-1 \tag{3}$$

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in a left neighbourhood of τ is posed and studied in [5,6] (in case $\alpha = 1$) for Emden-Fowler equation

$$y^{(n)} = r(t)|y|^{\lambda} \operatorname{sgn} y, \quad r \neq 0, \tag{4}$$

see [1] and [2], too. For Eq. (1) the results are generalized in [7,8]. The existence of oscillatory singular solution is proved only for Eq. (4) in [3]. Note that singular solutions of (4) (with all derivatives) are unbounded, see e.g. [9].

On the other hand singular solutions with different asymptotic behaviour than (3) may exist. Jaroš and Kusano announced that in [4] they studied a special case of (1), the second order equation

$$y'' = r(t)|y|^{\sigma}|y'|^{\lambda} \operatorname{sgn} y, \quad \sigma > 0, r < 0 \quad \text{on} \quad \mathbb{R}_+.$$

They proved that the necessary and sufficient condition for the existence of a singular solution y fulfilling

$$\lim_{t \to \tau_-} y(t) = c \in [0, \infty), \ \lim_{t \to \tau_-} y'(t) = -\infty \tag{5}$$

is $\lambda > 2$; solutions fulfilling (5) are called black hole solutions.

In our paper we generalize this result for (1).

We will study the existence of a singular solution y fulfilling the conditions:

$$\begin{aligned} \tau \in (0, \infty), \lim\nolimits_{t \to \tau_-} y^{(i)}(t) = c_i \in \mathbb{R}, \ i = 0, 1, \ldots, n-2, \\ \lim\nolimits_{t \to \tau_-} |y^{(n-1)}(t)| = \infty. \end{aligned} \tag{6}$$

This solution is nonoscillatory. Moreover the sign of $y^{(n-1)}, \alpha$ and c_0 cannot be arbitrary.

Lemma 1. *Let y be a solution of (1) fulfilling (6).*
(a) If $\lim_{t \to \tau_-} y^{(n-1)}(t) = \infty$ then $\alpha c_0 \geq 0$.
(b) If $\lim_{t \to \tau_-} y^{(n-1)}(t) = -\infty$ then $\alpha c_0 \leq 0$.

Proof. (a) Let $\alpha = 1$ for simplicity and suppose $c_0 < 0$. Then according to (1) and (2) $y^{(n)}(t) \leq 0$ for large t that contradicts $\lim_{t \to \tau_-} y^{(n-1)}(t) = \infty$. Hence $c_0 \geq 0$.
(b) The proof is similar.

Denote by $[[x]]$ the entire part of x.

Theorem 1. *Let $\tau \in (0, \infty), \lambda > 2, c_0 \neq 0, c_i \in \mathbb{R}$ for $i = 1, \ldots, n-2$ and $M \in (0, \infty)$. Let $\beta = \alpha \operatorname{sgn} c_0$ and*

$$g(x) \geq |x|^{\lambda} \qquad for \qquad \beta x \geq M. \tag{7}$$

Then there exists a singular solution y of (1) fulfilling (6) that is defined in a left neighbourhood of τ.

If, moreover, $\varepsilon > 0$,

$$n + \frac{1-\alpha}{2} \quad \text{is odd}, \quad (-1)^i c_i c_0 \geq 0 \quad \text{for} \quad i = 1, 2, \ldots, n-2 \tag{8}$$

and

$$\left| \int_0^{\beta\varepsilon} \frac{ds}{g(s)} \right| = \infty \tag{9}$$

then y *is defined on* $[0, \tau)$.

Proof. We prove the statement for $\alpha = 1$ and $c_0 > 0$ (thus $\beta = 1$). For the other cases the proof is similar.

Let $N > 2\max(c_0, |c_1|, \ldots, |c_{n-2}|)$. Consider the auxilliary problem

$$\begin{gathered} y^{(n)} = f\left(t, \chi_0, (y), \chi(y'), \ldots, \chi(y^{(n-2)})\right) g(y^{(n-1)}), \\ y^{(i)}(\tau) = c_i, i = 0, 1, \ldots, n-2, \quad y^{(n-1)}(\tau) = k \end{gathered} \tag{10}$$

where $k \in \{k_0, k_0+1, \ldots\},\ k_0 \geq [[2M]]$,

$$\begin{aligned} \chi_0(s) &= s && \text{for} \quad \tfrac{c_0}{2} \leq s \leq N, \\ &= N && \text{for} \quad s > N, \\ &= c_0/2 && \text{for} \quad s < c_0/2, \\ \chi(s) &= s && \text{for} \quad |s| \leq N, \\ &= N && \text{for} \quad s > N, \\ &= -N && \text{for} \quad s < -N. \end{aligned} \tag{11}$$

Denote by y_k a solution of (10) and by J_1 the penetration of its definition interval and $[0, \tau]$. Note, that (2), (10) and (11) yield

$$y_k^{(n)}(t) \geq 0 \quad \text{on} \quad J_1. \tag{12}$$

Put

$$\begin{aligned} M_1 =& \min\{f(t, x_1, \ldots, x_{n-1}) : t \in [0, \tau], \frac{c_0}{2} \leq x_1 \leq N, \\ & |x_j| \leq N, j = 2, \ldots, n-1\} > 0, \\ M_2 =& \max\{f(t, x_1, \ldots, x_{n-1}) : t \in [0, \tau], \frac{c_0}{2} \leq x_1 \leq N, \\ & |x_j| \leq N, j = 2, \ldots, n-1\}, \\ M_3 =& [(\lambda - 1)M_1]^{-\frac{1}{\lambda-1}}. \end{aligned}$$

Further, let $J = [T, \tau] \subset J_1$ be such that $T < \tau$,

$$\sum_{j=i}^{n-2} \frac{|c_j|}{(j-i)!} (\tau - T)^{j-i} + \frac{\lambda - 1}{\lambda - 2} M_3 (\tau - T)^{n-i-1-\frac{1}{\lambda-1}} \leq N, \quad i = 0, 1, \ldots, n-2, \tag{13}$$

$$\sum_{j=1}^{n-2} \frac{|c_j|}{j!} (\tau - T)^j + \frac{\lambda - 1}{\lambda - 2} M_3 (\tau - T)^{n-1-\frac{1}{\lambda-1}} \leq \frac{c_0}{2} \tag{14}$$

and

$$M_2(\tau - T) < \int_M^{2M} \frac{ds}{g(s)}. \tag{15}$$

As (7), $\lambda > 2$ and $n \geq 2, J$ exists.

We prove that

$$y_k^{(n-1)}(t) \geq M, t \in J. \tag{16}$$

Suppose, contrarily, that $T_1 \in [T, \tau)$ exists such that $y_k^{(n-1)}(T_1) = M$. Then with respect to (10) and (12) $y_k^{(n-1)}(t) \geq M$ for $t \in [T_1, \tau]$. From this and from (10) and (11)

$$y_k^{(n)}(t) \leq M_2 g\left(y_k^{(n-1)}(t)\right), t \in [T_1, \tau]$$

and hence, by the integration on $[T_1, \tau]$,

$$\int_M^{2M} \frac{ds}{g(s)} \leq \int_M^k \frac{ds}{g(s)} \leq M_2(\tau - T_1) \leq M_2(\tau - T).$$

The contradiction with (15) proves that $y^{(n-1)} \neq M$ for $t \in J$. From this, from (12) and $y_k^{(n-1)}(\tau) = k > M$ (16) holds.

Further, (7), (10), (11) and (16) yield

$$y_k^{(n)}(t) \geq M_1 g\left(y^{(n-1)}(t)\right) \geq M_1 \left(y^{(n-1)}(t)\right)^\lambda, \ t \in J$$

and by the integration on $[t, \tau] \subset J$ we have

$$(y_k^{(n-1)}(t))^{1-\lambda} - k^{1-\lambda} \geq M_1(\lambda - 1)(\tau - t),$$

$$y_k^{(n-1)}(t) \leq M_3(\tau - t)^{-\frac{1}{\lambda-1}}, \ t \in [T, \tau), \ k \geq k_0. \tag{17}$$

Hence, using the Taylor series formula at τ, (13), (17) and $\lambda > 2$, we have

$$\begin{aligned}
\left|y_k^{(i)}(t)\right| &\leq \sum_{j=i}^{n-2} \frac{|c_j|}{(j-i)!}(\tau - t)^{j-i} + \left|\int_\tau^t \frac{(t-s)^{n-i-2}}{(n-i-2)!} \, y_k^{(n-1)}(s) ds\right| \leq \\
&\leq \sum_{j=i}^{n-2} \frac{|c_j|}{(j-i)!}(\tau - T)^{j-i} + \frac{M_3(\tau - t)^{n-i-2}}{(n-i-2)!} \left|\int_\tau^t (\tau - s)^{-\frac{1}{\lambda-1}} ds\right| \\
&\leq N \qquad , i = 0, 1, \dots, n-2, t \in [T, \tau), k \geq k_0.
\end{aligned}$$

Similarly, using (14) and (17)

$$y_k(t) \geq c_0 - \sum_{j=1}^{n-2} \frac{|c_j|}{j!}(\tau - T)^j - \frac{\lambda - 1}{\lambda - 2} M_3(\tau - T)^{n-1-\frac{1}{\lambda-1}} \geq \frac{c_0}{2},$$

$$t \in [T, \tau), \ k \geq k_0.$$

From these estimations and (11) we can see that y_k is the solution of Eq. (1), too. Moreover, the sequences $\{y_k^{(i)}\}_{k_0}^{\infty}, i = 0, \dots, n-1$ are uniformly bounded and equipotentially continuous on every segment of $[T, \tau)$. Hence according to Arzel-Ascoli Theorem there exists a subsequence that converges uniformly to a solution y of (1). Evidently, the conditions (6) are fulfilled with $\lim_{t \to \tau_-} y^{(n-1)}(t) = \infty$.

Let (8) and (9) be valid. Let the above given solution y be defined on $(\bar{\tau}, \tau) \subset [0, \tau)$ and cannot be extended to $t = \bar{\tau}$. Then

$$\limsup_{t \to \bar{\tau}_+} \left| y^{(n-1)}(t) \right| = \infty. \tag{18}$$

First, we prove that

$$y^{(n-1)}(t) > 0 \quad \text{on} \quad (\bar{\tau}, \tau). \tag{19}$$

Thus, suppose that there exists $\tau_1 \in (\bar{\tau}, \tau)$ such that $y^{(n-1)}(\tau_1) = 0$ and $y^{(n-1)}(t) > 0$ on (τ_1, τ). It follows from this and from (6) that $y^{(j)}, j = 0, 1, \dots, n-2$ are bounded, $|y^{(j)}(t)| \leq K, j = 0, 1, \dots, n-2, t \in [\tau_1, \tau)$. Let $\tau_2 \in (\tau_1, \tau)$ be such that $y^{(n-1)}(\tau_2) = \varepsilon$. Then by the integration of (1) and by (9)

$$\infty = \int_0^{\varepsilon} \frac{ds}{g(s)} = \int_{\tau_1}^{\tau_2} f\left(t, y(t), \dots, y^{(n-2)}(t)\right) dt < \infty.$$

Hence, (19) is valid, and (8) and (19) yield $y(t) > 0$ on $(\bar{\tau}, \tau)$. From this and from (1) $y^{(n)}(t) > 0$ on $(\bar{\tau}, \tau)$, that, together with (19), contradicts (18). Thus y is defined at $t = \bar{\tau}$ and $\bar{\tau} = 0$.

Corollary 1. *Let $\lambda > 2$ and $M \in \mathbb{R}_+$ be such that*

$$g(x) \geq x^{\lambda} \quad \textit{for} \quad x \geq M.$$

Then (1) has a singular solution.

Remark 1. For $\alpha = 1$ the conclusion of Corollary 1 is known, see, e.g., [9, Theorem 11.3].

The following result shows that for the existence of a singular solution with (6) λ cannot be equal to 2.

Theorem 2. *Let $M \in (0, \infty)$ be such that $g(x) \leq x^2$ for $|x| \geq M$. Then Eq. (1) has no singular solution y fulfilling (6).*

Proof. Let y be singular and fulfil (6). Suppose, for simplicity, $\alpha = 1$ and $\lim_{t \to \tau_-} y^{(n-1)}(t) = \infty$. From this there exists a left neighbourhood $[\tau_1, \tau)$ of τ such that $|y^{(i)}(t)| \leq M_1 < \infty$ for $i = 0, 1, \dots, n-2$ and $y^{(n-1)}(t) \geq M$ on $[\tau_1, \tau)$

where M_1 is a suitable constant. Hence, using the assumptions of the theorem we have

$$\infty = ln\ \frac{y^{(n-1)}(\tau)}{y^{(n-1)}(\tau_1)} = \int_{\tau_1}^{\tau} \frac{y^{(n)}(s)}{y^{(n-1)}(s)} ds \le \int_{\tau_1}^{\tau} \left| f\left(s, y(s), \dots, y^{(n-2)}(s)\right) \right| y^{(n-1)}(s) ds$$
$$\le \left(c_{n-2} - y^{(n-2)}(\tau_1)\right) \max |f(s, x_1, \dots, x_{n-1})| < \infty$$

where the maximum is taken for $s \in [\tau_1, \tau], |x_i| \le M_1, i = 1, \dots, n-1$. The contradiction proves the conclusion.

Corollary 2. *Let $c_0 \ne 0, M \in (0, \infty)$ and $g(x) = |x|^\lambda$ for $|x| \ge M$. Then (1) has a singular solution y fulfilling (6) if and only if $\lambda > 2$.*

Proof. It follows from Theorems 1 and 2.

Remark 2. Note, that, especially, eq.

$$y^{(n)} = f(t, y, y', \dots, y^{(n-2)})$$

has no singular solutions satisfying (6).

In the next part of the paper the case $c_0 = 0$ will be investigated.

Theorem 3. *Let $\beta \in \{-1, 1\}, \sigma > 0,\ \varepsilon > 0, \tau \in (0, \infty),\ M \in (0, \infty), \alpha \in \{-1, 1\}$*

$$\lambda > \sigma(n-2) + 2, \tag{20}$$

$$c_0 = 0, (-1)^i \beta\ c_i \ge 0 \quad \textit{for} \quad i = 1, 2, \dots, n-2, \tag{21}$$

and

$$n + \frac{1-\alpha}{2} \quad \textit{be odd.} \tag{22}$$

Let (7) hold and a continuous function $r : \mathbb{R}_+ \to \mathbb{R}$ exist such that

$$\alpha r(t) > 0 \quad on \quad R_+,$$
$$|f(t, x_1, x_2, \dots, x_{n-1})| \ge |r(t)|\, |x_1|^\sigma \quad for \quad t \in [0, \tau],$$
$$\beta x_1 \in [0, \varepsilon], (-1)^j \beta x_{j+1} \in \left[(-1)^j \beta c_j, (-1)^j \beta c_j + \varepsilon\right],\ j = 1, 2, \dots, n-2.$$

Then there exists a singular solution y of (1) fulfilling (6) that is defined in a left neighbourhood of τ. If, moreover, (9) holds, then y is defined on $[0, \tau)$.

Proof. Let $\alpha = 1$ and $\beta = 1$; thus n is odd. For the other cases the proof is similar. Put for $i \in \{0, 1, \dots, n-2\}$

$$\begin{aligned} \chi_i(s) &= s && \text{for} \quad (-1)^i c_i \le (-1)^i s \le (-1)^i c_i + \varepsilon, \\ &= c_i + (-1)^i \varepsilon && \text{for} \quad (-1)^i s > (-1)^i c_i + \varepsilon, \\ &= c_i && \text{for} \quad (-1)^i s < (-1)^i c_i. \end{aligned} \tag{23}$$

Consider the Cauchy problem

$$y^{(n)} = f\left(t, \chi_0(y), \chi_1(y'), \dots, \chi_{n-2}(y^{(n-2)})\right) g\left(y^{(n-1)}\right), \tag{24}$$
$$y^{(i)}(\tau) = c_i, i = 0, 1, \dots, n-2, \quad y^{(n-1)}(\tau) = k$$

where $k \in \{k_0, k_0 + 1, \dots\}, k_0 \geq [[2M]]$.

Denote by y_k a solution of (24) and J_1 the penetration of its definition interval and $[0, \tau]$. Note, that $\alpha = 1$, (23), (24) yield

$$y_k^{(n)}(t) \geq 0 \quad \text{and} \quad y_k^{(n-1)} \quad \text{is nondecreasing on} \quad J_1. \tag{25}$$

Put $M_1 = \frac{1}{[(n-1)!]^\sigma} \min_{t \in [0,\tau]} r(t) > 0,\ M_2 = \left[\frac{M_1}{\sigma(n-1)+1}(\lambda + \sigma - 1)\right]^{-\frac{1}{\lambda+\sigma-1}}$,

$$\sigma_1 = \frac{\sigma(n-1)+1}{\lambda + \sigma - 1}, M_3 = \max f(t, x_1, \dots, x_{n-1}),$$

where the maximum is given for $t \in [0, \tau], 0 \leq x_1 \leq \varepsilon, (-1)^i c_i \leq (-1)^i x_{i+1} \leq (-1)^i c_i + \varepsilon, i = 1, \dots, n-2$. Then (20) yields $\sigma_1 \in (0, 1)$.

Further, let $J = [T, \tau] \subset J_1$ be such that $T < \tau$,

$$\sum_{j=i}^{n-2} \frac{|c_j|}{(j-i)!}(\tau - T)^{j-i} + \frac{M_2}{(n-i-2)!(1-\sigma_1)}(\tau - T)^{n-i-\sigma_1-1} \leq (-1)^i c_i + \varepsilon$$
$$i = 0, 1, \dots, n-2 \tag{26}$$

and

$$M_3(\tau - T) < \int_M^{2M} \frac{ds}{g(s)}. \tag{27}$$

Using (27), it can be proved similarly to the proof of Theorem 1, that (16) holds. Hence, using (21) and (22) we have

$$(-1)^i y_k^{(i)}(t) \geq (-1)^i c_i \geq 0 \quad \text{on} \quad J,\ i = 0, 1, 2, \dots, n-2. \tag{28}$$

The Taylor series formula at $t = \tau$, (16), (21), (25) and n be odd yield

$$y_k(t) = \sum_{j=0}^{n-2} c_j \frac{(t-\tau)^j}{j!} + \int_\tau^t \frac{(t-s)^{n-2}}{(n-2)!} y_k^{(n-1)}(s) ds \geq \int_\tau^t \frac{(t-s)^{n-2}}{(n-2)!} y_k^{(n-1)}(s) ds$$
$$\geq \frac{(\tau - t)^{n-1}}{(n-1)!} y_k^{(n-1)}(t), \qquad t \in J,$$

and from (24), (16), (25), (28) and the assumptions of the theorem

$$y_k^{(n)}(t) \geq r(t)(y_k(t))^\sigma [y_k^{(n-1)}(t)]^\lambda \geq M_1(\tau - t)^{\sigma(n-1)} \left(y_k^{(n-1)}(t)\right)^{\lambda+\sigma}, t \in J.$$

Hence, by the integration on $[t, \tau]$ we obtain similarly to the proof of Theorem 1

$$y_k^{(n-1)}(t) \le M_2(\tau - t)^{-\sigma_1}, t \in [T, \tau), k = k_0, k_0 + 1, \dots$$

From this, using the Taylor series formula at $t = \tau$, (26), (28) and $\sigma_1 < 1$ we have

$$0 \le (-1)^i c_i \le (-1)^i y_k^{(i)}(t) = \sum_{j=i}^{n-2} \frac{|c_j|}{(j-i)!}(\tau - t)^{j-i} +$$

$$(-1)^i \int_\tau^t \frac{(t-s)^{n-i-2}}{(n-i-2)!}\, y_k^{(n-1)}(s)ds$$

$$\le \sum_{j=i}^{n-2} \frac{|c_j|}{(j-i)!}(\tau - t)^{j-i} + \frac{M_2(\tau - t)^{n-i-1-\sigma_1}}{(n-i-2)!(1-\sigma_1)} \le (-1)^i c_i + \varepsilon,$$

$$i = 0, 1, \dots, n-2.$$

Thus, according to (23), y_k is a solution of Eq. (1), too and the rest of the proof is similar as in Theorem 1.

The following theorem shows that the condition (20) cannot be weaken.

Theorem 4. *Let $c_i = 0, i = 0, 1, \dots, n-2, \sigma > 0, n \ge 2, n + \frac{1-\alpha}{2}$ be odd, $\alpha \in \{-1, 1\}$ and let $r \in C^0(\mathbb{R}_+), \alpha r > 0$ on $\mathbb{R}_+$. Then the equation*

$$y^{(n)} = r(t)|y|^\sigma |y^{(n-1)}|^\lambda \operatorname{sgn} y \tag{29}$$

has a singular solution y fulfilling (6) if, and only if $\lambda > \sigma(n-2) + 2$.

Proof. In view of Theorem 3 we must prove the necessity only. Let $\lambda \le \sigma(n-2)+2$, y be singular and fulfilling (6). Suppose, for simplicity, that $r > 0$, $\lim_{t \to \tau_-} y^{(n-1)}(t) = \infty$ and thus n be odd. In the other cases the proof is similar. Then there exists $t_0 \in [0, \tau)$ such that

$$(-1)^i y^{(i)}(t) > 0,\ i = 0, 1, \dots, n-2,\ y^{(n-1)}(t) \ge 1,\ y^{(n)}(t) \ge 0 \quad \text{on} \quad J = [t_0, \tau). \tag{30}$$

Then using the Taylor series formula on $[t, \tau]$ and (6) we obtain

$$y(t) = \int_\tau^t \frac{(t-s)^{n-2}}{(n-2)!} y^{(n-1)}(s)ds \le \frac{(\tau - t)^{n-2}}{(n-2)!}|y^{n-2}(t)|, t \in J. \tag{31}$$

Further,

$$|y^{(n-2)}(t)| = \int_t^\tau y^{(n-1)}(s)ds \ge y^{(n-1)}(t)(\tau - t), t \in J$$

and hence, using (31)

$$y(t)[y^{(n-1)}(t)]^{n-2} \le \frac{[y^{(n-2)}(t)]^{n-1}}{(n-2)!} \le M_1, t \in J$$

where M_1 is a suitable number. From this,(30) and from $\lambda \le \sigma(n-2)+2$

$$\infty = \ln \frac{y^{(n-1)}(\tau)}{y^{(n-1)}(t_0)} = \int_{t_0}^{\tau} \frac{y^{(n)}(s)}{y^{(n-1)}(s)} ds = \int_{t_0}^{\tau} r(s)\ y^{\sigma}(s)[y^{(n-1)}(s)]^{\lambda-1} ds \le$$
$$\le M_1^{\sigma} \int_{t_0}^{\tau} r(s)[y^{(n-1)}(s)]^{\lambda-1-\sigma(n-2)} ds$$
$$\le M_1^{\sigma} \int_{t_0}^{\tau} r(s) y^{(n-1)}(s) ds \le M_1^{\sigma} \max_{0\le s\le \tau} r(s) \left| y^{(n-2)}(t_0) \right| < \infty.$$

The contradiction proves the conclusion.

The following proposition shows that condition (22) in Theorem 3 cannot be weaken.

Proposition 1. *Let $\beta \in \{-1, 1\}$, (21), $c_{n-2} = 0$ and $n + \frac{1-\alpha}{2}$ be even. Then (1) has no singular solution fulfilling (6).*

Proof. Let for the simplicity $\alpha = 1$ and $\beta = -1$; for the other cases the proof is similar. Hence, n is even. Let y be a singular solution of (1) fulfilling (6). Then (1) and (21) yield $y(t) < 0, y^{(n-1)}(t) > 0$. Thus $y^{(n)}(t) > 0$ in a left neighbourhood J of τ that contradicts (1), (2) and $\alpha = 1$.

Remark 3. The following conclusion follows from Corollary 2 and Theorem 4. *Let $n = 2$. Then Eq. (29) has a singular solution y, fulfilling (6) if, and only if $\lambda > 2$.* Hence our results generalize the above mentioned one of Jaroš and Kusano.

Open problem. It is possible to look for sufficient and (or) necessary conditions under which there is a singular solution y of (1) satisfying

$$\begin{aligned} &\tau \in (0, \infty), \quad k \in \{0, 1, \dots, n-2\}, \\ &\lim_{t\to\tau_-} y^{(i)}(t) = c_i \in \mathbb{R}, \quad i = 0, 1, \dots, k, \\ &\lim_{t\to\tau_-} |y^{(j)}(t)| = \infty, \quad j = k+1, \dots, n-1. \end{aligned}$$

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CONVERGENCE TESTS FOR ONE SCALAR DIFFERENTIAL EQUATION WITH VANISHING DELAY

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Abstract. In the present paper the differential equation

$$\dot{y}(t) = \alpha(t)[y(t) - y(t - \tau(t))]$$

with positive coefficient α and with positive bounded delay τ (which can have the property $\tau(+\infty) = 0$) is considered. Explicit tests for convergence of all its solutions (for $t \to +\infty$) are proved.

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1. Introduction

We will deal with the linear homogeneous differential equation with delay

$$\dot{y}(t) = \alpha(t)\left[y(t) - y(t - \tau(t))\right], \tag{1}$$

where $\alpha \in C(I, \mathbb{R}^+)$, $I = [t_0, \infty)$, $t_0 \in \mathbb{R}$, $\mathbb{R}^+ = (0, +\infty)$, $\tau \in C(I, \mathbb{R}^+)$; $\tau(t) \le \tau_0 =$ const and the difference $t - \tau(t)$ is increasing on I. Let us denote $I_1 = [t_0 + \tau(t_0), \infty)$.

A function y is called a *solution of Eq.* (1) *corresponding to initial point* $t^* \in I$ if y is defined and is continuous on $[t^* - \tau(t^*), \infty)$, differentiable on $[t^*, \infty)$ and satisfies (1) for $t \geq t^*$. By a *solution* of (1) we mean a solution corresponding to some initial point $t^* \in I$. We denote $y(t^*, \varphi)(t)$ a solution of Eq. (1) which is generated by continuous *initial* function $\varphi : [t^* - \tau(t^*), t^*] \mapsto \mathbb{R}$ and which corresponds to initial point $t^* \in I$.

In the case of the linear Eq. (1) the solution $y(t^*, \varphi)(t)$ is unique on its maximal existence interval $[t^*, \infty)$ ([13]). We say that a solution of Eq. (1) corresponding to initial point t^* is *convergent* or *asymptotically convergent* if it has a finite limit at $+\infty$.

The main goal of this paper is to formulate and prove several explicit tests for convergence of all solutions of Eq. (1).

Problems concerning asymptotic constancy of solutions, asymptotic convergence of solutions or existence of so called asymptotic equilibrium of various classes of retarded functional differential equations were investigated, e.g., by O. Arino, I. Györi and M. Pituk [1], O. Arino and M. Pituk [2], F.V. Atkinson and J.R. Haddock [3], R. Bellman and K.L. Cooke [4], I. Györi and M. Pituk [11], [12], K. Murakami [17], and T. Krisztin [14]–[16].

So called nonconvergence case (i.e. the case when there exists a monotone increasing divergent solution of Eq. (1)) was considered e.g. by S.N. Zhang [18] and by J. Diblík [10]. Some closely connected questions were discussed in the cycle of recent papers by J. Čermák [5]–[8] as well.

In the paper [9] the equation

$$\dot{y}(t) = \sum_{j=1}^{n} \alpha_j(t)[y(t) - y(t - \tau_j(t))], \tag{2}$$

was considered, where $\alpha_j \in C(I, \overline{\mathbb{R}^+})$, $\sum_{j=1}^{n} \alpha_j(t) > 0$ on I, $\tau_j \in C(I, \mathbb{R}^+)$, functions $t - \tau_j(t), j = 1, 2, \ldots, n$ are increasing on I and τ_j are bounded on I.

The following theorem is the main result of [9]:

Theorem 1. *For the convergence of all solutions of Eq.* (2)*, corresponding to the initial point t_0, a necessary and sufficient condition is that there exist functions $k_i \in C(I, \mathbb{R}^+), i = 1, 2, \ldots, n$ satisfying the system of integral inequalities*

$$1 + k_i(t) \geq \exp\left[\int_{t-\tau_i(t)}^{t} \sum_{j=1}^{n} \alpha_j(s) k_j(s)\, ds\right], \quad i = 1, 2, \ldots, n$$

on interval I_1.

In this paper the following partial case of this result with respect to Eq. (1) will be used:

Theorem 2. *All solutions of Eq.* (1)*, corresponding to the initial point t_0, converge if and only if there exists a function $k \in C(I, \mathbb{R}^+)$, such that*

$$1 + k(t) \geq \exp\left[\int_{t-\tau(t)}^{t} \alpha(s) k(s)\, ds\right] \tag{3}$$

on the interval I_1.

Theorem 2 serves as a source for several explicit convergence tests. In the sequel we will prove one test which uses the values of the function $\alpha(t)$ itself (point test) and two tests which use an integral weighted average of the function $\alpha(t)$. Note that the case $\tau(+\infty) = 0$ is not excluded from our investigation.

2. Point test of convergence

Theorem 3. *If for a sufficiently large t*

$$\alpha(t) \leq \frac{1}{\tau(t)} - \frac{L}{t}\,, \tag{4}$$

where $L > 1/2$ is a constant, then each solution of Eq. (1) is convergent.

Proof. Without loss of generality, let us suppose t sufficiently large. Let us put

$$k(t) \equiv \frac{\varepsilon}{t} \cdot \left(\frac{1}{\tau(t)} - \frac{L}{t}\right)^{-1}, \tag{5}$$

where ε is a positive number (then $k(t) > 0$ for $t \to +\infty$), and verify inequality (3). Develop the asymptotic expansion $\mathcal{L}(t)$ of the left hand side of (3). We get

$$\mathcal{L}(t) = 1 + k(t) = 1 + \frac{\varepsilon}{t} \cdot \left(\frac{1}{\tau(t)} - \frac{L}{t}\right)^{-1} = 1 + \frac{\varepsilon\tau(t)}{t(1 - L\tau(t)/t)} =$$
$$1 + \frac{\varepsilon\tau(t)}{t} \cdot \left(1 + \frac{L\tau(t)}{t} + \frac{L^2\tau^2(t)}{t^2} \cdot (1 + o(1))\right).$$

Here and throughout this paper "o" is the Landau symbol "small" o. The symbol "O" used in the sequel is the Landau symbol "big" O. These symbols are used in the neighbourhood of the point $t = \infty$.

Now estimate the right hand side $\mathcal{R}(t)$ of (3). With the aid of (4) and (5) we get

$$\mathcal{R}(t) = \exp\left[\int_{t-\tau(t)}^{t} \alpha(s)k(s)ds\right] \leq$$
$$\exp\left[\int_{t-\tau(t)}^{t} \left(\frac{1}{\tau(s)} - \frac{L}{s}\right)\frac{\varepsilon}{s} \cdot \left(\frac{1}{\tau(s)} - \frac{L}{s}\right)^{-1} ds\right] =$$
$$\exp\left[\varepsilon \int_{t-\tau(t)}^{t} \frac{1}{s}\, ds\right] = \exp\left[\varepsilon \ln \frac{t}{t-\tau(t)}\right] = \left(\frac{t-\tau(t)}{t}\right)^{-\varepsilon} = \left(1 - \frac{\tau(t)}{t}\right)^{-\varepsilon} =$$
$$1 + \binom{-\varepsilon}{1}\left(-\frac{\tau(t)}{t}\right) + \binom{-\varepsilon}{2}\left(-\frac{\tau(t)}{t}\right)^2 \cdot (1 + o(1)) =$$
$$1 + \frac{\varepsilon\tau(t)}{t} + \frac{\varepsilon(\varepsilon+1)}{2} \cdot \frac{\tau^2(t)}{t^2} \cdot (1 + o(1)).$$

We conclude that (3) will hold (supposing t_0 sufficiently large) if

$$\mathcal{L}(t) = 1 + \frac{\varepsilon\tau(t)}{t} + \frac{\varepsilon L\tau^2(t)}{t^2} + \frac{\varepsilon L^2\tau^3(t)}{t^3} \cdot (1 + o(1)) \geq$$
$$1 + \frac{\varepsilon\tau(t)}{t} + \frac{\varepsilon(\varepsilon+1)}{2} \cdot \frac{\tau^2(t)}{t^2} \cdot (1 + o(1)) \geq \mathcal{R}(t).$$

Comparing corresponding terms, we can see that this will hold when $L > (\varepsilon+1)/2$. Since ε is a positive number and may be chosen arbitrarily small, we get $L > 1/2$. Theorem 3 is proved.

3. Integral tests of convergence

Theorem 4. *If for a sufficiently large t*

$$\frac{1}{\tau(t)} \cdot \int_{t-\tau(t)}^{t} \tau(s)\alpha(s)ds \leq 1 - \frac{\tau(t)}{t-\tau(t)} - \frac{\tau(t)}{(t-\tau(t))\ln(t-\tau(t))} - \frac{L\tau(t)}{(t-\tau(t))\ln(t-\tau(t))\ln_2(t-\tau(t))}, \tag{6}$$

with $L > 1, L = \text{const}$ and $\ln_2 t = \ln\ln t$, then each solution of Eq. (1) is convergent.

Proof. Without loss of generality, let us suppose t sufficiently large. Let us put

$$k(t) \equiv \frac{\tau(t)}{t\ln t(\ln_2 t)^{\varepsilon}},$$

where $\varepsilon > 1$ is a constant. Obviously, the inequality (3) will be valid if

$$\mathcal{L}(t) \equiv 1 + k(t) \geq \mathcal{R}(t) \equiv \exp\left[\int_{t-\tau(t)}^{t} \alpha(s)k(s)\,ds\right], \quad t \in I_1. \tag{7}$$

We estimate the expression $\mathcal{R}(t)$. With the aid of the inequality (6), we have

$$\mathcal{R}(t) \leq \exp(\mathcal{R}^{\star}(t))$$

where

$$\mathcal{R}^{\star}(t) \equiv \frac{\tau(t)}{(t-\tau(t))\ln(t-\tau(t))(\ln_2(t-\tau(t)))^{\varepsilon}} \cdot \Bigg(1 - \frac{\tau(t)}{t-\tau(t)} -$$
$$\frac{\tau(t)}{(t-\tau(t))\ln(t-\tau(t))} - \frac{L\tau(t)}{(t-\tau(t))(\ln(t-\tau(t)))(\ln_2(t-\tau(t)))}\Bigg).$$

Let us develop the asymptotic expansion of the expression $\mathcal{R}^{\star}(t)$. At first, it is trivial to verify that the following asymptotic expansions hold:

$$\frac{1}{t-\tau(t)} = \frac{1}{t}\left(1 + \frac{\tau(t)}{t} + \frac{\tau^2(t)}{t^2} + o\left(\frac{\tau^2(t)}{t^2}\right)\right),$$

$$\frac{1}{\ln(t-\tau(t))} = \frac{1}{\ln t}\left(1 + \frac{\tau(t)}{t\ln t} + \frac{\tau^2(t)}{2t^2\ln t} + o\left(\frac{\tau^2(t)}{t^2\ln t}\right)\right)$$

and

$$\frac{1}{\ln_2(t-\tau(t))} = \frac{1}{\ln_2 t}\left(1 + \frac{\tau(t)}{t\ln t\ln_2 t} + \frac{\tau^2(t)}{2t^2\ln t\ln_2 t} + o\left(\frac{\tau^2(t)}{t^2\ln t\ln_2 t}\right)\right).$$

Thus

$$\begin{aligned}\mathcal{R}^\star(t) = \frac{\tau(t)}{t\ln t(\ln_2 t)^\varepsilon}\cdot\left(1 + \frac{\tau(t)}{t} + \frac{\tau^2(t)}{t^2} + o\left(\frac{\tau^2(t)}{t^2}\right)\right)\times\\ \left(1 + \frac{\tau(t)}{t\ln t} + \frac{\tau^2(t)}{2t^2\ln t} + o\left(\frac{\tau^2(t)}{t^2\ln t}\right)\right)\times\\ \left(1 + \frac{\varepsilon\tau(t)}{t\ln t\ln_2 t} + \frac{\varepsilon\tau^2(t)}{2t^2\ln t\ln_2 t} + o\left(\frac{\tau^2(t)}{t^2\ln t\ln_2 t}\right)\right)\times\\ \left[1 - \frac{\tau(t)}{t} - \frac{\tau^2(t)}{t^2} + o\left(\frac{\tau^2(t)}{t^2}\right) - \frac{\tau(t)}{t\ln t}\left(1 + \frac{\tau(t)}{t} + o\left(\frac{\tau(t)}{t}\right)\right)\times\right.\\ \left(1 + \frac{\tau(t)}{t\ln t} + o\left(\frac{\tau(t)}{t\ln t}\right)\right) -\\ \frac{L\tau(t)}{t\ln t\ln_2 t}\left(1 + \frac{\tau(t)}{t} + o\left(\frac{\tau(t)}{t}\right)\right)\cdot\left(1 + \frac{\tau(t)}{t\ln t} + o\left(\frac{\tau(t)}{t\ln t}\right)\right)\times\\ \left.\left(1 + \frac{\tau(t)}{t\ln t\ln_2 t} + o\left(\frac{\tau(t)}{t\ln t\ln_2 t}\right)\right)\right] =\\ \frac{\tau(t)}{t\ln t(\ln_2 t)^\varepsilon}\cdot\left(1 + \frac{\varepsilon\tau(t) - L\tau(t)}{t\ln t\ln_2 t} - \frac{\tau^2(t)}{t^2} + o\left(\frac{\tau^2(t)}{t^2}\right)\right).\end{aligned}$$

At the end we get

$$\begin{aligned}\exp(\mathcal{R}^\star(t)) = 1 + \frac{\tau(t)}{t\ln t(\ln_2 t)^\varepsilon}\cdot\left(1 + \frac{\varepsilon\tau(t) - L\tau(t)}{t\ln t\ln_2 t} - \frac{\tau^2(t)}{t^2} + o\left(\frac{\tau^2(t)}{t^2}\right)\right) +\\ \frac{\tau^2(t)}{2t^2\ln^2 t(\ln_2 t)^{2\varepsilon}}\cdot\left(1 + \frac{\varepsilon\tau(t) - L\tau(t)}{t\ln t\ln_2 t} - \frac{\tau^2(t)}{t^2} + o\left(\frac{\tau^2(t)}{t^2}\right)\right)^2\cdot(1+o(1)) =\\ 1 + \frac{\tau(t)}{t\ln t(\ln_2 t)^\varepsilon} + \frac{\varepsilon\tau^2(t) - L\tau^2(t)}{t^2\ln^2 t\ln_2^{1+\varepsilon} t} + \frac{\tau^2(t)}{2t^2\ln^2 t\ln_2^{2\varepsilon} t} + o\left(\frac{\tau^3(t)}{t^3}\right).\end{aligned}$$

For the validity of the inequality (7) it is sufficient to suppose that (for sufficiently large t)

$$\mathcal{L}(t) \geq \exp(\mathcal{R}^\star(t)),$$

i.e. that

$$1 + \frac{\tau(t)}{t\ln t(\ln_2 t)^\varepsilon} \geq 1 + \frac{\tau(t)}{t\ln t(\ln_2 t)^\varepsilon} + \frac{\varepsilon\tau^2(t) - L\tau^2(t)}{t^2\ln^2 t\ln_2^{1+\varepsilon} t} + \frac{\tau^2(t)}{2t^2\ln^2 t\ln_2^{2\varepsilon} t} + o\left(\frac{\tau^3(t)}{t^3}\right).$$

This will hold (we take into account the supposition $\varepsilon > 1$) if $L > \varepsilon$. Since ε may be chosen arbitrarily close to 1 – this assumption is necessary for the asymptotic dominance of the third term in the right-hand side in above inequality, we obtain $L > 1$. Theorem 4 is proved.

Theorem 5. *If for sufficiently large t*

$$\frac{1}{\tau(t)} \cdot \int_{t-\tau(t)}^{t} \frac{\tau(s)\alpha(s)}{s^m} ds \le \frac{\tau(t)}{(t-\tau(t))^m} - \frac{L\tau(t)}{(t-\tau(t))^{m+1}} ,$$

where $m, L = \text{const}, m \ge 1, L > m$ then each solution of Eq. (1) is convergent.

Proof. Without loss of generality, let us suppose t sufficiently large. Let us put

$$k(t) \equiv \frac{\tau(t)}{t^{m+p}},$$

where p is a positive constant. Obviously, the of inequality (3) will be valid if, as above, inequality (7) holds. Let us develop (suppposing t sufficiently large) the asymptotic expansion of $\mathcal{R}(t)$. We get

$$\begin{gathered}
\mathcal{R}(t) \equiv \exp\left[\int_{t-\tau(t)}^{t} \frac{\tau(s)\alpha(s)}{s^{m+p}}\, ds\right] \le \exp\left[\frac{1}{(t-\tau(t))^p}\int_{t-\tau(t)}^{t} \frac{\tau(s)\alpha(s)}{s^m} ds\right] \le \\
\exp\left[\frac{\tau(t)}{(t-\tau(t))^{m+p}} - \frac{L\tau^2(t)}{(t-\tau(t))^{m+p+1}}\right] = \\
1 + \frac{\tau(t)}{(t-\tau(t))^{m+p}} - \frac{L\tau^2(t)}{(t-\tau(t))^{m+p+1}} + \frac{\tau^2(t)}{2(t-\tau(t))^{2(m+p)}}\,(1+o(1)) = \\
1 + \frac{\tau(t)}{t^{m+p}}\left(1-\frac{\tau(t)}{t}\right)^{-(m+p)} - \frac{L\tau^2(t)}{t^{m+p+1}}\left(1-\frac{\tau(t)}{t}\right)^{-(m+p+1)} + O\left(\frac{\tau^2(t)}{t^{2(m+p)}}\right) = \\
1 + \frac{\tau(t)}{t^{m+p}}\left(1+\frac{(m+p)\tau(t)}{t} + O\left(\frac{\tau^2(t)}{t^2}\right)\right) - \\
\frac{L\tau^2(t)}{t^{m+p+1}}\left(1+O\left(\frac{\tau(t)}{t}\right)\right) + O\left(\frac{\tau^2(t)}{t^{2(m+p)}}\right) = \\
1 + \frac{\tau(t)}{t^{m+p}} + \frac{(m+p)\tau^2(t) - L\tau^2(t)}{t^{m+p+1}} + O\left(\frac{\tau^3(t)}{t^{m+p+2}}\right) + O\left(\frac{\tau^2(t)}{t^{2(m+p)}}\right).
\end{gathered}$$

Now, for

$$\begin{gathered}
\mathcal{L}(t) \equiv 1 + \frac{\tau(t)}{t^{m+p}} \ge \\
1 + \frac{\tau(t)}{t^{m+p}} + \frac{(m+p)\tau^2(t) - L\tau^2(t)}{t^{m+p+1}} + O\left(\frac{\tau^3(t)}{t^{m+p+2}}\right) + O\left(\frac{\tau^2(t)}{t^{2(m+p)}}\right) \ge \mathcal{R}(t)
\end{gathered}$$

is $m + p > 1$ (this assumption is necessary for the asymptotic dominance of the third term in the right-hand side in above inequality) and $L > (m + p)$ sufficient. Since p may be arbitrarily small positive number, we have $L > m$. Theorem 5 is proved.

4. Concluding remarks

4.1. Sharpness of Theorem 3.

Let Eq. (1) has the form

$$\dot{y}(t) = \left(t - \frac{a}{t}\right)[y(t) - y(t - 1/t)] \tag{8}$$

with $a > 1/2$, a =const; i.e. $\alpha(t) = t - a/t$ and $\tau(t) = 1/t$. Note that $\tau(+\infty) = 0$. It is easy to see that the inequality (4) holds since the inequality

$$\alpha(t) = t - \frac{a}{t} \le t - \frac{L}{t} = \frac{1}{\tau(t)} - \frac{L}{t}$$

is valid for $1/2 < L \le a$. In accordance with Theorem 3, each solution of Eq. (8) is convergent.

In the sequel we will show that the interval $a > 1/2$ is the best possible. Namely, we will show that the property of convergence of all solutions of equation (8) is not valid for $a = 1/2$. In paper [10] (see Theorem 2 in [10]), the following is proved:

Theorem 6. *Equation (1) has a solution $y(t)$ with property $y(+\infty) = +\infty$ if and only if the inequality*

$$\dot{\omega}(t) \le \alpha(t)[\omega(t) - \omega(t - \tau(t))]$$

has a solution $\omega(t)$ with property $\omega(+\infty) = +\infty$.

So, it is sufficient to show that, in the case of equation (8) with $a = 1/2$, the inequality

$$\dot{\omega}(t) \le \left(t - \frac{1}{2t}\right)[\omega(t) - \omega(t - 1/t)] \tag{9}$$

has a solution $\omega(t)$ with the property $\omega(+\infty) = +\infty$. It is easy to verify that the function

$$\omega(t) = \ln t$$

is such solution of inequality (9).

4.2. A comparison with Atkinson – Haddock's results.

Let us use the equation (8) as a concrete example of the equation (1) again and let us show with its aid that our convergence results are in some sense more general than the results given in [3].

Really, by Theorem 3.3 in [3] all solutions of equation (1) will converge if for a sufficiently large t

$$\int_t^{t+r} \alpha(s)ds \le 1 - \frac{r}{t} - \frac{K}{t \ln t} \tag{10}$$

with some $K > r$ where r is a positive *constant* which bounds delay. In the case of the equation (8) the left hand side of the inequality (10) equals

$$\int_t^{t+r} \alpha(s)ds = \int_t^{t+r} \left(s - \frac{a}{s}\right) ds = \left[\frac{1}{2}s^2 - a\ln s\right]_t^{t+r} = rt + \frac{r^2}{2} - a\ln\left(1 + \frac{r}{t}\right)$$

and

$$\lim_{t\to+\infty} \int_t^{t+r} \alpha(s)ds = \infty.$$

So inequality (10) does not hold for a positive r. Nevertheless in this case our Theorem 4 holds for $a > 1$ since the left hand side of inequality (6)

$$\frac{1}{\tau(t)} \cdot \int_{t-\tau(t)}^t \tau(s)\alpha(s)ds = t\int_{t-1/t}^t \frac{1}{s}\left(s - \frac{a}{s}\right) ds = 1 - \frac{a}{t^2 - 1}$$

is not greater than the right hand side of inequality (6) which equals

$$1 - \frac{1}{t^2-1} - \frac{1}{(t^2-1)\ln(t-1/t)} - \frac{L}{(t^2-1)\ln(t-1/t)\ln_2(t-1/t)}.$$

Note that, except this, Theorem 4 generalizes Theorem 3.3 even in the case when the delay is constant, i.e. in the case when $\tau(t) \equiv \tau_0 > 0$.

4.3. Comparisons with sufficient conditions of convergence given in [9].

Sufficient conditions of convergence given in [9] (Theorems 8 – 10 in [9]), with respect to the equation (1), are:

Theorem 7. *If for a sufficiently large t*

$$\alpha(t) \le \frac{1}{\tau_0} - \frac{M_1}{t}$$

where $M_1 > 1/2$ is a constant, then each solution of Eq. (1) is convergent.

Theorem 8. *If for a sufficiently large t*

$$\int_t^{t+\tau_0} \alpha(s)ds \le 1 - \frac{\tau_0}{t} - \frac{\tau_0}{t\ln t} - \frac{M_2}{t\ln t\ln_2 t}$$

where $M_2 > \tau_0, M_2 =$const, then each solution of Eq. (1) is convergent.

Theorem 9. *If for a sufficiently large t*

$$\int_t^{t+\tau_0} \frac{\alpha(s)}{s^m}ds \le \frac{1}{t^m} - \frac{M_3}{t^{m+1}}$$

where $m, M_3 =$const, $m > 1, M_3 > \tau_0 m$, then each solution of Eq. (1) is convergent.

The Theorems 7, 8, 9 are at the same time consequences of our Theorems 3, 4, 5 if the delay is constant. Really, putting $\tau(t) \equiv \tau_0 > 0$ and $L = M_1$ in Theorem 3; $L = M_2/\tau_0$ in Theorem 4, and $L = M_3/\tau_0$ in Theorem 5 we get (after the shift $t \to t + \tau_0$), consequently, Theorems 7, 8, 9.

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A NOTE ON DIFFERENTIAL AND INTEGRAL EQUATIONS IN LOCALLY CONVEX SPACES

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Abstract. In this survey paper we consider differential and integral equations in locally convex spaces (in particular, in these sequentially complete spaces which contain a compact barrel). We present recently obtained by us results concerning the existence and topological structure of some basic nonlinear equations and we accent applications in our results some theorems of functional analysis.

AMS Subject Classification. 34G20

Keywords. existence theorems, Kneser type theorems, locally convex spaces containing a compact barrel, Sadovski measure of noncompactness.

1. Introduction

Consider the initial value problem

$$x' = f(t,x), \qquad x(0) = x_0, \tag{1}$$

where f is a bounded continuous function taking values in a quasicomplete locally convex space E. The idea to consider problem (1) in these spaces goes back to Millionščikov [13] and Hukuhara [8] who proved that (1) has a solution if the function f is compact or it satisfies the Kamke condition. The existence of solutions of (1) under different assumptions on E or f has been investigated later by many authors (see e.g. [1], [12] and [18]).

Moreover, there have appeared recently papers concerning the existence and topological structure of solutions of nonlinear integral equations in locally convex spaces (see e.g. [9] and [18]).

In Section 2 we present recently obtained by us Kneser type theorems for the equation of nth order in quasicomplete locally convex spaces. The main conditions in these results are formulated in terms of the Sadovski measure of noncompactness (see [16] for the definition and properties).

In Section 3 we consider sequentially complete locally convex spaces. Moreover, we assume that these spaces contain a compact barrel. In [1] Astala gave the following characterization of these spaces.

Lemma 1. *E is a sequentially complete locally convex space containing a compact barrel iff*

$$E = (X', \tau),$$

where X' is the dual of a barrelled normed space X and τ is a locally convex topology of X' that is stronger than the $w*$-topology but weaker than the topology of precompact convergence; briefly

$$\sigma(X', X) \leq \tau \leq \lambda(X', X).$$

By the above lemma we can use in the space E the notion of the norm.

Moreover, in [1] Astala proved that for each continuous mapping $f : [0, a] \times E \to E$, where E is as above, there exists a local solution of the problem (1). Additionally, he noted that applying the method from [17] one can prove that there exists an interval $J \subset I$ such that the set of all solutions of (1), defined on J, and considered as a subset of the space $C(J, E)$ of all continuous functions $J \to E$ with the topology of uniform convergence is compact and connected; shortly: it has the Kneser property.

Here we present results concerning the existence of continuous solutions of the nonlinear Volterra integral equation

$$x(t) = g(t) + \int\limits_{A(t)} f(t, s, x(s))ds, \qquad t \in A, \tag{2}$$

and the Urysohn integral equation

$$x(t) = g(t) + \lambda \int\limits_{A} f(t, s, x(s))ds, \qquad t \in A, \ \lambda \in \mathbb{R}, \tag{3}$$

considered in the space E, where $A = [0, a_1] \times [0, a_2] \times \ldots \times [0, a_n]$ ($a_i > 0$, $i = 1, \ldots, n$) and $A(t) = \{s \in \mathbb{R}^n : \ 0 \leq s_i \leq t_i, \ i = 1, \ldots, n\}$. In the above equations the sign "$\int$" stands for the Riemann integral.

Moreover, we characterize the topological structure of the solutions of (2) and the Darboux problem for the hyperbolic type equation.

2. Differential equation of nth order

Let E be a quasicomplete locally convex space and let $\mathcal{P}$ be a family of seminorms which generate the topology of E. Moreover, let $I = [0, a]$ be a compact interval in $\mathbb{R}$ and $B = \{x \in E : p_i(x) \leq b,\ i = 1, \dots, k\}$, $b > 0$, $k \in \mathbb{N}$ and $p_1, \dots, p_k \in \mathcal{P}$.

Consider the problem

$$x^{(n)} = f(t, x) \tag{4}$$

$$x^{(j)}(0) = x_j, \quad j = 0, \dots, n-1,$$

where $x_j \in E$ for $j = 0, \dots, n-1$, $x_0 = 0$ and $f : I \times B \to E$ is a bounded, continuous function.

Denote by $(\beta_p(\cdot))_{p \in \mathcal{P}}$ the Sadovski measure of noncompactness. Define

$$\varphi_p(t, X) = \lim_{r \to 0^+} \beta_p(f(I_{tr} \times X)) \quad \text{for } t \in (0, a) \text{ and } X \subset B,$$

where $I_{tr} = (t - r, t + r) \cap I$ (cf. [14]). Moreover, let $B_p(0, r) = \{x \in E : \ p(x) \leq r\}$.

Theorem 1. *([3]) Assume that for every seminorm $p \in \mathcal{P}$ there exists a continuous function u_p, defined on I and such that $u_p(t) > 0$ for $t > 0$, $u_p(0) = \dots = u_p^{(n-1)}(0) = 0$, $u_p^{(n)}(t)$ is positive , integrable in Lebesgue sense and*

$$\varphi_p(t, X) \leq \frac{u_p^{(n)}(t)}{u_p(t)} \beta_p(X) \tag{5}$$

for $t \in (0, a)$ and for every bounded set $X \subset B$, and

$$\lim_{t \to 0^+} \lim_{r \to 0^+} \frac{\beta_p(f(t, B_p(0, r)))}{u_p^{(n)}(t)} = 0. \tag{6}$$

Then there exists an interval $J = [0, d] \subset I$ such that the set of all solutions of (4), defined on J and considered as a subset of the space $C(J, E)$ is nonempty, compact and connected.

Note that the assumption (5) in Th. 2 is inspirated by the paper [7]. In the case of separable spaces Th. 2 has a simpler form, namely, the following theorem holds.

Theorem 2. *([3]) If the space E is separable, then Theorem 2 remains true, if one replaces the assumption (5) by the following one*

$$\beta_p(f(t, X)) \leq \frac{u_p^{(n)}(t)}{u_p(t)} \beta_p(X). \tag{7}$$

where $X \subset B$ is any bounded set, $t \in (0, a]$ and $p \in \mathcal{P}$.

Using another method of a proof as in the case of Th. 1 and Th. 2, namely Reichert's connectness principle from [15], the first author of this paper proved the following

Theorem 3. *([2]) In the assumptions of Th.1 instead of (6) assume that*

$$\lim_{t\to 0^+\ r\to 0^+} \frac{\varphi_p(t, B_p(0,r))}{u_p^{(n)}(t)} = 0.$$

Then there exists an interval $J = [0,d] \subset I$ such that the set of all solutions of (4), defined on J, is nonempty, compact and connected in $C(J,E)$.

3. Nonlinear integral equations

Consider first the equation (2). Arguing similarly as in [1] we obtain the following

Theorem 4. *([4]) Assume that the functions $g : A \to E$ and $f : A^2 \times E \to E$ are continuous. Then the equation (2) has a local continuous solution.*

To prove the above theorem we construct the sequence of the approximate solutions of the problem (2) and applying generalized Ascoli's theorem ([10], p.81) we show that this sequence has a convergent subsequence to the solution of (2).

The following Kneser-type theorem extends Th. 4.

Theorem 5. *([5]) Under the above assumptions there exists a set*

$$J = [0,d_1] \times [0,d_2] \times \ldots \times [0,d_n] \subset A$$

such that the set S of all continuous solutions of (2), defined on J, is nonempty, compact and connected in the space $C(J,E)$.

To prove Th. 5 one can not apply the method from [17]. Now, we sketch the idea of the proof of Th. 5. Let r be any positive number. Since the ball $B_r = \{x \in E : \ \|x\| \le r\}$ is convex, ballanced, closed, bounded and sequentially complete, in view of the Banach-Mackey theorem ([10], p.91) it is absorbing by the barrel and therefore it is compact. Hence for every number $r > 0$ there exists a number $m_r > 0$ such that

$$\|f(t,s,x)\| \le m_r \quad \text{for} \quad (t,s) \in A \quad \text{and} \quad x \in B_r$$

(cf. Lemma 1). Now, knowing that f is locally bounded we can define $J = [0,d_1] \times [0,d_2] \times \ldots \times [0,d_n]$ in the classical way. Denote by $\widetilde{B}$ the set of all continuous functions $J \to B_b$, where b is the suitably choosen number. We consider $\widetilde{B}$ as a subspace of $C(J,E)$. Set

$$G(x)(t) = g(t) + \int\limits_{A(t)} f(t,s,x(s))ds, \qquad t \in J,\ x \in \widetilde{B}.$$

One can easy show that $G(\widetilde{B}) \subset \widetilde{B}$ and the family $G(\widetilde{B})$ is equiuniformly continuous. Moreover, in view of the Krasnoselski-Krein-type lemma (cf. [11]) we deduce that G is continuous.

For any $\varepsilon > 0$ denote by S_ε the set of all $x \in \widetilde{B}$ such that $\|x(t) - G(x)(t)\| < \varepsilon$ for every $t \in J$. It can be proved (cf. [6]) that for sufficiently small $\varepsilon > 0$, the set S_ε is nonempty and connected. Using this fact, the generalized Ascoli theorem and the continuity of G we infer that S is nonempty and compact.

To prove that S is connected it is enough to apply standard arguments as e.g. in [6].

Now, we pass to the equation (3). As in the above theorems we assume that the functions $g : A \to E$ and $f : A^2 \times E \to E$ are continuous. Our next result is the following

Theorem 6. *([5]) Under the above assumptions there exists $\eta > 0$ such that for $\lambda \in \mathbb{R}$ with $\mid \lambda \mid< \eta$, the equation (3) has a continuous solution defined on A.*

Analogously as in the proof of Th. 5 we deduce that f is locally bounded, next we define η and the subset $\widetilde{B} \subset C(A, E)$ in the classical way. Put

$$G(x)(t) = g(t) + \lambda \int\limits_A f(t, s, x(s))ds, \quad t \in A, \quad x \in \widetilde{B}.$$

The operator G maps continuously $\widetilde{B}$ into itself. Let $V = \overline{\mathrm{conv}} G(\widetilde{B})$. By the generalized Ascoli theorem we deduce that V is compact and we can apply the Schauder-Tychonoff theorem for the mapping $G\mid_V$.

Now, let pass on to the Darboux problem for the hyperbolic partial differential equation.

Let $B = \{z \in E : \ \|z\| \le b\}$, $A = [0, a_1] \times [0, a_2]$ $(a_1, a_2 > 0)$ and let $f : A \times B \to E$ be a continuous mapping. Again, by the Banach-Mackey theorem the mapping f is norm-bounded on $A \times B$. In view of this, we choose a subrectangle $J = [0, d_1] \times [0, d_2]$ in the classical way and consider the following Darboux problem

$$\frac{\partial^2 z}{\partial x \partial y} = f(x, y, z), \qquad (x, y) \in J,$$

$$z(x, 0) = 0, \quad 0 \le x \le d_1, \qquad z(0, y) = 0, \quad 0 \le y \le d_2. \tag{8}$$

It can be easily seen that the problem (8) is equivalent to the following integral equation

$$z(x, y) = \int\limits_0^x \int\limits_0^y f(\xi, \eta, z(\xi, \eta))d\xi d\eta, \quad (x, y) \in J,$$

where the sign"$\int\int$" stands for the Riemann integral. In view of this equivalence, as a corollary from Th. 5 we obtain the following Kneser-type characterization for the problem (8).

Theorem 7. *([5]) Under the above assumptions the set of all solutions of* (8)*, defined on J, is nonempty, compact and connected in $C(J, E)$.*

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LINEAR DIFFERENTIAL EQUATIONS WITH SEVERAL UNBOUNDED DELAYS

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Abstract. The paper is concerned with the asymptotic estimate of the solutions of the delay differential equation

$$\dot{x}(t) = -a(t)x(t) + b_1(t)x(\tau_1(t)) + b_2(t)x(\tau_2(t))$$

with the continuous coefficients $a(t)$, $b_1(t)$, $b_2(t)$ and the unbounded lags. We derive the conditions under which each solution of this equation can be estimated in the terms of a solution of the system of Schröder's functional equations.

AMS Subject Classification. 34K25, 39B22

Keywords. Delay differential equation, functional equation, asymptotic behaviour of the solutions.

1. Introduction

We study the functional differential equation

$$(1) \qquad \dot{x}(t) = -a(t)x(t) + b_1(t)x(\tau_1(t)) + b_2(t)x(\tau_2(t)), \qquad t \in I = [t_0, \infty),$$

where $a(t)$ is a positive continuous function on I, $b_i(t)$ are continuous functions on I, $\tau_i(t)$ are continuously differentiable and unbounded functions on I such that $\tau_i(t_0) = t_0$, $\tau_i(t) < t$ for every $t > t_0$, $\dot{\tau}_i(t_0) < 1$ and $\dot{\tau}_i(t)$ are nonincreasing on I, $i = 1, 2$. We assume that all these conditions are fulfilled throughout the whole paper.

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The investigation of these equations has been motivated by the equation

$$\dot{x}(t) = a\,x(t) + b\,x(\lambda t), \qquad 0 < \lambda < 1 \tag{2}$$

arising in the problem of the motion of a pantograph head on an electric locomotive. Equation (2) and its modifications has been subject of numerous investigations (for the methods and results see, e.g., G. Derfel [4] , A. Iserles [6], T. Kato and J. B. McLeod [7], E. B. Lim [9], Y. Liu [10], G. Makay and J. Terjéki [11], L. Pandolfi [13] and papers [2], [3]). In this paper, we wish to extend some of the asymptotic results discussed in these papers to the case of the equation (1).

2. Preliminaries

Choose any $\sigma \in I$ and let $\sigma^* = \min\{\tau_i(\sigma),\, i = 1, 2\}$. By a solution of (1) we understand a real valued function $x(t) \in C^0([\sigma^*, \infty)) \cap C^1([\sigma, \infty))$ such that $x(t)$ satisfies (1) for every $t \geq \sigma$.

The key tool in our investigations is the theory of functional equations in a single variable. The survey of the methods and results concerning this theory can be found in the book M. Kuczma, B. Choczewski, R. Ger [8]. In this section, we mention the problem of the existence of the simultaneous solution of the system of the Schröder's equations

$$\begin{aligned} \varphi(\tau_1(t)) &= \lambda_1 \varphi(t), \\ \varphi(\tau_2(t)) &= \lambda_2 \varphi(t), \end{aligned} \tag{3}$$

where $t \in I$, λ_1 and λ_2 are suitable reals parameters. We have the following

Proposition 1. *Let $\lambda_1 = \dot{\tau}_1(t_0)$, $\lambda_2 = \dot{\tau}_2(t_0)$ and $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1$ on I. Then the system (3) has a solution $\varphi(t) \in C^1(I)$ with a positive and bounded derivative on I.*

Proof. First we consider a single equation of the system (3), e.g.,

$$\varphi(\tau_1(t)) = \lambda_1 \varphi(t), \qquad t \in I. \tag{4}$$

The existence of the solution $\varphi(t) \in C^1(I)$ having a positive derivative on I follows from the classical result of the theory of functional equations (see, e.g., [8]). Differentiating (4) we obtain

$$\dot{\varphi}(\tau_1(t)) = \frac{\lambda_1}{\dot{\tau}_1(t)} \dot{\varphi}(t).$$

The inequality $\lambda_1 / \dot{\tau}_1(t) \geq 1$ now implies the boundedness of $\dot{\varphi}(t)$ on I.

It remains to show that $\varphi(t)$ defines also a solution of the latter equation of (3). This problem has been dealt with in [1] (see also F. Neuman [12] and M. Zdun [14]). By Proposition 3 of [1], the necessary and sufficient condition for the existence of the simultaneous solution $\varphi(t)$ of (3) is the commutativity of the couple $\tau_1(t)$, $\tau_2(t)$. □

Remark 1. The required solution of (3) can be given in several important cases explicitly. These cases are discussed in Section 4.

3. Asymptotic behaviour of the solutions

In this section, we mention the main result concerning equation (1).

Theorem 1. *Let $\tau_1(t)$, $\tau_2(t)$ be commutable functions on the interval I. Let $\lambda_1 = \dot{\tau}_1(t_0)$, $\lambda_2 = \dot{\tau}_2(t_0)$ and let $\varphi(t) \in C^1(I)$ be a solution of (3) with a positive and bounded derivative on I. Let $x(t)$ be a solution of (1), where $a(t) \geq K/(\varphi(t))^\beta$ and $0 < |b_1(t)| + |b_2(t)| \leq La(t)$ for every $t \in I$ and suitable reals $K, L > 0$, $\beta < 1$. Then*

$$x(t) = O((\varphi(t))^\alpha) \qquad \text{as } t \to \infty, \qquad \alpha = \frac{\log L}{\log \lambda^{-1}}, \qquad \lambda = \max(\lambda_1, \lambda_2). \tag{5}$$

Proof. The function $\varphi(t)$ is obviously positive for all $t > t_0$. Then the substitution

$$s = \log \varphi(t), \qquad z(s) = (\varphi(t))^{-\alpha} x(t), \tag{6}$$

where $t > t_0$, converts equation (1) into the form

$$z'(s) = -(a(h(s))h'(s) + \alpha)z(s) + b_1(h(s))\lambda^\alpha h'(s) z(s - c_1) + b_2(h(s))\lambda^\alpha h'(s) z(s - c_2),$$

where $s \in J = [s_0, \infty)$. Here "$'$" means $\mathrm{d}/\mathrm{d}s$, $h(s) \equiv \varphi^{-1}(\mathrm{e}^s)$ on J, $c_1 = \log \lambda_1^{-1}$, $c_2 = \log \lambda_2^{-1}$ and $s_0 > \log \varphi(t_0)$. Then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left[\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s)\right] = \\
&b_1(h(s))\lambda^\alpha h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s - c_1) \\
&+ b_2(h(s))\lambda^\alpha h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s - c_2).
\end{aligned} \tag{7}$$

Due to the boundedness of $\dot{\varphi}(t)$ on I

$$\frac{1}{h'(s)} = \frac{\dot{\varphi}(h(s))}{\varphi(h(s))} = O(\mathrm{e}^{-s}) \qquad \text{as } s \to \infty.$$

From here we get

$$a(h(s))h'(s) \geq M\mathrm{e}^{(1-\beta)s} \tag{8}$$

for a suitable real $M > 0$ and every $s \geq s_0$. Then we can choose $d_0 \geq s_0$ such that $\alpha + a(h(s))h'(s) > 0$ for every $s \geq d_0$. Put $c = \min(c_1, c_2)$, $d_i = d_0 + ic$, $J_i = [d_{i-1}, d_i]$ and $M_i = \max\{|z(s)|, s \in \cup_{k=1}^i J_k\}$, $i = 1, 2, \ldots$. If we choose any $s^* \in J_{i+1}$, then we can integrate (7) over $[d_i, s^*]$ to obtain

$$\begin{aligned}
&\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s)\Big|_{d_i}^{s^*} = \\
&\int_{d_i}^{s^*} b_1(h(s))\lambda^\alpha h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s - c_1)\,\mathrm{d}s \\
&+ \int_{d_i}^{s^*} b_2(h(s))\lambda^\alpha h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s - c_2)\,\mathrm{d}s.
\end{aligned}$$

Then

$$\begin{aligned} z(s^*) = &\ \exp\{\alpha(d_i - s^*) - \int_{h(d_i)}^{h(s^*)} a(u)\,\mathrm{d}u\} z(d_i) \\ &+ \exp\{-\int_{s_0}^{h(s^*)} a(u)\,\mathrm{d}u - \alpha s^*\} \\ &\times (\int_{d_i}^{s^*} b_1(h(s))\lambda^\alpha h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s - c_1)\,\mathrm{d}s \\ &\quad + \int_{d_i}^{s^*} b_2(h(s))\lambda^\alpha h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s - c_2)\,\mathrm{d}s). \end{aligned}$$

Consequently,

$$\begin{aligned} (9)\ |z(s^*)| \le &\ M_i \exp\{\alpha(d_i - s^*) - \int_{h(d_i)}^{h(s^*)} a(u)\,\mathrm{d}u\} \\ &+ M_i \exp\{-\int_{s_0}^{h(s^*)} a(u)\,\mathrm{d}u - \alpha s^*\} \\ &\times \int_{d_i}^{s^*} (|b_1(h(s))| + b_2(h(s))|)\lambda^\alpha h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\,\mathrm{d}s \\ \le &\ M_i \exp\{\alpha(d_i - s^*) - \int_{h(d_i)}^{h(s^*)} a(u)\,\mathrm{d}u\} \\ &+ M_i \exp\{-\int_{s_0}^{h(s^*)} a(u)\,\mathrm{d}u - \alpha s^*\} \\ &\times \int_{d_i}^{s^*} a(h(s))h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\,\mathrm{d}s. \end{aligned}$$

Now we estimate the last integral as

$$\begin{aligned} &\int_{d_i}^{s^*} a(h(s))h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\,\mathrm{d}s \le \\ &\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\Big|_{d_i}^{s^*} + |\alpha| \int_{d_i}^{s^*} \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\,\mathrm{d}s. \end{aligned}$$

Rewrite the last term as

$$\begin{aligned} &|\alpha| \int_{d_i}^{s^*} \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\,\mathrm{d}s = \\ &\int_{d_i}^{s^*} \frac{|\alpha|}{\alpha + a(h(s))h'(s)} \frac{\mathrm{d}}{\mathrm{d}s}[\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}]\,\mathrm{d}s. \end{aligned}$$

Notice that due to (8)

$$\frac{|\alpha|}{\alpha + a(h(s))h'(s)} = O(\exp\{(\beta - 1)s\}) \qquad \text{as } s \to \infty.$$

Put $\gamma = 1-\beta > 0$. Then

$$\begin{aligned}
&\int_{d_i}^{s^*} \frac{|\alpha|}{\alpha + a(h(s))h'(s)} \frac{\mathrm{d}}{\mathrm{d}s}[\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}]\,\mathrm{d}s \le \\
&N \int_{d_i}^{s^*} \mathrm{e}^{-\gamma s} \frac{\mathrm{d}}{\mathrm{d}s}[\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}]\,\mathrm{d}s \le \\
&N\mathrm{e}^{-\gamma d_i} \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\big|_{d_i}^{s^*}
\end{aligned}$$

for a suitable $N > 0$. Consequently,

$$\begin{aligned}
&\int_{d_i}^{s^*} a(h(s))h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\,\mathrm{d}s \le \\
&\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\big|_{d_i}^{s^*} (1 + N\,\mathrm{e}^{-\gamma d_i}).
\end{aligned}$$

Substituting this back into (9) we obtain

$$\begin{aligned}
|z(s^*)| \le\ & M_i \exp\{\alpha(d_i - s^*) - \int_{h(d_i)}^{h(s^*)} a(u)\,\mathrm{d}u\} \\
&+ M_i \exp\{-\int_{s_0}^{h(s^*)} a(u)\,\mathrm{d}u - \alpha s^*\} \\
&\times \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\big|_{d_i}^{s^*} (1 + N\,\mathrm{e}^{-\gamma d_i}) \\
\le\ & M_i(1 + N\,\mathrm{e}^{-\gamma d_i}).
\end{aligned}$$

Consequently,

$$M_{i+1} \le M_i(1 + N\,\mathrm{e}^{-\gamma d_i}) \le M_1 \prod_{k=1}^{i} (1 + N\,\mathrm{e}^{-\gamma d_k}), \qquad i = 1, 2, \ldots.$$

Letting $i \to \infty$ we can see that the infinite product

$$\prod_{k=1}^{\infty} (1 + N\,\mathrm{e}^{-\gamma d_k})$$

converges. This implies that (M_i) is bounded as $i \to \infty$, hence $z(s)$ is bounded as $s \to \infty$. Substituting this back into (6) we obtain the asymptotic property (5). This completes the proof. □

Remark 2. The validity of the previous statement can be easily generalized to the case when equation (1) with m delayed arguments is considered.

Remark 3. It is easy to verify that the function $\omega(t) = \big(\varphi(t)\big)^\alpha$ occuring in (5) defines the solution of the functional equation

$$\omega\big(\tau(t)\big) = \frac{\omega(t)}{L},$$

where $\tau(t) = \max\big(\tau_1(t), \tau_2(t)\big)$, $t > t_0$.

4. Applications

In this section, we specify delays $\tau_1(t)$, $\tau_2(t)$ in (1) to illustrate our asymptotic result.

Example 1. We consider the equation

$$\dot{x}(t) = -a(t)x(t) + b_1(t)x(\lambda_1 t) + b_2(t)x(\lambda_2 t), \qquad t \in I = [0, \infty), \tag{10}$$

where $0 < \lambda_1 < \lambda_2 < 1$, $a(t)$, $b_1(t)$, $b_2(t) \in C^0(I)$. The corresponding system of Schröder's equations is

$$\begin{aligned}\varphi(\lambda_1 t) &= \lambda_1 \varphi(t),\\ \varphi(\lambda_2 t) &= \lambda_2 \varphi(t)\end{aligned}$$

and admits the identity function $\varphi(t) = t$ as the required solution. Then we can reformulate the main result as follows:

Let $a(t) \geq K\big/t^\beta$, $0 < |b_1(t)| + |b_2(t)| \leq La(t)$ for every $t \in I$ and suitable reals K, $L > 0$ and $\beta < 1$. If $x(t)$ is a solution of (10), then

$$x(t) = O(t^\alpha) \qquad \text{as } t \to \infty, \qquad \alpha = \frac{\log L}{\log \lambda_2^{-1}}.$$

This asymptotic estimate generalizes some parts of [7], [11] and [3]. Particularly, if we consider the equation

$$\dot{x}(t) = \beta_1(t)[x(\lambda_1 t) - x(t)] + \beta_2(t)[x(\lambda_2 t) - x(t)], \qquad t \in I \tag{11}$$

(i.e. $L = 1$), where $\beta_1(t)$, $\beta_2(t) \geq K\big/t^\beta$ for every $t \in I$ and suitable reals $K > 0$, $\beta < 1$, then all the solutions of (11) are bounded. We note that equation (11) with $\beta_i(t) < 0$ and constant delays has been investigated by J. Diblík [5].

Example 2. Now we investigate the asymptotic behaviour of the solutions of the equation

$$\dot{x}(t) = -a(t)x(t) + b_1(t)x(t^{\gamma_1}) + b_2(t)x(t^{\gamma_2}), \qquad t \in I = [1, \infty), \tag{12}$$

where $0 < \gamma_1 < \gamma_2 < 1$, and $a(t)$, $b_1(t)$, $b_2(t) \in C^0(I)$. It is easy to verify that the corresponding system of Schröder's equations

$$\begin{aligned}\varphi(t^{\gamma_1}) &= \gamma_1 \varphi(t),\\ \varphi(t^{\gamma_2}) &= \gamma_2 \varphi(t)\end{aligned}$$

has the solution $\varphi(t) = \log t$. Substituting this into (5) we get that if $a(t) \geq K\big/(\log t)^\beta$ and $0 < |b_1(t)| + |b_2(t)| \leq La(t)$ for every $t \in I$ and suitable reals K, $L > 0$, $\beta < 1$, then

$$x(t) = O\big((\log t)^\alpha\big) \qquad \text{as } t \to \infty, \qquad \alpha = \frac{\log L}{\log \gamma_2^{-1}}$$

for all the solutions $x(t)$ of (12).

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MULTISTRUCTURES DETERMINED BY DIFFERENTIAL RINGS

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Abstract. Multistructures namely hypergroups are playing very essential role in contemporary mathematics. This contribution aims at some natural constructions of such multistructures defined on differential rings with differentiation operators which can be especially applied to rings of continuously differentiable functions.

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The systematic study of algebraic aspects of transformations of differential and difference operators applied to investigation of differential and difference equations is of a persistent interest. General algebraic approach to the transformation theory is described in [13], more in detail see also [12] and other related papers of Professor Neuman. This fruitful direction has been iniciated by Professor O. Borůvka in the 1950s in the framework of his intensive research of linear differential transformations of the second order - [3]. The theory dominating by high level of algebraization and geometrization is developing by the Borůvka's school and his succesors up to present times.

In contemporary investigations of algebraic and geometrical structures an important role is playing by hyperstructures, formerly called multistructures, which occur very naturally in convexity theory, harmonic analysis, in projective and affine geometry, in the decomposition theory of noncommutative algebraic structures and elsewhere, cf. [2,4,6,7,8,11,14,15,16].

In this contribution we give construction of multistructures determined by quasi-orders defined by means of derivation operators on differential rings. Some constructions based on results of [4], par.1 chapter IV and of paper [5] are possible for general differential rings, the other are specialized.

Recall basic concepts overtaken e.g. from [6]. A *multigroupoid* or a *hypergroupoid* (in recent literature) is a pair $(M, \cdot)$, where M is a nonempty set and a mapping $\cdot : M \times M \to \mathcal{P}^*(M)$ (the system of all nonempty subsets of M) is a binary *multioperation* called also a *hyperoperation*. This multioperation is usually extended onto the powerset $\mathcal{P}(M)$ by the rule $A.B = \bigcup\{a.b; a \in A, b \in B\}$ for any pair $A \neq \emptyset \neq B$, where $A, B \subset M$ and moreover by $\emptyset.A = \emptyset = A.\emptyset$. It is to be noted that operations on powersets of carriers of ternary relational structures were used by Professor M. Novotný in a series of his papers - started by [14] - including also investigations of relationships between ternary structures and multistructures. If this multioperation is associative (here $A.B = \bigcup\{a.b; a \in A, b \in B\}$ for any pair $A \neq \emptyset \neq B$, $A, B \subseteq M$) then $(M, \cdot)$ is called a *semihypergroup*, if $(M, \cdot)$, moreover, satisfies the reproduction axiom - $a.M = M = M.a$ for any $a \in M$ - then $(M, \cdot)$ is said to be a *multigroup* or a *hypergroup*. We will use the latter terms. A hypergroupoid satisfying the reproduction axiom is called a *quasi-hypergroup*.

Let $(R, +, ., \Delta_R)$ be a commutative *differential ring*, i.e. $(R, +, .)$ is a commutative ring, Δ_R is a set of derivations on the set R, which means that Δ_R is a subset of the endomorphism monoid $\mathrm{End}(R, +)$ of the additive abelian group of the ring $(R, +, .)$ satisfying the differentiation rule. Thus for $d \in \Delta_R$ and any pair of elements $x, y \in R$ we have $d(x + y) = d(x) + d(y)$ and $d(x.y) = d(x).y + x.d(y)$. Moreover we suppose that any $d : R \to R$ is surjective. A differential structure Δ_R of a ring can be endowed with the Lie multiplication $d_1 \diamond_L d_2 = d_1d_2 - d_2d_1$; then $(R, +, \diamond_L)$ is a Lie ring of derivations. If $\Delta_R = \{d\}$ is a singleton we say that this differential structure is *monogeneous*.

By $\mathbf{R}, \mathbf{R}^+, \mathbf{N}$ we denote the set of all real, positive real numbers, positive integers, respectively.

Examples 1. Let $J = (a, b) \subseteq \mathbf{R}$ (possibly $J = \mathbf{R}$) and $\mathbf{C}^\infty(J)$ - as usually - be the ring of real functions $f : J \to \mathbf{R}$ with continuous derivatives of all orders. If $\Delta = \{\frac{d}{dx}\}$, where $\frac{df}{dx} = f'$ is the usual derivative of a function $f \in \mathbf{C}^\infty(J)$, then $(\mathbf{C}^\infty(J), +, ., \Delta)$ is a differential ring with a monogenous differential structure.

2. Let $\mathbf{R}[x_1, \dots, x_n]$, $[x_1, \dots, x_n] \in \mathbf{R}^n$ (for a fixed integer n) be the ring of all polynomials with coefficients in the field $(\mathbf{R}, +, .)$. Denoting $\Delta = \{\sum_1^n \lambda_k.\frac{\partial}{\partial x_k}; [\lambda_1, \dots, \lambda_n] \in \mathbf{R}^n\}$ we obtain $(\mathbf{R}[x_1, \dots, x_n], +, ., \Delta)$ as an example of a differential ring.

Other examples can be found e.g. in [9,10] The join operation $\cdot$ in a hypergroupoid $(M, \cdot)$ has two inverses - right extension and left extension - defined by $a/b = \{x; a \in x \cdot b\}$ and $b\backslash a = \{x; a \in b \cdot x\}$ called also *right* and *left fractions*, respectively. The reproductive axiom for $(M, \cdot)$ is easily seen to be equivalent to the condition that fractions $a/b, b\backslash a$ are nonempty for any pair $a, b \in M$. In the case of a commutative join operation $\cdot$ evidently $a/b = b\backslash a$. Now, a hypergroup $(M, \cdot)$ is called a *join space* if it is commutative and satisfies the *transposition axiom*: For any quadruple $a, b, c, d \in M$ the implication $a/b \cap c/d \neq \emptyset \Rightarrow a.d \cap c.d \neq \emptyset$ is valid -

[6,7,8]. The concept of a join space has been introduced by W. Prenowitz and used by him and afterwards by him and J. Jantosciak to build again several branches of geometry. Recall that a self-map f of a hypergroupoid $(M, .)$ is called a *good endomorphism* of $(M, .)$ if it satisfies these set equalities $f(x, y) = f(x).f(y)$ for any pair $x, y \in M$.

Let $(R, +, \cdot, \Delta_R)$ be a differential (non necessary commutative) ring, $M(\Delta_R)$ be the free monoid over Δ_R within the full transformation monoid of R (i.e. $M(\Delta_R)$ is the set of all finite words $d_1 \dots d_n$, $d_k \in \Delta_R$ including the empty word Λ, identified with the identity operator id_R, endowed with the binary operation of concatenation). We define $d_1 \dots d_n(x) = d_n(\dots(d_1(x))\dots)$ which means application of the composition of operators $d_1, \dots, d_n$- in this order - to the element $x \in R$.

Theorem 1. *Let $(R, +, \cdot, \Delta_R)$ be a differential ring. Let $x * y = \{d_1 \dots d_n(z); z \in \{x, y\}, d_k \in \Delta_R, n \in \mathbf{N}\} = \{\delta(z); z \in \{x, y\}, \delta \in M(\Delta_R)\}$. Then we have*

1° *$(R, *)$ is a commutative hypergroup such that any differential endomorphism of the ring $(R, +, \cdot, \Delta_R)$ (i.e. $f \in \mathrm{End}(R, +, \cdot)$ with $f(d_k(x)) = d_k(f(x)), x \in R$) is a good endomorphism of $(R, *)$.*

2° *The hypergroup $(R, *)$ satisfies the transposition law, hence it is a join space if and only if for any pair of elements $x, y \in R$ such that there exists a pair of words $(\delta, \sigma) \in M(\Delta_R) \times M(\Delta_R)$ and a suitable element $z \in R$ with $\delta(z) = x, \sigma(z) = y$, we have $\tau(x) = \omega(y)$ for some pair of words $\tau \in M(\Delta_R)$, $\omega \in M(\Delta_R)$.*

Proof. Define a binary relation $r \subset R \times R$ by xry whenever there exists an m-tuple of derivations operators $d_1, \dots, d_m \in \Delta_R$, i.e. a word $\delta = d_1 \dots d_m \in M(\Delta_R)$ such that $y = \delta(x)$. The relation r is reflexive (if $d_1 = \dots = d_m = id_R$) and transitive: For $x, y, z \in R$ such that xry, yrz, i.e. $y = \delta(x), z = \sigma(y)$ for suitable words $\delta, \sigma \in M(\Delta_R)$ we get $z = \delta\sigma(x) = \sigma(\delta(x))$, with $\delta\sigma \in M(\Delta_R)$, thus xrz. If for arbitrary pair $x, y \in R$ we define

$$x * y = \{\delta(z); z \in \{x, y\}, \delta \in M(\Delta_R)\} =$$
$$= \{\delta(x); \delta \in M(\Delta_R)\} \cup \{\delta(y); \delta \in M(\Delta_R)\} = r(x) \cup r(y)$$

then by the fundamental construction [4], or [5] and [16] we have that $(R, *)$ is a commutative hypergroup. Further, if $f : R \to R$ is a differential endomorphism of the ring $(R, +, ., \Delta_R)$ which means $f \in \mathrm{End}(R, +, .)$ and $f(d(x)) = d(f(x))$ for any $d \in \Delta_R$ and any $x \in R$, then by the induction $f(\delta(x)) = \delta(f(x))$ for any word $\delta \in M(\Delta_R)$ and each element $x \in R$, thus for any pair $x, y \in R$ we have

$$f(x * y) = \{f(\delta(z)); z \in \{x, y\}, \delta \in M(\Delta_R)\} =$$
$$= \{\delta(f(z)); z \in \{x, y\}, \delta \in M(\Delta_R)\} = f(x) * f(y).$$

Hence the assertion 1° is true.

Finally, the monoid $M(\Delta_R)$ acts on the set R. By [5] Theorem 6 the hypergroup $(R, *)$ is a join space iff for every pair of elements $x, y \in R$ such that there exists a pair of words $\delta_1, \sigma_1 \in M(\Delta_R)$ and an element $z \in R$ with $\delta(z) = x, \sigma(z) = y$, we have $\tau(x) = \omega(y)$ for suitable words $\tau, \omega \in M(\Delta_R)$, thus we obtain the assertion 2°.

Remark. Using principal ideals within differential images of the carrier set R of a differential ring $(R, +, ., \{d\})$ with a monogeneous differential structure, we can construct a countable set (in general) of commutative extensive hypergroups $(R, \circ_m)$ with the same carrier R. (Extensivity of a hyperoperation $\circ$ means $x, y \in x \circ y$ for all $x, y \in R$.) This construction is based on [4], chapt.IV, Theorem 2.1 which is generalized in [15] - Propositions 2,3. More in detail, for a given positive integer $m \in \mathbf{N}$ we define

$$x \circ_m y = \{z \in R; x.d^m(R) \subseteq z.d^m(R) \text{ or } y.d^m(R) \subseteq z.d^m(R)\},$$

where $d^m(R) = \{d^m(x); x \in R\}$. Then by the above mentioned theorems we obtain that $(R, \circ_m)$ is a commutative extensive hypergroup.

Theorem 2. *Let $(R, +, \cdot, \Delta_R)$ be a commutative differential ring with a monogeneous differential structure $\Delta_R = \{d\}$. Let $(R, *_d)$ be a commutative hypergroupoid defined by the indefinite integral $x *_d y = d^{-1}(x + y)$ for all $x, y \in R$. Then $(R, *_d)$ is a commutative quasi-hypergroup such that $(x + y)/(u + v) = x/u + y/v$ for any quadruple $x, y, u, v \in R$ and for arbitrary triad $x, y, z \in R$ we have*

1° $x/y = d(x) - y$,

2° $d(x) = (x + y)/z - y/z$,

3° $d(x/y) = d(x)/d(y)$,

4° $d(x *_d x + y *_d y) = d(x *_d y) + d(x *_d y)$.

Proof. We show first that the hypergroupoid $(R, *_d)$ satisfies the reproduction axiom.

Let $a \in R$ be an arbitrary element. Since $a *_d R \subseteq R$ and $(R, *_d)$ is commutative if suffices to prove the inclusion $R \subseteq a *_d R$. For any $x \in R$ then $d^{-1}(x) = I(x) = \{y \in R; d(y) = x\}$ is called the indefinite integral of x. Now, for arbitrary $b \in R$ we denote $x_b = d(b) - a$. Then $d(b) = a + x_b$, i.e. $b \in d^{-1}(a + x_b) = I(a + x_b) = a *_d x_b \subseteq \bigcup_{x \in R} a *_d x = a *_d R$, hence $a *_d R = R = R *_d a$ for any $a \in R$. It is easy to see that $(R, *_d)$ is not associative in general, thus $(R, *_d)$ is a commutative quasi-hypergroup. Further, for $x, y, u, v \in R$ arbitrary we have

1° $x/y = \{z \in R; x \in z *_d y\} = \{z \in R; x \in I(z + y)\}$, thus $x \in z *_d y$ iff $d(x) = z + y$, thus $z = d(x) - y$, hence we get that $z \in x/y$ iff $z = d(x) - y$ consequently $x/y = d(x) - y$ which is a singleton.

Now $x/u + y/v = d(x) - u + d(y) - v = d(x + y) - (u + v) = (x + y)/(u + v)$.

2° For any $x, y, z \in R$ we have $(x + y)/z = d(x + y) - z = d(x) + d(y) - z = d(x) + y/z$, therefore $d(x) = (x + y)/z - y/z$. Similarly,

3° $d(x/y) = d(d(x) - y) = d(d(x)) - d(y) = d(x)/d(y)$ and

4° $d(x *_d y) + d(x *_d y) = d(d^{-1}(x + y)) + d(d^{-1}(x + y)) = x + y + x + y = x+x+y+y = d(d^{-1}(x+x+y+y)) = d(d^{-1}(x+x)+d^{-1}(y+y)) = d^{-1}(x*_d x+y*_d y)$.

Now we specialize our considerations to the classical differential rings of real functions $f \in \mathbf{C}^\infty(J), J = (a, b) \subseteq \mathbf{R}$ (not excluding the case $J = \mathbf{R}$) with the usual differentiation. For any $f \in \mathbf{C}^\infty(J)$ we denote by $\int f(x)dx$ the set of all primitive functions to f, i.e.$\int f(x)dx = \{F : J \to \mathbf{R}; F'(x) = f(x), x \in J\}$. For any pair of function $\varphi, \psi \in \mathbf{C}^\infty(J)$ we define a hyperoperation $\star$ on the ring $\mathbf{C}^\infty(J)$ by

$$f \star_{(\varphi,\psi)} g = \int (\varphi'(x)f(x) + \psi'(x)g(x))dx, \quad f, g \in \mathbf{C}^\infty(J).$$

Evidently, $(\mathbf{C}^\infty(J), \star_{(\varphi,\psi)})$ is a hypergroupoid (noncommutative in general).

Theorem 3. *Let $J \subseteq \mathbf{R}$ be an open interval, $\varphi, \psi \in \mathbf{C}^\infty(J)$ be a pair of strictly monotone functions (i.e. $\varphi'(x).\psi'(x) \neq 0$ for all $x \in J$). Then the hypergroupoid $(\mathbf{C}^\infty(J), \star_{(\varphi,\psi)})$ is a quasi-hypergroup (i.e. it satisfies the reproduction axiom) which is commutative if and only if the difference $\varphi - \psi$ on the interval J is a constant function.*

Proof. Clearly, for any pair $f, g \in \mathbf{C}^\infty(J)$ and any function $h \in f \star_{(\varphi_1,\varphi_2)} g$ we have $h \in \mathbf{C}^\infty(J)$. Suppose $f \in \mathbf{C}^\infty(J)$ is an arbitrary function. Then evidently

$$f \star_{(\varphi_1,\varphi_2)} \mathbf{C}^\infty(J) = \bigcup\{f \star_{(\varphi_1,\varphi_2)} g; g \in \mathbf{C}^\infty(J)\} \subseteq \mathbf{C}^\infty(J)$$

and

$\mathbf{C}^\infty(J) \star_{(\varphi_1,\varphi_2)} f \subseteq \mathbf{C}^\infty(J)$, as well. We prove the opposite inclusions.

Suppose that $g \in \mathbf{C}^\infty(J)$ is an arbitrary function. Define

$$h_1(x) = \tfrac{1}{\varphi_2'(x)}(g'(x) - \varphi_1'(x)f(x)), \quad x \in J.$$

Since $\varphi_1'(x).\varphi_2'(x) \neq 0$ for each $x \in J$, then $\varphi_2'(x) \neq 0$ for any $x \in J$, thus the function $\frac{1}{\varphi_2'(x)}$ is defined on the interval J and $\frac{1}{\varphi_2'(x)} \in \mathbf{C}^\infty(J)$, $g'(x) - \varphi_1'(x)f(x) \in \mathbf{C}^\infty(J)$, hence $h_1 \in \mathbf{C}^\infty(J)$. Then

$$f \star_{(\varphi_1,\varphi_2)} h_1 = \int(\varphi_1'(x)f(x) + \varphi_2'(x)h_1(x))dx = \int g'(x)dx = \{g(x) + c; c \in \mathbf{R}\},$$

thus

$$g \in f \star_{(\varphi_1,\varphi_2)} h_1 \subseteq \bigcup\{f \star_{(\varphi_1,\varphi_2)} h; h \in \mathbf{C}^\infty(J)\}.$$

Similarly if we define

$$h_2(x) = \tfrac{1}{\varphi_1'(x)}(g'(x) - \varphi_2'(x)f(x)), \quad x \in J,$$

then the assumption $\varphi_1'(x) \neq 0$ for any $x \in J$ and $f, g, \varphi_1, \varphi_2 \in \mathbf{C}^\infty(J)$ implies $h_2 \in \mathbf{C}^\infty(J)$. Further,

$$h_2 \star_{(\varphi_1,\varphi_2)} f = \int(\varphi_1'(x)h_2(x) + \varphi_2'(x)f(x))dx = \int g'(x)dx = \{g(x) + c; c \in \mathbf{R}\},$$

thus - similarly as above - we have

$$g \in h_2 \star_{(\varphi_1,\varphi_2)} f \subseteq \bigcup\{h \star_{(\varphi_1,\varphi_2)} f; h \in \mathbf{C}^\infty(J)\} = \mathbf{C}^\infty(J) \star_{(\varphi_1,\varphi_2)} f.$$

Hence

$$\mathbf{C}^\infty(J) \subseteq (f \star_{(\varphi_1,\varphi_2)} \mathbf{C}^\infty(J)) \cap (\mathbf{C}^\infty(J) \star_{(\varphi_1,\varphi_2)} f),$$

consequently the hypergroupoid $(\mathbf{C}^\infty(J), \star_{(\varphi_1,\varphi_2)})$ satisfies the reproduction axiom. Therefore it is a quasi-hypergroup.

Now suppose $\varphi_1(x) - \varphi_2(x) = c$ for some real number $c \in \mathbf{R}$. Then $\varphi_1' = \varphi_2'$ and $f \star_{(\varphi_1,\varphi_2)} g = g \star_{(\varphi_1,\varphi_2)} f$ for any pair of functions $f, g \in \mathbf{C}^\infty(J)$. On the contrary, if the hyperoperation $\star_{(\varphi_1,\varphi_2)}$ is commutative then $\int(\varphi_1'(x)f(x) + \varphi_2'(x)g(x))dx = \int(\varphi_1'(x)g(x) + \varphi_2'(x)f(x))dx$ which is equivalent to

$$\int(\varphi_1'(x) - \varphi_2'(x))(f(x) - g(x))dx = 0. \tag{1}$$

Especially for $f(x) = g(x) + 1, x \in J$ the equality (1) gives $\int(\varphi_1'(x) - \varphi_2'(x))dx = 0$, which implies $\varphi_1'(x) - \varphi_2'(x) = 0$ thus $\varphi_1(x) - \varphi_2(x)$ is a constant function.

Remark. It is easy to see that the hyperoperation

$$\star_{(\varphi_1,\varphi_2)} : \mathbf{C}^\infty(J) \times \mathbf{C}^\infty(J) \to \mathcal{P}^*(\mathbf{C}^\infty(J))$$

is not associative. In a special case $\varphi_1(x) = \varphi_2(x) = x, x \in J$, i.e. within the commutative quasi-hypergroup $(\mathbf{C}^\infty(J), *)$, where $f * g = \int(f(x) + g(x))dx$ for any pair $f, g \in \mathbf{C}^\infty(J)$, we get from Theorem 2 $(1^\circ, 2^\circ, 3^\circ)$ the following rules:

$$f(x)/g(x) = \tfrac{df(x)}{dx} - g(x),\ \tfrac{d}{dx}(f(x)/g(x)) = \tfrac{df(x)}{dx}/\tfrac{dg(x)}{dx},$$
$$\tfrac{df(x)}{dx} = (f(x) + g(x))/h(x) - g(x)/h(x)$$

for arbitrary $f, g, h \in \mathbf{C}^\infty(J)$. Moreover, for any quadruple $f, g, u, v \in \mathbf{C}^\infty(J)$ then we have $(f(x)+g(x))/(u(x)+v(x)) = f(x)/u(x)+g(x)/v(x)$. Using derivatives of functions from $\mathbf{C}^\infty(J)$ we can expressed certain sufficient conditions for validity of transposition law for the quasi-hypergroup $(\mathbf{C}^\infty(J), \star_{(\varphi_1,\varphi_2)})$. Moreover, transposition hypergroups, forming an important class of hypergroups, can be constructed from quasi-ordered groups and monoids of some transformation operators of rings of continuously differentiable functions. These operators yielding substitutions for some classes of ordinary differential equations will be investigated in a forthcoming paper.

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EXISTENCE OF POSITIVE SOLUTIONS OF N-DIMENSIONAL SYSTEM OF NONLINEAR DIFFERENTIAL EQUATIONS ENTERING INTO A SINGULAR POINT

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Abstract. An n-dimensional system of nonlinear differential equations is considered. It is shown that a singular initial problem has at least one solution (or infinitely many solutions) with positive coordinates. Moreover, asymptotic behaviour of these solutions is described by means of the curves that are defined implicitly.

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1. Introduction

This paper deals with the existence of solutions of the singular initial problem, stated for n-dimensional nonlinear systems of differential equations, entering into a singular point. Namely, we will consider the initial problems ($S_\pm$), (IP) where

$$g_i(x)\, y_i' = \pm \left[\sum_{j=1}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x) \right], \quad i = 1, 2, \dots, n, \tag{S$_\pm$}$$

$$y_i(0^+) = 0, \quad i = 1, 2, \dots, n. \tag{IP}$$

Let us denote as I_{x_0} an interval of the form $I_{x_0} = (0, x_0]$ and I_{y_0} an interval of the form $I_{y_0} = (0, y_0]$ with $x_0, y_0 > 0$. The systems ($S_\pm$) will be considered under the following main assumptions:

(C1) $g_i \in C(I_{x_0}, \mathbb{R}^+)$, $i = 1, 2, \dots, n$, $\mathbb{R}^+ = (0, \infty)$;

(C2) $\alpha_i \in C^1(I_{y_0}, \mathbb{R})$, $\alpha_i > 0$ on I_{y_0}, $\alpha_i' > 0$ on $I_{y_0}, \alpha_i(0^+) = 0$, $i = 1, 2, \dots, n$;

(C3) $\omega_i \in C^1(I_{x_0}, \mathbb{R})$, $\omega_i > 0$ on I_{x_0}, $\omega_i' > 0$ on $I_{x_0}, \omega_i(0^+) = 0$, $i = 1, 2, \dots, n$;

(C4) $a_{ij} = \text{const}$, $i, j = 1, 2, \dots, n$; $a_{ii} > 0, a_{ij} \le 0$, $i, j = 1, 2, \dots, n$, $i \ne j$, $\Delta = \det A > 0$, $A = (a_{ij})_{i,j=1}^n$, and cofactors $C_{ij} = (-1)^{i+j} A_{ij} \ge 0$ where A_{ij} are minors of the elements a_{ij} of the matrix A.

We will consider the systems ($\mathrm{S}_\pm$) in the domain $Q \equiv I_{x_0} \times \underbrace{I_{y_0} \times \cdots \times I_{y_0}}_{n}$, i.e. the corresponding results will concern the existence of solutions of this problem having positive coordinates. More precisely, we define a solution of the problems ($\mathrm{S}_\pm$), (IP) in the sense of the following definition:

Definition 1. A function $y = y(x) = (y_1(x), \dots, y_n(x)) \in C^1(I_{x^*}, \mathbb{R}^n)$ with $0 < x^* \le x_0$ is said to be a *solution of the singular problem* (S_+), (IP) (or (S_-), (IP)) on interval I_{x^*} if:

1) $(x, y(x)) \in Q$ for $x \in I_{x^*}$;
2) y satisfies (S_+) (or (S_-)) on I_{x^*};
3) $y_i(0^+) = 0$, $i = 1, 2, \dots, n$.

The origin of coordinates $O = (0, 0, \dots, 0)$ is a boundary point of introduced domain Q. The possibility $g_i(0^+) = 0$, $i = 1, 2, \dots, n$ is not excluded from our investigation (note that, for validity of assumptions of the theorems formulated below, this condition is often tacitly assumed). So the problems ($\mathrm{S}_\pm$), (IP), in view of assumptions (C1)–(C4), are really the singular problems and known theorems about existence of solution of initial problems cannot be used.

Various initial singular problems for ordinary differential equations were widely considered (let us cite e.g. the works of K. Balla [1], J. Baštinec and J. Diblík [2], J. Diblík [3]–[5], I.T. Kiguradze [11], N.B. Konyukhova [12], Chr. Nowak [13], D. O'Regan [14], M. Růžičková [15]), namely after the appearance of the pioneering work of V.A. Chechyk [10], the solvability of the considered problems ($\mathrm{S}_\pm$), (IP) cannot be established by using the results which are known to the authors of this paper.

Let us explain the scheme of our investigation. We consider the implicit system of nondifferential equations with respect to unknowns $z_1, \dots, z_n$ which arise if in the systems ($\mathrm{S}_\pm$) the left-hand sides equal zero (i.e. if $g_i(x) \equiv 0, i = 1, 2, \dots, n$):

$$\sum_{j=1}^{n} a_{ij}\alpha_j(z_j) = \omega_i(x), \quad i = 1, 2, \dots, n. \tag{1}$$

Considering this system we conclude that it is equivalent to the system consisting of separated scalar equations:

$$\alpha(z) = \Omega(x) \equiv A^{-1}\omega(x)$$

with $\alpha(z) = (\alpha_1(z_1), \dots, \alpha_n(z_n))^T$, $\varOmega = (\varOmega_1, \dots, \varOmega_n)^T$, $\omega = (\omega_1, \dots, \omega_n)^T$ or

$$\alpha_i(z_i) = \varOmega_i(x), \quad i = 1, 2, \dots, n$$

with (see (C4))

$$\varOmega_i(x) \equiv \frac{1}{\varDelta} \cdot \sum_{j=1}^{n} C_{ji}\omega_j(x), \ \ i = 1, 2, \dots, n. \tag{2}$$

Note that in the view of our assumptions $\varOmega_i(x) > 0$ on I_{x_0} and $\varOmega_i(0^+) = 0$, $i = 1, 2, \dots, n$. Solving these scalar equations with respect to z_i, $i = 1, 2, \dots, n$ we get

$$z_i = \alpha_i^{-1}[\varOmega_i(x)], \quad i = 1, 2, \dots, n,$$

where α_i^{-1} is the inverse function of the function α_i (existence of it follows from the condition (C2)).

It can be expected that under appropriate conditions the asymptotic behaviour of a solution $y(x) = (y_1(x), \dots, y_n(x))$ of the problem (S$_+$), (IP) (or (S$_-$), (IP)) for $x \to 0^+$ will be in a sense similar to the asymptotic behaviour of $z(x) = (z_1(x), \dots, z_n(x))$ for $x \to 0^+$, i.e. it can be expected that the asymptotic formulae

$$y_i(x) \approx z_i(x) \quad \text{if} \quad x \to 0^+, \quad i = 1, 2, \dots, n$$

will hold.

The proofs of Theorem 1 and Theorem 2 below are based on known qualitative properties of solutions of differential equations. Besides, in the proof of Theorem 2 the topological method of T. Ważewski is used (see, e.g., [9], [16]). Except this, properties of functions that are defined implicitly are applied in these proofs. Let us note that one of the advantages of our results is the fact that although properties of implicit functions are used, the assumptions of them are easily verifiable and do not use any supposition which cannot be verified immediately. Moreover, it is easy to get corresponding linear cases as a consequence of our results. Results of this paper generalize previous ones, given in the work [8].

2. Auxiliary Lemmas

Let us state a lemma on existence and differentiability of a function given implicitly by the equation

$$\tilde{\alpha}(y) = \tilde{\omega}(x) \qquad \text{if} \quad (x, y) \in I_{x_0} \times I_{y_0}. \tag{3}$$

Lemma 1. *Let the following assumptions be valid:*

$$\tilde{\alpha} \in C^1(I_{y_0}, \mathbb{R}),\ \tilde{\alpha} > 0 \text{ on } I_{y_0},\ \tilde{\alpha}' > 0 \text{ on } I_{y_0} \text{ and } \tilde{\alpha}(0^+) = 0;$$
$$\tilde{\omega} \in C^1(I_{x_0}, \mathbb{R}),\ \tilde{\omega} > 0 \text{ on } I_{x_0},\ \tilde{\omega}' > 0 \text{ on } I_{x_0} \text{ and } \tilde{\omega}(0^+) = 0.$$

Then there exists a unique solution

$$y = \varphi(x) \equiv \tilde{\alpha}^{-1}[\tilde{\omega}(x)]$$

of equation (3) *on an interval* $I_{\delta_0} \subset I_{x_0}$ *with properties:*

$$\varphi \in C^1(I_{\delta_0}, \mathbb{R}),\ \ \varphi \in I_{y_0} \ \ \textit{and} \ \ \varphi' > 0 \ \ \textit{on} \ \ I_{\delta_0};$$
$$\varphi(0^+) = 0;$$
$$\varphi'(x) \equiv \frac{\tilde{\omega}'(x)}{\tilde{\alpha}'[\varphi(x)]}\,,\ \ x \in I_{\delta_0}.$$

The proof of Lemma 1 can be made in an elementary way and is therefore omitted.

Remark 1. The next obvious property will be used in the sequel: let $\varepsilon_1, \varepsilon_2$ be two positive constants and $\varepsilon_1 < \varepsilon_2$. Then there exists an interval $I_{\delta_1} \subset I_{\delta_0}$ (determined by the requirement $\varphi(\varepsilon_2 x) \le y_0$ on I_{δ_1}, i.e. $\delta_1 = \min\{\delta_0 \varepsilon_2^{-1}, \delta_0\}$) such that the inequality $\varphi(\varepsilon_1 x) < \varphi(\varepsilon_2 x)$ holds on I_{δ_1}.

Lemma 2. *Let all assumptions of Lemma* 1 *be valid and, moreover, there exist a constant* $M \in \mathbb{R}^+$ *such that*

$$\tilde{\alpha}(y) \le M \tilde{\alpha}'(y), \quad y \in I_{y_0}.$$

Then the unique solution $y = \varphi(x)$ *of equation* (3) *defined on an interval* $I_{\delta_0} \subset I_{x_0}$ *satisfies the inequality:*

$$\varphi'(x) \le M \cdot \frac{\tilde{\omega}'(x)}{\tilde{\omega}(x)},\ \ x \in I_{\delta_0}.$$

Proof. In view of equation (3) and the affirmation of Lemma 1 we get

$$\varphi'(x) = \frac{\tilde{\omega}'(x)}{\tilde{\alpha}'[\varphi(x)]} = \frac{\tilde{\omega}'(x)}{\tilde{\omega}(x)} \cdot \frac{\tilde{\alpha}[\varphi(x)]}{\tilde{\alpha}'[\varphi(x)]} \le M \cdot \frac{\tilde{\omega}'(x)}{\tilde{\omega}(x)}, \quad x \in I_{\delta_0}.$$

Lemma 3. *Let the assumptions* (C2)–(C4) *be valid. Then the implicit equations*

$$\alpha_i(z_i) = \Omega_i(x),\ \ i = 1, 2, \ldots, n \tag{4}$$

define on an interval $I_{\delta_2} \subset I_{x_0}$ *implicit functions*

$$z_i = \varphi_i(x) \equiv \alpha_i^{-1}[\Omega_i(x)],\ \ i = 1, 2, \ldots, n \tag{5}$$

satisfying the properties

$$\varphi_i \in C^1(I_{\delta_2}, \mathbb{R}),\ \ \varphi_i \in I_{y_0} \ \ \textit{and} \ \ \varphi_i' > 0 \ \ \textit{on} \ \ I_{\delta_2},\ i = 1, 2, \ldots, n; \tag{6}$$

$$\varphi_i(0^+) = 0, \quad i = 1, 2, \ldots, n;$$

$$\varphi_i'(x) \equiv \frac{\Omega_i'(x)}{\alpha_i'[\varphi_i(x)]}\,,\ \ i = 1, 2, \ldots, n. \tag{7}$$

If, moreover, there exists a constant $M \in \mathbb{R}^+$ such that

$$\alpha_i(y_i) \le M\alpha_i'(y_i),\ \ y \in I_{y_0},\ \ i = 1, 2, \dots, n,$$

then

$$\varphi_i'(x) \le M \cdot \frac{\Omega_i'(x)}{\Omega_i(x)},\ \ \ i = 1, 2, \dots, n. \tag{8}$$

Proof. The proof follows immediately from Lemmas 1 and 2 if in their formulations $\tilde{\alpha} \equiv \alpha_i$ and $\tilde{\omega} \equiv \Omega_i$, $i = 1, 2, \dots, n$ are put since all Ω_i, $i = 1, 2, \dots, n$, defined by (2) satisfy necessary conditions. The value of δ_2 can be taken as minimal value of all corresponding δ_{0i}, $i = 1, 2, \dots, n$.

3. Singular problem (S_+), (IP)

Let us consider the singular problem (S_+) and (IP), i.e. the problem

$$g_i(x)\, y_i' = \sum_{j=1}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x),\ \ i = 1, 2, \dots, n, \tag{S_+}$$

$$y_i(0^+) = 0,\ \ i = 1, 2, \dots, n. \tag{IP}$$

Theorem 1. *Suppose that conditions* (C1)–(C4) *are satisfied, there exist constants $k \in \mathbb{R}^+$, $M \in \mathbb{R}^+$, $k > 1$ and an interval $I_{x^{**}}$ with $x^{**} \le \min\left\{x_0 k^{-1}, y_0\right\}$ such that for $x \in I_{x^{**}}$:*

$$\omega_i(kx) > \omega_i(x) + kMg_i(x) \cdot \frac{\sum_{j=1}^{n} C_{ji}\,\omega_j'(kx)}{\sum_{j=1}^{n} C_{ji}\,\omega_j(kx)}\,,\ \ \ \ i = 1, 2, \dots, n; \tag{i}$$

$$\alpha_i(x) \le M\alpha_i'(x),\ \ \ \ i = 1, 2, \dots, n. \tag{ii}$$

Then there exist infinitely many solutions of the problem (S_+), (IP) *on an interval $I_{x^*} \subseteq I_{x^{**}}$.*

Proof. Let $\varphi_i(x)$, $i = 1, 2, \dots, n$ be the implicit functions defined on the interval I_{δ_2} by means of relations (4) or (5) (see Lemma 3). Let us define a domain Ω_1^0 of the form

$$\Omega_1^0 = \{(x, y) \in Q : x \in (0, \delta_3), \varphi_i(x) < y_i < \varphi_i(kx),\ i = 1, 2, \dots, n\},$$

supposing, without loss of generality, that $\delta_3 \le \delta_2$ is so small that $\varphi_i(kx) < y_0$, $i = 1, 2, \dots, n$ on I_{δ_3}. (Note that in accordance with Remark 1, $\varphi_i(x) < \varphi_i(kx)$, $i = 1, 2, \dots, n$, $x \in I_{\delta_3}$.)

Let us define auxiliary functions

$$u_i(x,y) \equiv u_i(x,y_i) \equiv \big(y_i - \varphi_i(x)\big)\big(y_i - \varphi_i(kx)\big), \ \ i = 1,2,\dots,n$$

and

$$v(x,y) \equiv v(x) \equiv x - \delta_3.$$

Then the domain Ω_1^0 can be written as

$$\Omega_1^0 = \{(x,y) \in Q : u_i(x,y) < 0, \ \ i = 1,2,\dots,n, \ v(x,y) < 0\}.$$

In the next we will show that all points of the sets

$$U_{1i} = \{(x,y) \in Q : u_i(x,y) = 0, u_j(x,y) \le 0, \ \ j = 1,2,\dots,n, \ j \ne i, \\ v(x,y) \le 0\}, \quad i = 1,2,\dots,n,$$

are the points of strict egress of the set Ω_1^0 with respect to the system (S_+). (For the corresponding definitions of this notion and for further details here and in the sequel we refer, e.g., to the book [9]. The notation used in the proof is taken from this book as well. Except this, the technique used is punctually explained e.g. in the papers [3,6,7] and [15].)

For verifying this we will compute the full derivatives of the functions $u_i(x,y)$, $i = 1,2,\dots,n$ along the trajectories of the system (S_+) on corresponding sets U_{1i}, $i = 1,2,\dots,n$. Let the index i be fixed. Then

$$\begin{aligned} \frac{du_i(x,y)}{dx} &= (y_i' - \varphi_i'(x))\,(y_i - \varphi_i(kx)) + \big(y_i - \varphi_i(x)\big)\big(y_i' - k\varphi_i'(kx)\big) = \\ &= \left[\frac{1}{g_i(x)} \cdot \left(\sum_{j=1}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - \varphi_i'(x)\right] \big(y_i - \varphi_i(kx)\big) + \\ &\quad + \big(y_i - \varphi_i(x)\big) \left[\frac{1}{g_i(x)} \cdot \left(\sum_{j=1}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - k\varphi_i'(kx)\right]. \end{aligned}$$

If $(x,y) \in U_{1i}$ then either $y_i = \varphi_i(x)$ and $\varphi_j(x) \le y_j \le \varphi_j(kx)$, $j = 1,2,\dots,n$, $j \ne i$, or $y_i = \varphi_i(kx)$ and $\varphi_j(x) \le y_j \le \varphi_j(kx)$, $j = 1,2,\dots,n$, $j \ne i$. In the first case i.e. if

$$(x,y) \in U_{1i},\ y_i = \varphi_i(x), \ \ \varphi_j(x) \le y_j \le \varphi_j(kx),\ j = 1,2,\dots,n,\ j \ne i \tag{9}$$

we have

$$\left.\frac{du_i(x,y)}{dx}\right|_{(x,y)\in U_{1i}, y_i = \varphi_i(x)} =$$

$$= \left[\frac{1}{g_i(x)} \cdot \left(a_{ii}\alpha_i(\varphi_i(x)) + \sum_{j=1, j\ne i}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - \varphi_i'(x)\right] \times$$

$$\times \big(\varphi_i(x) - \varphi_i(kx)\big) = [\text{see (1) with } z_i = \varphi_i(x)] =$$

$$\left[\frac{1}{g_i(x)}\cdot\left(\omega_i(x)-\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j(\varphi_j(x))+\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j(y_j)-\omega_i(x)\right)-\varphi_i'(x)\right]\times$$
$$\times\big(\varphi_i(x)-\varphi_i(kx)\big)=$$
$$=\left[-\frac{1}{g_i(x)}\cdot\sum_{j=1,j\neq i}^{n}a_{ij}\left[\alpha_j(\varphi_j(x))-\alpha_j(y_j)\right]-\varphi_i'(x)\right]\cdot\big(\varphi_i(x)-\varphi_i(kx)\big)\geq$$
$$\geq[\text{due to (C2)}, (6) \text{ and } (9)]\geq-\varphi_i'(x)\big(\varphi_i(x)-\varphi_i(kx)\big)>0.$$

Thus, points $(x,y)\in U_{1i}$ if $y_i=\varphi_i(x)$ are points of strict egress. In the second case, i.e. if

(10) $\qquad (x,y)\in U_{1i},\ y_i=\varphi_i(kx),\ \varphi_j(x)\leq y_j\leq\varphi_j(kx),\ j=1,2,\dots,n,\ j\neq i,$

direct computation yields:

$$\left.\frac{du_i(x,y)}{dx}\right|_{(x,y)\in U_{1i},y_i=\varphi_i(kx)}=\big(\varphi_i(kx)-\varphi_i(x)\big)\times$$
$$\times\left[\frac{1}{g_i(x)}\cdot\left(a_{ii}\alpha_i(\varphi_i(kx))+\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j(y_j)-\omega_i(x)\right)-k\varphi_i'(kx)\right]=$$
$$=[\text{in view of (1) with } z_i=\varphi_i(x), (2), (4) \text{ and } (7)]=\big(\varphi_i(kx)-\varphi_i(x)\big)\times$$
$$\times\left[\frac{1}{g_i(x)}\cdot\left(\omega_i(kx)-\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j\big(\varphi_j(kx)\big)+\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j(y_j)-\omega_i(x)\right)-\right.$$
$$\left.-k\frac{\Omega_i'(kx)\alpha_i\big(\varphi_i(kx)\big)}{\Omega_i(kx)\alpha_i'\big(\varphi_i(kx)\big)}\right]=\big(\varphi_i(kx)-\varphi_i(x)\big)\times$$
$$\times\left[\frac{1}{g_i(x)}\cdot\left([\omega_i(kx)-\omega_i(x)]-\sum_{j=1,j\neq i}^{n}a_{ij}\left[\alpha_j(\varphi_j(kx))-\alpha_j(y_j)\right]\right)-\right.$$
$$\left.-k\frac{\Omega_i'(kx)\alpha_i\big(\varphi_i(kx)\big)}{\Omega_i(kx)\alpha_i'\big(\varphi_i(kx)\big)}\right]\geq[\text{in view of (C2)}, (2), (6), (8), (10) \text{ and (ii)}]\geq$$
$$\geq\big(\varphi_i(kx)-\varphi_i(x)\big)\left[\frac{\omega_i(kx)-\omega_i(x)}{g_i(x)}-kM\cdot\frac{\sum_{j=1}^{n}C_{ji}\omega_j'(kx)}{\sum_{j=1}^{n}C_{ji}\omega_j(kx)}\right]>$$
$$>[\text{in view of (i)}]>0.$$

This means that in both of the cases considered,

$$\left.\frac{du_i(x,y)}{dx}\right|_{(x,y)\in U_{1i}} > 0, \quad i = 1, 2, \ldots, n. \tag{11}$$

So, points of the sets U_{1i}, $i = 1, 2, \ldots, n$ are the points of strict egress. Inequality (11) simultaneously says that, if orientation of the x-axis is changed into reverse orientation, points $(x, y) \in U_{1i}$, $i = 1, 2, \ldots, n$ are points of strict ingress and every point of the set

$$S = \{(x, y) \in Q : x = \delta_3,\ \varphi_i(x) < y_i < \varphi_i(kx),\ i = 1, 2, \ldots, n\}$$

defines a unique solution $y = y^*(x)$ such that $(x, y^*(x)) \in \Omega_1^0$ on interval I_{δ_3}, i.e. this solution solves the problem (S_+), (IP). Put $x^* = \delta_3$. Now Theorem 1 is proved.

Corollary 1. *The affirmation of the Theorem 1 can be improved. Namely, as it follows from proof above, there exist infinitely many solutions $y = y^*(x)$ of the problem (S_+), (IP), each of which satisfies, on the interval I_{x^*}, the inequalities*

$$\varphi_i(x) < y_i^*(x) < \varphi_i(kx),\ \ i = 1, 2, \ldots, n.$$

4. Singular problem (S_-), (IP)

Now consider the singular problem (S_-), (IP), i.e. the problem

$$g_i(x)\, y_i' = -\sum_{j=1}^{n} a_{ij}\alpha_j(y_j) + \omega_i(x),\ \ i = 1, 2, \ldots, n, \tag{S_-}$$

$$y_i(0^+) = 0,\ \ i = 1, 2, \ldots, n. \tag{IP}$$

Theorem 2. *Suppose that conditions* (C1)–(C4) *are satisfied, there exist constants $k \in \mathbb{R}^+$, $M \in \mathbb{R}^+$, $k < 1$ and an interval $I_{x^{**}}$ with $x^{**} \le \min\{x_0, y_0\}$ such that for $x \in I_{x^{**}}$:*

$$\omega_i(x) > \omega_i(kx) + kM g_i(x) \cdot \frac{\sum\limits_{j=1}^{n} C_{ji}\, \omega_j'(kx)}{\sum\limits_{j=1}^{n} C_{ji}\, \omega_j(kx)}, \quad i = 1, 2, \ldots, n \tag{iii}$$

and the condition (ii) holds. Then there exists at least one solution $y = y^*(x)$ of the problem (S_-), (IP) on an interval $I_{x^*} \subseteq I_{x^{**}}$.

Proof. Introduce a domain Ω_2^0 of the form

$$\Omega_2^0 = \{(x, y) \in Q : x \in (0, \delta_2), \varphi_i(kx) < y_i < \varphi_i(x),\ i = 1, 2, \ldots, n\},$$

where δ_2 was defined in Lemma 3 and $\varphi_i(x)$, $i = 1, 2, \dots, n$ are defined as in the proof of Theorem 1. (Note that in the case considered $\varphi_i(kx) < \varphi_i(x)$, $i = 1, 2, \dots, n$, $x \in I_{\delta_2}$.)

Let us define auxiliary functions

$$u_i(x, y) \equiv u_i(x, y_i) \equiv \big(y_i - \varphi_i(x)\big)\big(y_i - \varphi_i(kx)\big), \ \ i = 1, 2, \dots, n$$

and

$$v(x, y) \equiv v(x) \equiv x - \delta_2.$$

Then the domain Ω_2^0 can be written as

$$\Omega_2^0 = \{(x, y) \in Q : u_i(x, y) < 0, \ i = 1, 2, \dots, n, \ v(x, y) < 0\}.$$

In the following we will show that all points of the sets

$$U_{2i} = \{(x, y) \in Q : u_i(x, y) = 0, u_j(x, y) \le 0, \ j = 1, 2, \dots, n, \ j \ne i, \\ v(x, y) \le 0\}, \quad i = 1, 2, \dots, n,$$

are the points of strict ingress of the set Ω_2^0 with respect to the system (S_-).

Analogously as in the proof of Theorem 1 we compute the full derivatives of the functions $u_i(x, y)$, $i = 1, 2, \dots, n$ along the trajectories of the system (S_-) on corresponding sets U_{2i}, $i = 1, 2, \dots, n$. Let the index i be fixed. Then

$$\begin{aligned}\frac{du_i(x,y)}{dx} &= \left(y_i' - \varphi_i'(x)\right)\left(y_i - \varphi_i(kx)\right) + \big(y_i - \varphi_i(x)\big)\big(y_i' - k\varphi_i'(kx)\big) = \\ &= \left[-\frac{1}{g_i(x)} \cdot \left(\sum_{j=1}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - \varphi_i'(x)\right]\left(y_i - \varphi_i(kx)\right) + \\ &\quad + \big(y_i - \varphi_i(x)\big)\left[-\frac{1}{g_i(x)} \cdot \left(\sum_{j=1}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - k\varphi_i'(kx)\right].\end{aligned}$$

If $(x, y) \in U_{2i}$ then either $y_i = \varphi_i(x)$ and $\varphi_j(kx) \le y_j \le \varphi_j(x)$, $j = 1, 2, \dots, n$, $j \ne i$, or $y_i = \varphi_i(kx)$ and $\varphi_j(kx) \le y_j \le \varphi_j(x)$, $j = 1, 2, \dots, n$, $j \ne i$. In the first case i.e. if

$$(x, y) \in U_{2i}, \ y_i = \varphi_i(x), \ \varphi_j(kx) \le y_j \le \varphi_j(x), \ j = 1, 2, \dots, n, \ j \ne i \tag{12}$$

we have

$$\left.\frac{du_i(x,y)}{dx}\right|_{(x,y)\in U_{2i}, y_i = \varphi_i(x)} =$$

$$= \left[-\frac{1}{g_i(x)} \cdot \left(a_{ii}\alpha_i(\varphi_i(x)) + \sum_{j=1, j\ne i}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - \varphi_i'(x)\right] \times$$

$$\times \big(\varphi_i(x) - \varphi_i(kx)\big) = [\text{see } (1) \text{ with } z_i = \varphi_i(x)] =$$

$$\left[\frac{-1}{g_i(x)} \cdot \left(\omega_i(x) - \sum_{j=1, j\neq i}^{n} a_{ij}\alpha_j(\varphi_j(x)) + \sum_{j=1, j\neq i}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - \varphi_i'(x)\right] \times$$

$$\times \big(\varphi_i(x) - \varphi_i(kx)\big) =$$

$$= \left[\frac{1}{g_i(x)} \cdot \sum_{j=1, j\neq i}^{n} a_{ij}\big[\alpha_j(\varphi_j(x)) - \alpha_j(y_j)\big] - \varphi_i'(x)\right] \cdot \big(\varphi_i(x) - \varphi_i(kx)\big) \leq$$

$$\leq [\text{due to (C2), (6) and (12)}] \leq -\varphi_i'(x)\big(\varphi_i(x) - \varphi_i(kx)\big) < 0.$$

Thus, points $(x, y) \in U_{2i}$ if $y_i = \varphi_i(x)$ are points of strict ingress. In the second case, i.e. if

$$(x, y) \in U_{2i},\ y_i = \varphi_i(kx),\ \varphi_j(kx) \leq y_j \leq \varphi_j(x),\ j = 1, 2, \dots, n,\ j \neq i, \tag{13}$$

direct computation yields:

$$\left.\frac{du_i(x, y)}{dx}\right|_{(x,y)\in U_{2i}, y_i=\varphi_i(kx)} = \big(\varphi_i(kx) - \varphi_i(x)\big) \times$$

$$\times \left[-\frac{1}{g_i(x)} \cdot \left(a_{ii}\alpha_i(\varphi_i(kx)) + \sum_{j=1, j\neq i}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - k\varphi_i'(kx)\right] =$$

$$= [\text{in view of (1) with } z_i = \varphi_i(x), (2), (4) \text{ and } (7)] = \big(\varphi_i(kx) - \varphi_i(x)\big) \times$$

$$\times \left[-\frac{1}{g_i(x)} \cdot \left(\omega_i(kx) - \sum_{j=1, j\neq i}^{n} a_{ij}\alpha_j\big(\varphi_j(kx)\big) + \sum_{j=1, j\neq i}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - \right.$$

$$\left. -k\frac{\Omega_i'(kx)\alpha_i\big(\varphi_i(kx)\big)}{\Omega_i(kx)\alpha_i'\big(\varphi_i(kx)\big)}\right] = \big(\varphi_i(kx) - \varphi_i(x)\big) \times$$

$$\times \left[\frac{1}{g_i(x)} \cdot \left([\omega_i(x) - \omega_i(kx)] + \sum_{j=1, j\neq i}^{n} a_{ij}\big[\alpha_j(\varphi_j(kx) - \alpha_j(y_j)\big]\right) - \right.$$

$$\left. -k\frac{\Omega_i'(kx)\alpha_i\big(\varphi_i(kx)\big)}{\Omega_i(kx)\alpha_i'\big(\varphi_i(kx)\big)}\right] \leq [\text{in view of (C2), (2), (6), (8), (13) and (ii)}] \leq$$

$$\leq \big(\varphi_i(kx) - \varphi_i(x)\big) \left[\frac{\omega_i(x) - \omega_i(kx)}{g_i(x)} - kM \cdot \frac{\sum_{j=1}^{n} C_{ji}\omega_j'(kx)}{\sum_{j=1}^{n} C_{ji}\omega_j(kx)}\right] <$$

$$< [\text{in view of (iii)}] < 0.$$

This means that in both of the cases considered

$$\left.\frac{du_i(x,y)}{dx}\right|_{(x,y)\in U_{2i}} < 0, \quad i = 1,2,\ldots,n. \tag{14}$$

So, points of the sets U_{2i}, $i = 1,2,\ldots,n$ are the points of strict ingress. Inequality (14) simultaneously says that, if orientation of the x-axis is changed into reverse orientation, points $(x,y) \in U_{2i}$, $i = 1,2,\ldots,n$ are points of strict egress. Let us define the set

$$S = \{(x,y) \in Q : x = \delta_2,\ \varphi_i(kx) \le y_i \le \varphi_i(x),\ \ i = 1,2,\ldots,n\}.$$

It is easy to show that its boundary

$$\partial S = \left\{(x,y) \in \bigcup_{i=1}^{n} U_{2i} : x = \delta_2\right\}$$

is not a retract of S but is a retract of the set $\bigcup\limits_{i=1}^{n} U_{2i}$. Then, according to Ważewski's principle, there is a point $(\delta_2, y^*) \in S \setminus \partial S$ such that the graph of corresponding solution $y = y^*(x)$ with $y^*(\delta_2) = y^*$ lies in the domain Ω_2^0 for $x \in (0,\delta_2]$. Therefore this solution solves simultaneously the problem (S_-), (IP). Put $x^* = \delta_2$. The theorem is proved.

Corollary 2. *The affirmation of the Theorem 2 can be improved. The solution $y = y^*(x)$ of the problem (S_-), (IP) as it follows from the proof of Theorem 2, satisfies the inequalities*

$$\varphi_i(kx) < y_i^*(x) < \varphi_i(x), \ \ i = 1,2,\ldots,n,\ x \in (0,x^*].$$

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THE GENERALIZED COINCIDENCE INDEX – APPLICATION TO A BOUNDARY VALUE PROBLEM

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Abstract. In this paper we investigate a general boundary value problem, which can be rewritten to the coincidence problem of the form $L(x) = F(x)$, where L is a Fredholm operator of nonnegative index and F is not necessarily compact map. We apply a homotopy invariant called a coincidence index.

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1. Introduction

Let $AC = AC([0,T], E)$ be the space of absolutely continuous functions $u : [0,T] \to E$ defined on the unit interval $[0,T]$ with values in a Banach space E and let $f : [0,1] \times E \times E \to E$ be a Caratheodory map, what means that $f(\cdot, u, v)$ is mesurable for every $(u,v) \in E \times E$ and $f(t, \cdot, \cdot)$ is continuous for a.a. $t \in [0,T]$. If we are to study the existence of solutions to the general boundary value problem

$$\begin{cases} u'(t) = f(t, u(t), u'(t)) \\ l_1(u(0)) + l_2(u(T)) = \alpha(u), \end{cases} \tag{1}$$

where $l_1, l_2 : E \to E'$ are linear bounded maps, $\alpha : AC \multimap E'$ is a continuous map, (E' is a Banach space) then we reformulate it to the following:

$$\begin{cases} y(t) = f(t, z + \int_0^t y(s)ds, y(t)) \\ l_1(z) + l_2(z + \int_0^T y(s)ds) = \alpha(z + \int_0^{\cdot} y(s)ds). \end{cases} \tag{2}$$

Obviously, if $(z, y) \in E \times AC$ is a solution to the problem (2), then $u(t) = z + \int_0^t y(s)ds$ is a solution to the problem (1).

Putting
$$x = (z, y),$$
$$L(z, y) = (y, l_1(z) + l_2(z))$$
and
$$F(z, y) = \left(f(\cdot, z + \int_0^{\cdot} y(s)ds, y(\cdot)), \alpha(z + \int_0^{\cdot} y(s)ds)) - l_2(\int_0^T y(s)ds) \right)$$
we arrive at a coincidence problem (a generalized fixed point problem) of the form

$$L(x) = F(x). \tag{3}$$

Such coincidence problems have been intensively studied by many authors, especially in case when F is a compact (single- or multivalued)map and L is the identity (the Leray-Schauder fixed point theory) or L is a Fredholm operator of index 0 (e.g. Mawhin [9], Pruszko [10]) or of nonnegative index (Kryszewski [8]). The situation when L is a Fredholm operator of nonnegative index and F belongs to a more general class of nonlinear (single- or multivalued) transformations, so called L–fundamentally contractible maps was investigated in [4]. We use some theoretical results from this paper, but not in the most general case (i.e. only for singlevalued maps) .

Observe that in case $E = E'$, $l_1 = id_E$, $l_2 = -id_E$ and $\alpha \equiv 0$, (1) becomes an ordinary periodic boundary value problem.

In Section 1 we introduce some notions and cite a few results and in Section 2 we carefully describe and solve our problem.

Throughout the paper we will use the following notation: if U is a subset of a Banach space E, then by $\mathrm{cl}\, U$ we mean the closure of U, by $\mathrm{bd}\, U$ - the boundary of U, $\mathrm{conv}\,(U)$ - the convex hull of U and $\overline{\mathrm{conv}}\,(U) = \mathrm{cl}\,\mathrm{conv}\,(U)$. Moreover, let $B^E(x_0, r) = \{x \in E; ||x_0 - x||_E \leq r\}$ and if $E = \mathbb{R}^n$, then $B^n(x_0, r) := B^{\mathbb{R}^n}(x_0, r)$.

2. Preliminaries

Let E, E' be Banach spaces with norms $|| \cdot ||_E$, $|| \cdot ||_{E'}$, respectively. A bounded linear map $L : E \to E'$ is a *Fredholm operator* if dimensions of its kernel ($\mathrm{Ker}\, L$) and cokernel ($\mathrm{Coker}\, L := E'/\mathrm{Im}\,(L)$, where $\mathrm{Im}\,(L)$ is the image of L), are finite. By the index of a Fredholm operator L we mean the number
$$i(L) := \dim \mathrm{Ker}\, L - \dim \mathrm{Coker}\, L.[1]$$

Since both $\mathrm{Ker}\,(L)$ and $\mathrm{Im}\,(L)$ are direct summands in E and E', respectively, we may consider continuous linear projections $P : E \to E$ and $Q : E' \to E'$, such

[1] Observe that if $L : \mathbb{R}^m \to \mathbb{R}^n$ is linear, then L is Fredholm and $i(L) = m - n$.

that $\operatorname{Ker} L = \operatorname{Im}(P)$ and $\operatorname{Ker} Q = \operatorname{Im}(L)$. Clearly E, E' split into (topological) direct sums

$$\operatorname{Ker}(P) \oplus \operatorname{Ker}(L) = E, \qquad \operatorname{Im}(Q) \oplus \operatorname{Im}(L) = E'.$$

Moreover, since $\operatorname{Im}(L)$ is a closed subspace of E', $L|_{\operatorname{Ker} P} : \operatorname{Ker} P \to \operatorname{Im} L$ is a linear homeomorphism onto $\operatorname{Im}(L)$. Denote by K_P the inverse isomorphism for $L|_{\operatorname{Ker} P}$. Note also that L is proper when restricted to a closed bounded set.

Consider a continuous map $F : X \to E'$, where $X \subset E$.

Definition 1. A closed convex and nonempty set $K \subset E'$ is called *L-fundamental* for F, provided

(i) $F(L^{-1}(K) \cap X) \subset K$; and
(ii) if for $x \in X$, $L(x) \in \overline{\operatorname{conv}}(F(x) \cup K)$, then $L(x) \in K$.

It is clear that for any F some L-fundamental set exists (for instance whole E' or $\overline{\operatorname{conv}}(F(X))$).

Observe that if $E = E'$ and $L = id_E$ is the identity on E, then K is nothing else but a fundamental set for F in the sense of e.g. [2] (see also references therein).

Some properties of L-fundamental sets are summarized in the following result (comp. [4] or [5]).

Proposition 1.

(i) If K is an L-fundamental set for F, then $\{x \in X \mid L(x) = F(x)\} \subset L^{-1}(K)$.
(ii) If K_1, K_2 are L-fundamental sets for F, then the set $K := K_1 \cap K_2$ is L-fundamental or empty.
(iii) If $P \subset K$ and K is an L-fundamental set for F, then so is $K' = \overline{\operatorname{conv}}(F(L^{-1}(K) \cap X) \cup P)$.
(iv) If K is the intersection of all L-fundamental sets for F, then

$$K = \overline{\operatorname{conv}}(F(L^{-1}(K) \cap X)).$$

(v) For any $A \subset E'$, there exists an L-fundamental set K such that $K = \overline{\operatorname{conv}}(F(L^{-1}(K) \cap X) \cup A)$.

Definition 2. We say that F is an *L-fundamentally restrictible map* if for any $y \in E'$ there exists a *compact* L-fundamental set for F, which contains y.

Let us collect some important examples of L-fundamentally restrictible maps.

Example 1. Let $L : E \to E'$ be an arbitrary Fredholm operator.

(i) if $F : X \to E'$ is compact (i.e. $\operatorname{cl} F(X)$ is compact), then $K = \overline{\operatorname{conv}}(F(X) \times \{y\})$ is a compact L- fundamental set for F; hence F is L-fundamentally restrictible.

(ii) Let μ be a measure of noncompactness in E' having usual properties (see e.g. [1]) and let F be L-condensing in the sense that, for any bounded set $A \subset X$, if $\mu(F(A)) \geq \mu(L(A))$, then A is compact. If F is bounded, then one shows that an L-fundamental set K, satisfying $K = \overline{\mathrm{conv}}\,(F(L^{-1}(K) \cap X) \cup \{y\})$ for some $y \in E'$ (see Proposition 1) is compact; hence F is L-fundamentally restrictible.

(iii) If F is an L-set contraction (i.e. there exists $k \in (0,1)$, such that for any bounded $A \subset X$, $\mu(F(A)) \leq k\mu(L(A))$), then F is L-condensing and therefore L-fundamentally restrictible.

Some other examples one can find in [4] and in [5].

Now we are going to sketch the construction of a generalized index of coincidence between L and an L-fundamentally restrictible map F. More details (in more general, multivalued case), one can find e.g. in [3] or in [5].

Let U be an open bounded subset of $\mathbb{R}^m$ and let $F : \mathrm{cl}\, U \to \mathbb{R}^n$, where $m \geq$ $\geq n \geq 1$ and suppose that $0 \notin F(x)$ for $x \in \mathrm{bd}\, U$. It implies that there is $\varepsilon > 0$ such that $F(\mathrm{bd}\, U) \subset \mathbb{R}^n \setminus B^n(0,\varepsilon)$.

We can of course define the Brouwer degree for such map, but if $m > n$ it is useless, because always equal to 0. Better homotopy invariant defined Kryszewski (comp. [8]), developing some ideas from [6]. In this definition he used cohomotopy sets. Consider the following sequence of maps:

$$\pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\varepsilon)) \xrightarrow{F^{\#}} \pi^n(\mathrm{cl}\, U, \mathrm{bd}\, U) \xleftarrow{i_1^{\#}} \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus U) \xrightarrow{i_2^{\#}}$$
$$\xrightarrow{i_2^{\#}} \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,r)),$$

where $r > 0$ is such that $U \subset B^m(0,r)$ and $i_1 : (\mathrm{cl}\, U, \mathrm{bd}\, U) \to (\mathbb{R}^m, \mathbb{R}^m \setminus U)$ and $i_2 : (\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,r)) \to (\mathbb{R}^m, \mathbb{R}^m \setminus U)$ are inclusions. Arrows denote maps between cohomotopy sets induced by respective maps (see [7]). By the excision property $i_1^{\#}$ is a bijection. Hence we have defined the transformation

$$\begin{aligned}\mathcal{K} := i_2^{\#} \circ (i_1^{\#})^{-1} \circ F^{\#} : \pi^n(S^n) = \pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\varepsilon)) \to \\ \to \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,r)) = \pi^n(S^m).\end{aligned} \tag{4}$$

Definition 3. By the *generalized degree* of the map F on U we understand the element

$$\deg((F, U, 0) := \mathcal{K}(\mathbf{1}) \in \pi^n(S^m).$$

($\mathbf{1}$ denotes the generator of $\pi^n(S^n) \cong \mathbb{Z}$, i.e. the homotopy class of the identity map $id : S^n \to S^n$.)

It is clear that this definition does not depend on the choice of ε and r.

Remark 1. One can check that if $n = m$, then $\deg(F, U, 0) \in \pi^n(S^n)$ is nothing else but the ordinary Brouwer degree of the map F (comp. the Hopf theorem [7], th.11.5).

Now we are going to define a generalized index of coincidence between a Fredholm operator L of index $i(L) = k$ and an L-fundamentally restrictible map $F : X \to E'$, where X is open subset of E and E, E' are Banach spaces. Suppose that $C := \{x \in X \;\;|\;\; L(x) \in F(x)\}$ is bounded and closed. Therefore there is an open bounded set U such that $C \subset U \subset \operatorname{cl} U \subset X$. Let K_0 be *any* compact L-fundamental set for F. In view of Proposition 1 (i), C is contained in $L^{-1}(K_0) \cap \operatorname{cl} U$. Since $L|_{\operatorname{cl} U}$ is proper, we gather that C being obviously closed is also compact. Now let consider a map

$$F_{|(L^{-1}(K_0)\cap X)} : L^{-1}(K_0) \cap X \to E',$$

According to Definition 1, the range of this map is contained in K_0. Hence it has a compact extension

$$\overline{F} : X \to K_0\text{[2]}.$$

It is clear that $\{x \in X \;\;|\;\; L(x) = \overline{F}(x)\} = C$.

There is $\varepsilon_0 > 0$ such that

$$\{y \in E' \;\;|\;\; \exists_{x\in \operatorname{bd} U} \;\; y = L(x) - F(x)\} \cap B^{E'}(0, 2\varepsilon_0) = \emptyset.$$

Take $\varepsilon \in (0, \varepsilon_0]$ and let $l_\varepsilon : \operatorname{cl} \overline{F}(U) \to E'$ be a Schauder projection of the compact set $\operatorname{cl} \overline{F}(U)$ into a finite dimensional subspace Z of E', such that $||l_\varepsilon(y) - y||_{E'} < \varepsilon$ for $y \in \operatorname{cl} \overline{F}(U)$. Denote by W' the finite dimensional subspace of $\operatorname{Im}(L)$ such that $Z \subset W = W' \oplus \operatorname{Im}(Q)$. Put $T := L^{-1}(W)$, $U_W = U \cap T$. It is clear that the closure $\operatorname{cl} U_W$ (in T) is contained in $\operatorname{cl} U \cap T$ and its boundary $\operatorname{bd} U_W$ (relative T) in $\operatorname{bd} U \cap T$. Further let $\overline{F_W} = l_\varepsilon \circ \overline{F}|_{\operatorname{cl} U_W}$ and $L_W = L|_T : T \to W$. Observe, that L_W is a Fredholm operator of index

$$i(L_W) = \dim T - \dim W = k = i(L).$$

Enlarging W' if necessary we may assume that $\dim W := n \geq k + 2$. Putting $m := \dim T = n + k$ we arrive in a finite dimensional situation discussed above.

Definition 4. By the *generalized index of the L-fundamentally restrictible map F* we understand the element

$$\operatorname{Ind}_L(F, X) := \deg(L_W - \overline{F_W}), U_W, 0) \in \Pi_k.$$

By definition, $\deg(L_W - \overline{F_W}, U_W, 0)$ belongs to $\pi^n(S^m)$ but since $m < 2n - 1$ we know that $\pi^n(S^m) \cong \Pi_k$.

One can check (see [5] or [3]) that the definition does not depend on the choice of a compact L-fundamental set K_0, an extension $\overline{F}$ of $F|_{L^{-1}(K_0)\cap X}$, an open subset U, a number $\varepsilon \in (0, \varepsilon_0]$, a projection l_ε and a space W'.

[2] For instance one can take any retraction $r : E' \to K_0$ and define $\overline{F} := r \circ F$.

Definition 5. Given L-fundamentally restrictible maps F_0, F_1 we say that they are (L,K)–*homotopic* (written $F_0 \simeq_K F_1$) if there is a homotopy $H : X \times [0,1] \to E'$ such that the set $\{x \in X \mid L(x) \in H(x,t))$ for some $t \in [0,1]\}$ is bounded and closed in E and K is a compact L-fundamental set for any map $X \ni x \mapsto H(x,t)$ where $t \in [0,1]$.

At the first glance the above definition of homotopic pairs is enough for our next considerations (comp. Theorem 1), but in applications we need the following more general one.

Definition 6. Two L-fundamentally restrictible maps F_0, F_1 are L-*homotopic* if there is a finite number of compact convex sets $K_1, \ldots, K_n$ and L-fundamentally restrictible maps $G_1, \ldots, G_{n-1}$ such that

$$F_0 \simeq_{K_1} G_1 \simeq_{K_2} \cdots \simeq_{K_{n-1}} G_{n-1} \simeq_{K_n} F_1.$$

Theorem 1. *The generalized index* Ind_L *on has the following properties (as above,* $C := \{x \in X \mid L(x) \in F(x)\}$*):*

(i) (Existence) *If* $\operatorname{Ind}_L(F,X) \neq 0$*, then there is* $x \in X$ *such that* $L(x) \in F(x)$*.*
(ii) (Localization) *If* $X' \subset X$ *is open and* $C \subset X'$*, then* $\operatorname{Ind}_L(F,X')$ *is defined and equal to* $\operatorname{Ind}_L(F,X)$*.*
(iii) (Homotopy Invariance) *If* F_0*,* F_1 *are* L*-homotopic, then* $\operatorname{Ind}_L(F_0,X) = \operatorname{Ind}_L(F_1,X)$*.*
(iv) (Additivity) *If* X_1*,* X_2 *are open disjoint subsets of* X *such that* $C \subset X_1 \cup X_2$*, then*

$$\operatorname{Ind}_L(F,X) = \operatorname{Ind}_L((F,X_1) + \operatorname{Ind}_L(F,X_2).$$

(v) (Restriction) *If* $F(X)) \subset Y$*, where* $Y \subset Y' \oplus \operatorname{Im}(Q)$ *is a closed subspace of* E'*, then* $\operatorname{Ind}_L(F,X) = \operatorname{Ind}_{L_Y}(F_Y, X \cap T)$*, where* $T := L^{-1}(Y' \oplus \operatorname{Im}(Q))$*,* $F_Y = F|_{X \cap T)}$ *and* $L_Y = L|_T$*.*

The proof can be found in [4] or in [3].

Applying the coincidence index constructed above, we present in the following theorem conditions sufficient for the existence of solutions to the abstract coincidence problem

$$L(x) = F(x), \tag{5}$$

where $L : E \to E'$ is a Fredholm linear operator of nonnegative index k (E, E' are Banach spaces) and F is a continuous map. This result is a slight modification of Theorem 4.1 in [4] (see Remark 4.2 therein), where the proof is included. Let P and Q be respective projections defined for L, I' be the identity map on E' and let $\operatorname{Im} Q \neq \{0\}$.

Theorem 2. *Let* $F : E \multimap E'$ *be a map such that*

(i) *there exists an open bounded subset* V *of* E *such that, for any* $x \in E \setminus V$ *and* $\lambda \in [0,1]$, $0 \notin ((1-\lambda)(I'-Q)+Q) \circ F(x)$, *and* $F|_{\mathrm{cl}\, V}$ *is an* L*-fundamentally restrictible map with some* L*-fundamental set containing* 0,

(ii) $\mathrm{Ind}_{\mathcal{O}}(Q \circ F|_{V \cap \mathrm{Im}\, P}, V \cap \mathrm{Im}\, P)$ *is nontrivial* ($\mathcal{O} : \mathrm{Im}\,(P) \to \mathrm{Im}\,(Q)$ *is a Fredholm operator such that* $\mathcal{O}(v) = 0$ *for all* $v \in \mathrm{Im}\,(P)$).

Then the problem $L(x) = F(x)$ *has a solution.*

3. Boundary value problem

Below we illustrate the above result by the boundary value problem.

Let E, E' be Banach spaces with Hausdorff measures of noncompactness [3] χ and χ' respectively and Z be the set of all positive numbers k such that the Fredholm linear operator $D : E \to E'$ is (k, χ, χ')-set contraction[4]. Following [1] we define

$$||D||^{(\chi,\chi')} := \inf Z.$$

Note that $||D||^{(\chi,\chi')} \leq ||D||$.

Denote $J = [0,T] \subset \mathbb{R}$ and let ξ be a Hausdorff measure of noncompactness in the space $\mathcal{L} = L^1(J,E)$ of integrable functions in the sense of Bochner with the norm $\|u\|_{\mathcal{L}} = \int_0^T \|u(s)\|_E \, ds$.

Let $f : J \times E \times E \to E$ be a map satisfying the following assumptions:

(f_1) $f(\cdot, u, v)$ is a measurable map for every $(u,v) \in E \times E$, and $f(t,\cdot,\cdot)$ is continuous for almost all $t \in J$,

(f_2) there are two continuous functions $\lambda_1, \lambda_2 : J \to [0,\infty)$ such that, for any $u_1, u_2, v_1, v_2 \in E$ and almost all $t \in J$,

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\|_E \leq \lambda_1(t)\|u_1 - u_2\|_E + \lambda_2(t)\|v_1 - v_2\|_E,$$

(f_3) there are integrable functions $m, n : J \to [0,\infty)$ such that $\|f(t,u,v)\|_E \leq m(t) + n(t)\|u\|_E$ for any $u, v \in E$ and almost all $t \in J$.

Let us consider the following boundary value problem

$$\begin{cases} u'(t) = f(t, u(t), u'(t)) & \text{for a.a. } t \in J, \\ A_1(u(0)) + A_2(u(T)) = \alpha(u(0)), \end{cases} \tag{6}$$

where f satisfies assumptions (f_1)-(f_3), α is a continuous compact map, and $A_1, A_2 : E \to E'$ are linear operators such that $A := A_1 + A_2$ is a Fredholm operator of nonnegative index. By a solution of problem (6) we mean an absolutely continuous map satisfying the equation for a.a. $t \in J$ an the boundary condition.

[3] Recall that χ is a Hausdorff measure of noncompactness on a space Banach E if for any bounded set $A \subset E$, $\chi(A) = \inf\{\varepsilon \mid A \text{ has a finite } \varepsilon\text{-net}\}$

[4] i.e. for any bounded set $B \in E$, the set $D(B)$ is bounded and $\chi'(D(B)) \leq k\chi(B)$.

The problem (6) is equivalent to the following one:

$$L(z,y) = F(z,y), \tag{7}$$

where $L, F : E \times \mathcal{L} \to E' \times \mathcal{L}$ and

$$L(z,y) = (A(z), y)$$

$$F(z,y) = \left(\alpha(z) - A_2\Big(\int_0^T y(s)ds\Big), f\Big(\cdot, z + \int_0^{(\cdot)} y(s)ds,\ \ y(\cdot)\Big)\right).$$

In fact, (z,y) is a solution of the coincidence problem (7) iff the map $u \in \mathcal{L}$, $u(t) := z + \int_0^t y(s)ds$ is a solution of (6).

Assume that in the spaces $E \times \mathcal{L}$ and $E' \times \mathcal{L}$ we have the norms $\|(z,y)\|_1 = \max(\|z\|_E, \|y\|_{\mathcal{L}})$ and $\|(z',y)\|_2 = \max(\|z'\|_{E'}, \|y\|_{\mathcal{L}})$, respectively. Denote by μ and μ' the Hausdorff measures of noncompactness in $E\times\mathcal{L}$ and $E'\times\mathcal{L}$, respectively, and by pr_E and $pr_{\mathcal{L}}$ (resp. $pr_{E'}$ and $pr'_{\mathcal{L}}$) projections of the space $E \times \mathcal{L}$ (resp. $E' \times \mathcal{L}$) onto E and onto $\mathcal{L}$ (resp. onto E' and $\mathcal{L}$). Observe that if S is a bounded subset of $E \times \mathcal{L}$, then $\mu(S) = \max(\chi(pr_E(S)), \xi(pr_{\mathcal{L}}(S)))$.

Let $N = \int_0^T n(s)ds$, $M = \int_0^T m(t)dt$, $\mathit{\Lambda}_1 = \int_0^T \lambda_1(s)ds$, $\mathit{\Lambda}_2 = \sup_{t\in J} \lambda_2$ and let P_A, Q_A i K_{P_A} be the respective projections and the right inverse for A.

Theorem 3. *Assume that f satisfies assumptions $(f_1) - (f_3)$, the maps α and A are as above, and $Q_A \not\equiv 0$. Moreover, let*
(f_4) $\mathit{\Lambda}_2 < 1$ and $\mathit{\Lambda}_1(1 + \|K_{P_A}\|^{(\chi',\chi)}) < 1 - \mathit{\Lambda}_2$,
(f_5) $\|A_2\| < 1$,
(f_6) $\|K_{P_A}\| \cdot N\exp(N) < 1$,
(f_7) $\operatorname{Im} A_2 \subset \operatorname{Im} A$,
(f_8) there exists $R > 0$ such that, for every $z \in E$ satisfying $\|P_A(z)\|_E \geq R$, $Q_A(\alpha(z)) \neq 0$ and $\operatorname{Ind}_{\mathcal{O}}(Q_A\circ\alpha, B^E(0,R)\cap\operatorname{Im} P_A) \neq 0$, where $\mathcal{O} : \operatorname{Im} P_A \to \operatorname{Im} Q_A$ and $\mathcal{O} \equiv 0$

Then problem (7) has a solution.

Assumptions (f_4) and (f_5) will secure that F is L-condensing, while (f_6)–(f_8) will allow us to check that a generalized index of F is nontrivial, which will imply the existence of a solution to problem (7).

Proof. We show that L and F satisfy assumptions of Theorem 1. For clarity we divide the proof into some steps but first of all, notice that L is a Fredholm operator of index $i(L) = i(A) \geq 0$. Respective projections and the right inverse of L will be denoted in a standard way by P, Q and K_P. The following equalities hold: $\operatorname{Ker} L = \operatorname{Ker} A \times \{0\}$, $\operatorname{Ker} P = \operatorname{Ker} P_A \times \mathcal{L}$, $\operatorname{Im} L = \operatorname{Im} A \times \mathcal{L}$ and $\operatorname{Im} Q = \operatorname{Im} Q_A \times \{0\}$.

STEP 1. We prove that F is continuous.

Let $(z_0, y_0) \in E \times \mathcal{L}$ and $\varepsilon > 0$ be arbitrary. By the continuity of α, there is $\delta_1 > 0$ such that $\|\alpha(z_0) - \alpha(z)\|_{E'} < \frac{\varepsilon}{4}$ for $\|z_0 - z\|_E < \delta_1$.

Take

$$\delta < \min(\delta_1, \frac{\varepsilon}{4\|A_2\|}, \frac{\varepsilon}{8\Lambda_1}, \frac{\varepsilon}{4\Lambda_2}) \tag{8}$$

and assume that for some $(z, y) \in E \times \mathcal{L}$,

$$\begin{aligned}\delta > \|(z_0, y_0) - (z, y)\|_{E \times \mathcal{L}} = \max(\|z_0 - z\|_E, \|y_0 - y\|_{\mathcal{L}}) = \\ = \max\left(\|z_0 - z\|_E, \int_0^T \|y_0(s) - y(s)\|_E ds\right).\end{aligned}$$

Since

$$\begin{aligned}\|F(z_0, y_0) - F(z, y)\|_{E' \times \mathcal{L}} = \\ \max\left(\left\|\alpha(z_0) - A_2\left(\int_0^T y_0(s)ds\right) - \alpha(z) + A_2\left(\int_0^T y(s)ds\right)\right\|_{E'},\right. \\ \left.\left\|f\left(\cdot, z_0 + \int_0^\cdot y_0(s)ds, y_0(\cdot)\right) - f\left(\cdot, z + \int_0^\cdot y(s)ds, y(\cdot)\right)\right\|_{\mathcal{L}}\right)\end{aligned}$$

and one can check, from (8), that

$$\left\|\alpha(z_0) - A_2\left(\int_0^T y_0(s)ds\right) - \alpha(z) + A_2\left(\int_0^T y(s)ds\right)\right\|_{E'} \leq \frac{\varepsilon}{2},$$

$$\left\|f(\cdot, z_0 + \int_0^\cdot y_0(s)ds, y_0(\cdot)) - f(\cdot, z + \int_0^\cdot y(s)ds, y(\cdot))\right\|_{\mathcal{L}} \leq \frac{\varepsilon}{2},$$

we obtain

$$\|F(z_0, y_0) - F(z, y)\|_{E' \times \mathcal{L}} < \varepsilon,$$

which implies a continuity of F.

STEP 2. We show that for any open bounded subset V of $E \times \mathcal{L}$, the set $F(V)$ is also bounded, and $F|_{\operatorname{cl} V}$ is L-condensing (so, L-fundamentally restrictible).

Let S be an arbitrary subset of V. We check that $\mu'(F(S)) < \mu'(L(S))$. Let $\chi(pr_E(S)) = \varepsilon$ and $\xi(pr_{\mathcal{L}}(S)) = \delta$. Then

$$\mu'(L(S)) = \max\left[\chi'\big(pr_{E'}(L(S))\big), \xi\big(pr_{\mathcal{L}}(L(S))\big)\right] = \max\left[\chi'\big(pr_{E'}L(S))\big), \delta\right].$$

Since $\operatorname{Ker} L = \operatorname{Im} P_A$ is a finite dimensional space,

$$\chi(pr_E(S)) = \chi((I_E - P_A) \circ pr_E(S)) = \chi(K_{P_A} \circ A \circ pr_E(S)).$$

One knows that

$$\chi(K_{P_A} \circ A \circ pr_E(S)) \leq \|K_{P_A}\|^{(\chi',\chi)} \chi'(A(pr_E(S)))$$

and

$$pr_{E'}(L(S)) = A(pr_E(S)),$$

thus

$$\chi'(pr_{E'}(L(S))) \geq \frac{\chi(pr_E(S))}{\|K_{P_A}\|^{(\chi',\chi)}} = \frac{\varepsilon}{\|K_{P_A}\|^{(\chi',\chi)}}.$$

This implies

$$\mu'(L(S)) \geq \max\left[\frac{\varepsilon}{\|K_{P_A}\|^{(\chi',\chi)}}, \delta\right].$$

Now, calculate $\mu'(F(S))$. Obviously,

$$\mu'(F(S)) = \max\Bigg(\chi'\Big(\Big\{\alpha(z) - A_2\Big(\int_0^T y(s)ds\Big); (z,y) \in S\Big\}\Big), \\ \xi\Big(\Big\{f\Big(\cdot, z + \int_0^{(\cdot)} y(s)ds, y(\cdot)\Big); (z,y) \in S\Big\}\Big)\Bigg).$$

Since α is a compact map, $\chi'(\{\alpha(z) | z \in pr_E(S)\}) = 0$, hence, by a suitable property of measures of noncompactness,

$$\chi'\Big(\Big\{\alpha(z) - A_2\Big(\int_0^T y(s)ds\Big) | (z,y) \in S\Big\}\Big) \leq \chi'\Big(\Big\{A_2\Big(\int_0^T y(s)ds\Big) | y \in \mathrm{pr}_{\mathcal{L}}(S)\Big\}\Big).$$

For every $\delta_1 > 0$ there is a finite $(\delta + \delta_1)$–net in $pr_{\mathcal{L}}(S)$. Let y_k be an arbitrary element of this net. If $\|y_k - y\|_{\mathcal{L}} \leq \delta + \delta_1$ for some $y \in pr_{\mathcal{L}}(S)$, then

$$\left\|A_2\left(\int_0^T y_k(s)ds\right) - A_2\left(\int_0^T y(s)ds\right)\right\|_{E'} = \left\|A_2\left(\int_0^T y_k(s) - y(s)ds\right)\right\|_{E'} \leq \\ \leq \|A_2\| \cdot \left\|\int_0^T (y_k(s) - y(s))ds\right\|_E \leq \|A_2\| \cdot \int_0^T \|y_k(s) - y(s)\|_E ds = \\ = \|A_2\| \cdot \|y_k - y\|_{\mathcal{L}} < \\ < \|A_2\|(\delta + \delta_1).$$

Therefore $\chi'(A_2(\{\int_0^T y(s)ds | y \in pr_{\mathcal{L}}(S)\})) \leq \|A_2\|\delta < \delta$, what implies that

$$\mu'(F(S)) \leq \max\Bigg(\delta, \xi\Big(\Big\{f\Big(\cdot, z + \int_0^{(\cdot)} y(s)ds, y(\cdot)\Big); (z,y) \in S\Big\}\Big)\Bigg).$$

Analogously, for every $\varepsilon_1 > 0$ there is a finite $(\varepsilon+\varepsilon_1)$–net in $pr_E(S)$. Let z_l be its arbitrary element. If $\|y_k - y\|_{\mathcal{L}} \le \delta + \delta_1$ and $\|z_l - z\| \le \varepsilon + \varepsilon_1$ hold for some $y \in pr_{\mathcal{L}}(S)$ and $z \in pr_E(S)$, then

$$\int_0^T \left\| f\left(t, z_l + \int_0^t y_k(s)ds, y_k(t)\right) - f\left(t, z + \int_0^t y(s)ds, y(t)\right)\right\|_E dt \le$$
$$\le \int_0^T \left(\lambda_1(t) \left\| z_l + \int_0^t y_k(s)ds - z - \int_0^t y(s)ds\right\|_E + \lambda_2(t)\|y_k(t) - y(t)\|_E\right) dt \le$$
$$\int_0^T\left(\lambda_1(t)\left(\|z_l - z\|_E + \|\int_0^t y_k(s)ds - \int_0^t y(s)ds\|_E\right) + \lambda_2(t)\|y_k(t) - y(t)\|_E\right) dt \le$$
$$\le \int_0^T \lambda_1(t)(\varepsilon+\varepsilon_1+\delta+\delta_1)dt + \int_0^T \lambda_2(t)\|y_k(t) - y(t)\|_E dt \le$$
$$\le \Lambda_1(\varepsilon+\varepsilon_1+\delta+\delta_1) + \Lambda_2(\delta+\delta_1).$$

Since ε_1 and δ_1 was arbitrary, we have

$$\xi(pr'_{\mathcal{L}}(F(S))) \le \Lambda_1(\varepsilon+\delta) + \Lambda_2\delta,$$

and consequently, using (f_4),

$$\mu'(F(S)) = \max\left(\chi'\left(pr_{E'}(F(S))\right), \xi\left(pr'_{\mathcal{L}}(F(S))\right)\right) < \max\left(\delta, \frac{\varepsilon}{\|K_{P_A}\|^{(\chi',\chi)}}\right) \le$$
$$\le \mu'(L(S)).$$

This implies that $F|_{\mathrm{cl}\,V}$ is L-condensing map, hence there exists a compact L-fundamental set for $F|_{\mathrm{cl}\,V}$ containing 0.

STEP 3. We prove that, for some open bounded set $V \subset E \times \mathcal{L}$, the map $((1-\lambda)(I-Q)+Q)\circ F$ has no coincidence points with L outside V (I denotes the identity map in $E' \times \mathcal{L}$). Let I_E, $I_{E'}$ be the identity maps on spaces E, E' respectively

Let $Z > 0$ be such that $\alpha(E) \subset B^E(0, Z)$. Choose $R_1 > 0$ such that

$$R_1 > \frac{\|K_{P_A}\|(Z + M\exp(N) + NR\exp(N))}{1 - \|K_{P_A}\|N\exp(N)}$$

and let

$$R_2 := (M + N(R+R_1))\exp(N).$$

Define

$$V := \Big\{(z,y) \in E \times \mathcal{L} \mid \; P_A(z) \in B^E(0,R) \cap \operatorname{Ker} A,$$
$$(I_E - P_A)(z) \in B^E(0,R_1) \cap \operatorname{Ker} P_A, \;\; y \in B^{\mathcal{L}}(0,R_2)\Big\}.$$

Suppose, on the contrary, that there is $\lambda \in [0,1]$ such that

$$L(z,y) = ((1-\lambda)(I-Q)+Q) \circ F(z,y).$$

It follows that $Q \circ F(z,y) = 0$, since $L(z,y) \in (I-Q)(E' \times \mathcal{L})$. Moreover,

$$((1-\lambda)(I-Q)+Q) \circ F(z,y) =$$
$$= \left(\Big((1-\lambda)(I_{E'}-Q_A)+Q_A\Big)\left(\alpha(z) - A_2(\int_0^T y(s)ds)\right), f(\cdot, z+\int_0^{\cdot} y(s)ds, y(\cdot))\right),$$

so we obtain that:

$$y(\cdot) = f\left(\cdot, z+\int_0^{\cdot} y(s)ds, y(\cdot)\right), \tag{9}$$

$$A(z) = (1-\lambda)(I_{E'}-Q_A)\left(\alpha(z) - A_2(\int_0^T y(s)ds)\right), \tag{10}$$

and

$$Q_A\left(\alpha(z) - A_2(\int_0^T y(s)ds)\right) = 0. \tag{11}$$

The last equality and assumption (f_7) imply $Q_A(\alpha(z)) = 0$, so by (f_8),

$$\|P_A(z)\|_E < R. \tag{12}$$

Consider the continuous map $[0,T] \ni t \mapsto \int_0^t \|y(s)\|ds$. From equality (9) and assumption (f_3) it follows that

$$\int_0^t \|y(s)\|_E ds = \int_0^t \left\| f\left(s, z+\int_0^s y(\tau)d\tau, y(s)\right)\right\|_E ds \leq$$
$$\leq \int_0^t \left(m(s)+n(s)\left\|z+\int_0^s y(\tau)d\tau\right\|_E\right)ds \leq \int_0^t m(s)ds +$$
$$+\int_0^t n(s)(\|P_A(z)\|_E + \|(I_E-P_A)(z)\|_E)ds + \int_0^t (n(s)\int_0^s \|y(r)\|_E dr)ds,$$

and, by the Gronwall inequality,

$$\int_0^t \|y(s)\|_E ds \leq (M + N(\|P_A(z)\|_E + \|(I_E-P_A)(z)\|_E)) \exp(\int_0^t n(s)ds) \leq$$
$$\leq (M + N(\|P_A(z)\|_E + \|(I_E-P_A)(z)\|_E)) \exp(N).$$

Combining this with (12) one obtains

$$\|y\|_{\mathcal{L}} \leq (M + N(R + \|(I_E - P_A)(z)\|_E)) \exp(N). \tag{13}$$

Since $(I_E - P_A)(z) = K_{P_A} \circ A(z)$, conditions (10), (13) and assumption (f_5) imply that

$$\|(I_E - P_A)(z)\|_E = (1-\lambda) \left\| K_{P_A} \circ (I_{E'} - Q_A) \left(\alpha(z) - A_2(\int_0^T y(s)ds) \right) \right\|_E \leq$$

$$\leq (1-\lambda) \left(\|K_{P_A} \circ (I_{E'} - Q_A)(\alpha(z))\|_E + \|K_{P_A} \circ (I_{E'} - Q_A) \circ A_2(\int_0^T y(s)ds)\|_E \right) \leq$$

$$\leq (1-\lambda) \Big(\|K_{P_A}\| \cdot \|(I_{E'} - Q_A)(\alpha(z))\|_{E'} +$$

$$+ \|K_{P_A}\| \cdot \|I_{E'} - Q_A\| \cdot \|A_2\| \cdot \| \int_0^T y(s)ds\|_E \Big) \leq$$

$$\leq ((1-\lambda)\|K_{P_A}\| \cdot (Z + (M + N(R + \|(I_E - P_A)(z)\|_E)) \exp(N)).$$

Now, if $\lambda = 1$, then $\|(I_E - P_A)(z)\| = 0 < R_1$ and if $0 \leq \lambda < 1$, then also (using the above inequalities)

$$\|(I_E - P_A)(z)\|_E \leq \frac{\|K_{P_A}\|(Z + M\exp(N) + NR\exp(N))}{1 - \|K_{P_A}\| N \exp(N)} \leq R_1, \tag{14}$$

which jointly with (13) implies

$$\|y\|_{\mathcal{L}} < (M + N(R + R_1)) \exp(N) = R_2. \tag{15}$$

By inequalities (12), (14) and (15) we can conclude that all coincidence points of L and maps $((1-\lambda)(I - Q) + Q) \circ F$, where $\lambda \in [0,1]$, are contained in V.

STEP 4. We use assumptions (f_7) and (f_8) to obtain that, for every $(z, y) \in V$,

$$Q \circ F|_{V \cap \operatorname{Im} P}(z, y) = Q(\alpha(z), 0) = Q_A(\alpha(z)),$$

and hence, $\operatorname{Ind}_{\mathcal{O}}(Q \circ F|_{V \cap \operatorname{Im} P}, V \cap \operatorname{Im} P)$ is nontrivial.

Resuming, in succeeding steps we have proved that the Fredholm operator L and the map F satisfy the assumptions of Theorem 1, so problem (6) has a solution.

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ON CERTAIN THIRD ORDER EIGENVALUE PROBLEM

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Abstract. In this paper a singular third order eigenvalue problem is studied. The motivation was given by the paper [2] of Á. Elbert, T. Kusano and M. Naito for linear second order nonoscillatory differential equation.

AMS Subject Classification. 34B05, 34B24, 34C10

Keywords. Singular eigenvalue problem, third order oscillation theory, zeros of nonoscillatory solutions.

1. The aim of this paper is to study the following eigenvalue problem

$$y''' + 2A(t)y' + [A'(t) + \lambda b(t)]y = 0 \tag{a}$$

$$y(a,\lambda) = y(b,\lambda) = y(c,\lambda) = 0,\ a \le b < c < \infty \tag{1}$$

as well as the boundary condition at infinity

$$y(t,\lambda) = o\big(t[k_1u_1(t)u_2(t) + k_2u_2^2(t)]\big) \text{ for } t \to \infty \tag{2}$$

together with the requirement that

$$y(t,\lambda) \ne 0$$

in a certain neighborhood of infinity (t_0,∞), where $c \le t_0 < \infty$, and $u_1,\ u_2$ form a fundamental set of solutions of the second order differential equation

$$u'' + \frac{1}{2}A(t)u = 0 \tag{3}$$

with initial conditions $u_1(t_0) = 1,\ u_1'(t_0) = 0,\ u_2(t_0) = 0,\ u_2'(t_0) = 1,\ k_1,\ k_2$ are certain positive constants.

The basic suppositions on A and b in this paper are such that $A',\ b$ are continuous on $[a, \infty),\ b(t) > 0$ for (a, ∞) and the differential equation (a) is strongly nonoscilatory for each real positive λ.

2. In this section we introduce certain auxiliary statements on the linear third order differential equation, given in monograph [1].

Consider equation (a) and the third order differential equation

$$y''' + 2A(t)y' + [A'(t) + b(t)]y = 0. \tag{a_1}$$

Lemma 1 (2, Theorem 2.1). *Let $A(t) < 0,\ b(t) > 0$ for $t \in [a, \infty)$ and let $|A(t)| \geq \int_a^t b(\tau)d\tau$ for $t \geq a$. Then the differential equation (a_1) is disconjugate in the interval $[a, \infty)$.*

Lemma 2. *Let the suppositions of Lemma 1 be fulfilled and let $\int_a^\infty b(\tau)d\tau < \infty$. Then to each $\bar\lambda \in [1, \infty)$ there exists $t_0 > a$ such that $|A(t)| > \bar\lambda \int_{t_0}^t b(\tau)d\tau$ holds for $t \geq t_0$ and the differential equation (a) is disconjugate for $\lambda = \bar\lambda$ on the interval $[t_0, \infty)$.*

The proof follows immediately from Lemma 1.

Lemma 3 (2, Theorem 2.14). *Let $A(t) < 0,\ b(t) > 0$ and $A'(t) + b(t) > 0$ for $t \in [a, \infty)$. If, moreover*

$$\int_T^\infty \left[A'(t) + b(t) - \frac{4}{3}\sqrt{\frac{2}{3}}\sqrt{-A^3(t)}\right]dt = +\infty,$$

$a < T < \infty$, then the differential equation (a_1) is oscillatory in $[a, \infty)$.

Lemma 4. *Let $A(t) < 0,\ b(t) > 0$ and $|A(t)| < K,\ |A'(t)| < K,\ b(t) > K,\ K > 0$, for $t \in [a, \infty)$. Then there exists $\tilde\lambda > 0$ such that the differential equation (a) is oscillatory in $[a, \infty)$ for all $\lambda \geq \tilde\lambda$.*

The proof of this lemma follows immediately from Lemma 3.

Consider, moreover, the second order differential equation

$$y'' + \frac{1}{2}A(t)y = 0 \tag{3}$$

Lemma 5. *Let $A(t) < 0$ for $t \in [a, \infty)$. Let $u_1,\ u_2$ be independent solutions of (3) and let $u_1(t_0) = 1,\ u_1'(t_0) = 0,\ u_2(t_0) = 0,\ u_2'(t_0) = 1,\ a < t_0 < \infty$. Then there is $u_1(t) > 0,\ u_2(t) > 0$ for $t > t_0$ and $u_1(t) \to \infty,\ u_2(t) \to \infty$ for $t \to \infty$.*

The proof follows from equation (3).

Lemma 6. *Let $A(t) < 0,\ b(t) > 0$ for $t \in [a,\infty)$ and let $\lambda > 0$. Let y be a solution of (a) and let for $\lambda = \bar\lambda$ be $y(t_0,\bar\lambda) = 0,\ y'(t_0,\bar\lambda) \neq 0,\ y''(t_0,\bar\lambda) \neq 0$ and let $y(t,\bar\lambda) \neq 0$ for $t > t_0$. Then*

$$y(t,\bar\lambda) = u_2(t)\left[\frac{y''(t_0,\bar\lambda)}{2}u_2(t) + y'(t_0,\bar\lambda)u_1(t)\right] - \tag{4}$$

$$\frac{1}{2}\bar\lambda\int_{t_0}^{t} b(\tau)\begin{vmatrix} u_1(t) & u_2(t)\\ u_1(\tau) & u_2(\tau)\end{vmatrix}^2 y(\tau,\bar\lambda)d\tau.$$

where $u_1,\ u_2$ form a fundamental set of solutions of (3) with the properties as in the formulation of Lemma 5.

The proof of Lemma 6 is given in [2], Chap. I, §3 at the beginning of section 3 by method of variation of constants for

$$y''' + 2A(t)y' + A'(t) = -\bar\lambda b(t)y.$$

Remark 1. If in (4) $y(t,\bar\lambda) > 0$ $[y(t,\bar\lambda) < 0]$ for $t > t_0$, then $y'(t_0,\bar\lambda) > 0$ $[y'(t_0,\bar\lambda) < 0]$ and $u_2(t) > 0,\ u(t) = y'(t_0,\bar\lambda)u_1(t) + \frac{y''(t_0,\bar\lambda)}{2}u_2(t) > 0$ $[u(t) = y'(t_0,\bar\lambda)u_1(t) + \frac{y''(t_0,\bar\lambda)}{2}u_2(t) < 0]$ for $t > t_0$.

Corollary 1. *Let the supposition of Lemma 6 be fulfilled. Then there exist constants $k_1 > 0,\ k_2 > 0$ such that $|y(t,\bar\lambda)| \leq u_2(t)[k_1u_1(t) + k_2u_2(t)]$ for $t > t_0$ where $k_1 = |y'(t_0,\bar\lambda)|,\ k_2 = \frac{|y''(t_0,\bar\lambda)|}{2}$, or*

$$y(t,\bar\lambda) = o\big(tu_2(t)[k_1u_1(t) + k_2u_2(t)]\big) \quad \textit{for} \quad t \to \infty. \tag{2}$$

Adaptation of oscillation theorem [2, Theorem B, or Theorem 4.5 in the same section] to (a) in our case yields the following lemma.

Lemma 7. *Suppose that $|A(t)| \leq K,\ |A'(t)| \leq K,\ K > 0$ and $b(t) \geq k > 0$ for $t \in [a,\infty)$. Let $\lambda \in (0,\infty)$ and let $y(t,\lambda)$ be a nontrivial solution of (a) with $y(a,\lambda) = 0$. Then for any fixed $b > a$, the number of zeros of y on $[a,b]$ increases to infinity as $\lambda \to \infty$, and the distance between any consecutive zeros of y converges to zero.*

The continuous dependence of zeros of solutions of (a) upon the parameter λ is given in following lemma.

Lemma 8 (2, Lemma 4.2). *Let y be a nontrivial solution of (a) on $[a,\infty)$ such that $y(a,\lambda) = 0$. Then, the zeros of y on (a,∞) (if they exist) are continuous functions of the parameter $\lambda \in (0,\infty)$.*

With the help of results given in the preceding lemmas and Corollary 1 one can prove the following theorem regarding the singular eigenvalue problem (a), (1), (2).

Theorem 1. *Let $A(t) < 0$, $b(t) > 0$ and $|A(t)| < K$, $|A'(t)| < K$, $K > 0$ for $t \in [a, \infty)$. Let, further, $\int_a^\infty b(t)dt < \infty$ and $|A(t)| \geq \int_a^t b(\tau)d\tau$ for $t \in [a, \infty)$ and let $a \leq b < c < \infty$ be arbitrary, but fixed. Then there exists a natural number ν, a sequence of the values of the parameter λ, $\{\lambda_{\nu+p}\}_{p=0}^\infty$ (eigenvalues) such that $\lambda_{\nu+p} < \lambda_{\nu+p+1}$, $p = 0, 1, 2, \dots$ and $\lim_{p\to\infty} \lambda_{\nu+p} = \infty$ and a corresponding sequence of functions $\{y_{\nu+p}\}_{p=0}^\infty$ (eigenfunctions) such that $y_{\nu+p} = y(t, \lambda_{\nu+p})$ is a solution of (a) for $\lambda = \lambda_{\nu+p}$, has a finite number of zeros on (a, ∞) with the last zero at $t_0^{\nu+p}$, fulfills the boundary conditions (1), (2) and has exactly $\nu + p$ zeros in (b, c).*

Proof. Let $a < b < c < \infty$. Let $y = y(t, \lambda)$, $\lambda > 0$ be a nontrivial solution of (*a*) such that $y(a, \lambda) = y(b, \lambda) = 0$ for all $\lambda > 0$. Construct, now, on $[a, \infty)$ differential equation

$$\text{(A)} \qquad Y''' + 2A(t)Y' + [A'(t) + \lambda B(t)]Y = 0,$$

where

$$B(t) = \begin{cases} b(t) & \text{for } t \in [a, c] \\ b(c) & \text{for } t \geq c. \end{cases}$$

Let $Y = Y(t, \lambda)$ be a solution of (A) on $[a, \infty)$ such that $Y(a, \lambda) = Y(b, \lambda) = 0$ and $Y(t, \lambda) = y(t, \lambda)$ for $t \in [a, c]$ and $\lambda \in (0, \infty)$.

By Lemma 4, there exists $\bar\lambda$ such that the differential equation (A) is oscillatory in $[a, \infty)$ for all $\lambda > \bar\lambda$. Let $Y(t, \lambda^*)$, $\lambda^* \geq \bar\lambda$ have exactly ν zeros in (b, c). Let $t_\nu(\lambda)$ be the ν-th zero of $Y(t, \lambda)$. Then there is $t_\nu(\lambda^*) < c \leq t_{\nu+1}(\lambda^*)$. By Lemma 7 there exists $\bar{\lambda^*}$ such that $t_{\nu+1}(\bar{\lambda^*}) < c$ and by Lemma 8 (continuous dependence of zeros) there exists λ_ν, $\lambda^* \leq \lambda_\nu < \bar{\lambda^*}$ such that $t_{\nu+1}(\lambda_\nu) = c$ and $Y(t, \lambda_\nu)$ has exactly ν zeros in (b, c). But, we know that $Y(t, \lambda_\nu) = y(t, \lambda_\nu)$ on $[a, c]$. By Lemma 2 to λ_ν there exists $t_0^\nu \geq c$ such that $y(t, \lambda_\nu)$ has finite numbers of zeros to the right of c. Let t_0^ν be its last zero on $[c, \infty)$. Then by Corollary 1 the inequality (2) holds.

Continuing in the same manner we can find a sequence of values

$$\lambda_\nu, \lambda_{\nu+1}, \dots, \lambda_{\nu+p}, \dots$$

and the corresponding sequence of functions $\{y_{\nu+p}\}_{p=0}^\infty$ (eigenfunctions) with the prescribed properties and the theorem is proved.

Remark 2. If we take in consideration the fact, that equation (*a*) is for $\lambda = 1$ disconjugate on $[a, \infty)$, the oscillation Lemma 7 and Lemma 8 (continuous dependence of zeros on λ) then it is possible to prove Theorem 1 for $\nu = 0$.

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ON CERTAIN THIRD ORDER BOUNDARY VALUE PROBLEMS ON INFINITE INTERVAL

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Abstract. In this paper certain third order eigenvalue problems are studied. The motivation was given by the theory of the third order linear eigenvalue problems on finite intervals and asymptotic properties of solutions of the third order linear differential equations [1].

AMS Subject Classification. 34B10, 34B40

Keywords. linear third order differential equation, eigenvalue, eigenfunction, boundary value problems.

1. The aim of this paper is to study these two boundary value problems

$$y''' + 2A(t)y' + [A'(t) + \lambda b(t)]y = 0, \tag{a}$$

$$y(a,\lambda) = y(b,\lambda) = y(c,\lambda) = 0,\ a \le b < c < \infty \tag{1}$$

$$\lim_{t\to+\infty} y(t,\lambda) = 0 \tag{2}$$

and the problem (a), (1) and ($2'$) where

$$|y(t,\lambda)| < K,\ |y'(t,\lambda)| < K,\ K > 0,\ \lambda > 0, \tag{2'}$$

under certain suppositions on the functions A, A', b on $[a,\infty)$.

The result of this paper complete those which are given in monograph [1] in the case that equation (a) is oscillatory on $[a,\infty)$ for each $\lambda > 0$, i.e. every solution of (a) with one zero has infinite number of zeros on $[a,\infty)$.

2. In this section we introduce certain auxiliary statements on the third order differential equation given in [1].

In this and in the next section we will suppose that $A'(t)$ and $b(t)$ are continuous functions on $[a, \infty)$ and $b(t) > 0$ for all $t \in [a, \infty)$.

Have the linear third order differential equation

$$y''' + 2A(t)y' + [A'(t) + b(t)]y = 0, \tag{a_1}$$

(i.e. equation (a) with $\lambda = 1$).

Lemma 1 (1, Theorem 3.17). *Assume that every solution of the second order differential equation*

$$y'' + \frac{1}{2}A(t)y = 0 \tag{3}$$

converges to zero as $t \to +\infty$ and $\int_a^t b(\tau)d\tau$ converges. Then every solution of the differential equation (a_1) converges to zero as $t \to \infty$.

This Lemma was formulated and proved by M. Ráb in [2].

One of the sufficient conditions for the solutions of (3) to converge to zero as $t \to \infty$ is given in the following lemma [3].

Lemma 2. *Let $A(t) > 0$ be non decreasing on $[a, \infty)$ and let $A'(t) \geq l > 0$ and $\int_a^\infty \frac{dt}{A(t)} = +\infty$. Then every solution y of (3) has the property $\lim\limits_{t\to\infty} y(t) = 0$.*

Lemma 3 (1, Theorem 3.18). *Let the following assumptions in $[a, \infty)$ hold:*

1. *$A(t) > 0$, $\lim\limits_{t\to\infty} A(t) = \infty$.*
2. *The function $A^{-\frac{1}{4}}(t)$ is convex.*
3. *The integral $\int\limits_a^t \frac{b(\tau)}{A(\tau)}d\tau$ converges.*

Then every solution of (a_1) and its first derivative are bounded in $[a, \infty)$.

This lemma was formulated and proved by M.Zlámal [4].

Lemma 4 (1, Corollary 2.3). *Let the second order differential equation (3) be oscillatory in $[a, \infty)$.Then (a_1) is oscillatory in $[a, \infty)$ too, i.e. its every solution having a zero is oscillatory in $[a, \infty)$.*

Adaptation of the oscillation theorem [1, Theorem 4.5] to (a) yields the following lemma.

Lemma 5. *Suppose that $A \geq \frac{p}{2}$ for all $t \in [a, \infty)$, where p is a real constant and moreover $b(t) \geq k > 0$ on $[a, \infty)$ for some positive constant k.*
Let $\lambda \in (0, \infty)$ and let $y(t, \lambda)$ be a nontrivial solution of (a) with $y(a, \lambda) = 0$. Then, for any fixed $b > a$, the number of zeros of y on $[a, b]$ increases to infinity as $\lambda \to \infty$, in which case the distance between any consecutive zeros of y converges to zero.

The continuous dependence of zeros of solutions of (a) upon the parameters λ is given in following lemma.

Lemma 6 (1, Lemma 4.2). *Let y be a nontrivial solution of (a) on $[a, \infty)$ such that $y(a, \lambda) = 0$. Then, the zeros of y on $[a, \infty)$ (if they exist) are continuous functions of the parameter $\lambda \in (0, \infty)$.*

With the help of the results given in the preceding lemmas one can prove the following two theorems regarding the multipoint eigenvalue problems (a), (1), (2) and (a), (1), ($2'$).

Theorem 1. *Let the suppositions on A, A', b given in Lemma 1, be fulfilled and let $A(t) > 0$ and equation (3) be oscillatory on $[a, \infty)$. Let $a \le b < c < \infty$ be arbitrary, but fixed. Then there exists a natural number ν, a sequence of the values of parameter λ $\left\{\lambda_{\nu+p}\right\}_{p=0}^{\infty}$ (eigenvalues) such that $\lambda_{\nu+p} < \lambda_{\nu+p+1}$ and $\lim_{p\to\infty} \lambda_{\nu+p} = \infty$, and a sequence of functions $\left\{y(t, \lambda_{\nu+p})\right\}_{p=0}^{\infty}$ (eigenfunctions) such that $y(t, \lambda_{\nu+p})$, ${}_{p=0,1,\ldots}$ is a solution of (a) with $\lambda = \lambda_{\nu+p}$, which fulfills the conditions (1), (2) and has exactly $\nu + p - 1$ zeros on (b, c).*

Proof. Let $a < b < c < \infty$. Let $y(t, \lambda)$, $\lambda > 0$ be a nontrivial solution of (a) such that $y(a, \lambda) = y(b, \lambda) = 0$. Such a solution of ($a$) evidently exists (see e.g. properties of bands of solutions of (a), [1]). Solution $y(t, \lambda)$ is oscillatory on $[b, \infty)$. Construct now on $[a, \infty)$ the differential equation

$$Y''' + 2A(t)Y' + [A'(t) + \lambda B(t)]Y = 0, \tag{A}$$

where

$$B(t) = \begin{cases} b(t) & \text{for } t \in [a, c] \\ b(c) & \text{for } t > c. \end{cases}$$

On account of Lemma 4 equation (A) is oscillatory on $[a, \infty)$ for all $\lambda > 0$. Let $Y(t, \lambda)$ be a solution of (A) on $[a, \infty)$ with the property $Y(a, \lambda) = Y(b, \lambda) = 0$. If we denote $Y'(b, \lambda) = y'(b, \lambda)$, $Y''(b, \lambda) = y''(b, \lambda)$ for all $\lambda > 0$, then clearly $Y(t, \lambda) = y(t, \lambda)$ is the solution of (a) for $t \in [a, b]$ and $\lambda > 0$, too.

The function $Y(t, \lambda)$ as a solution of (A) is oscillatory on $[b, \infty)$ for all $\lambda > 0$.

Let for $\lambda = \bar{\lambda} > 0$ the solution $Y(t, \bar{\lambda})$ have exactly ν zeros in (b, c). Then clearly, for the ν-th zero $t_\nu(\bar{\lambda})$ and the $(\nu + 1)$-st zero $t_{\nu+1}(\bar{\lambda})$ of $Y(t, \bar{\lambda})$ we have $t_\nu(\bar{\lambda}) < c \le t_{\nu+1}(\bar{\lambda})$.

It follows from Lemma 5 (oscillation lemma), that for some $\bar{\bar{\lambda}} > \bar{\lambda}$ we have $t_{\nu+1}(\bar{\bar{\lambda}} < c$. Since $t_{\nu+1}(\lambda)$ is a continuous function of the parameter λ (Lemma 6), there exist $\lambda_\nu \in [\bar{\lambda}, \bar{\bar{\lambda}})$ such that for $\lambda = \lambda_\nu$ we have $t_{\nu+1}(\lambda_\nu) = c$, i.e $Y(c, \lambda_\nu) = y(c, \lambda_\nu) = 0$ and $Y(t, \lambda_\nu)$ has exactly ν zeros in (b, c).

Continuing in this manner, we can find a sequence of values of the parameters $\lambda > 0$

$$\lambda_\nu < \lambda_{\nu+1} < \cdots < \lambda_{\nu+p} < \cdots,$$

to which there corresponds a sequence of functions

$$Y_\nu,\ Y_{\nu+1}, \dots,\ Y_{\nu+p}, \dots$$

such that $Y_{\nu+p} = y(t, \lambda_{\nu+p})$ is a solution of (A) satisfying conditions (1) and $Y(t, \lambda_{\nu+p})$ has exactly $\nu + p - 1$ zeros in (b, c).

But $Y(t, \lambda_{\nu+p}) = y(t, \lambda_{\nu+p})$ on $[a, c]$ with the same initial conditions in c. Therefore the solution $y(t, \lambda_{\nu+p})$ of (a) fulfills the boundary condition (1) and by Lemma 1 (where instead of $\int_a^\infty b(\tau)d\tau < \infty$ we have $\lambda_{\nu+p} \int_a^\infty b(\tau)d\tau < \infty$) the solution $y(t, \lambda_{\nu+p})$ has the property (2), too and Theorem 1 is proved in the case $a < b < c < \infty$.

If $a = b < c < \infty$, the proof is similar, but for the solution y with the double zero at a for $\lambda > 0$.

By the same argument we can prove the following

Theorem 2. *Let the suppositions on A, A', b given in Lemma 3 be fulfilled on $[a, \infty)$. Then the assertion of Theorem 1 holds with the exception that $y(t, \lambda_{\nu+p})$ fulfills the condition (2′), (not the condition (2)).*

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THE USE OF LYAPUNOV FUNCTIONS IN UNIQUENESS AND NONUNIQUENESS THEOREMS

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Abstract. The contribution is devoted to using Lyapunov functions in uniqueness and nonuniqueness theorems. The survey of nonuniqueness results utilizing Lyapunov functions is given.

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1. Introduction

The question of the existence and uniqueness for the solutions of ordinary differential equations is an old problem of great importance. There is an enormous amount of literature offering various sufficient conditions for the uniqueness. We shall mention here only several mathematicians that have contributed to this problem.

The first result on the uniqueness of a scalar initial value problem

$$x' = f(t,x), \quad x(t_0) = x_0 \tag{1}$$

where $f = (f_1, f_2, \dots, f_n)$, $x = (x_1, x_2, \dots, x_n)$, $x_0 = (x_{01}, x_{02}, \dots, x_{0n})$, was given by A. Cauchy in 1820–1830. The result was improved by R. Lipschitz in 1876, who introduced so called Lipschitz condition of the form

$$|f(t,x) - f(t,y)| \le L|x-y|. \tag{2}$$

The Lipschitz condition was generalized by many authors such as W. F. Osgood (1898), P. Montel (1926), L. Tonelli (1925), M. Nagumo (1926). Very general is a condition of Perron's type

$$|f(t,x) - f(t,y)| \le g(t, |x-y|). \tag{3}$$

Perron's result (1926) was improved by E. Kamke (1930). His well-known theorem (see e. g. [1, pp. 56–57]) can be formulated for vector differential equations.

Theorem 1 (Kamke). *Assume that*

(i) $g \in C(R_0, \mathbb{R}^+)$, *where* $R_0 = \{(t,u) \in \mathbb{R}^2 : t_0 < t \le t_0 + a, 0 \le u \le 2b\}$, $\mathbb{R}^+ = [0,\infty)$ *and for every* $t_1 \in (t_0, t_0+a)$, *the function* $u(t) \equiv 0$ *is the only solution of* $u' = g(t,u)$ *defined on* (t_0, t_1) *and satisfying* $\lim_{t\to t_0}[u(t)/(t-t_0)] = 0$.

(ii) $f : R \to \mathbb{R}^n$, $R = \{(t,x) \in \mathbb{R}^{n+1} : t_0 \le t \le t_0 + a, |x - x_0| \le b\}$ *and*

$$|f(t,x) - f(t,y)| \le g(t, |x-y|) \quad \textit{for} \quad (t,x),(t,y) \in R, \quad t \ne t_0. \tag{4}$$

Then the initial value problem (1) *has at most one solution in* $[t_0, t_0 + a]$.

2. The use of Lyapunov functions

Kamke's theorem was generalized in several manners. One of the fruitful ways is the use of Lyapunov functions method. This approach allows to obtain very general and flexible results. These results contain the most of previous results as special cases and, by special choices, new interesting criteria for the uniqueness can be obtained. There exists a lot of variants of criteria utilizing Lyapunov functions. We can mention here the results of H. Okamura (1934–42), T. Sato (1936), O. Borůvka (1956), J. Chrastina (1969), S. C. Chu and J. B. Diaz (1970), T. Roger (1972), F. Brauer and S. Sternberg (1958), R. D. Moyer (1966), S. R. Bernfeld - R. D. Driver - V. Lakshmikantham (1976), Z. Tesařová - O. Došlý (1980), H. A. Antosiewicz (1962), V. Lakshmikantham - M. Samimi (1983).

The interesting and powerful uniqueness criteria for the Cauchy problem were derived by I. Kiguradze (1965). We shall remind a criterion for a singular Cauchy problem formulated for $t_0 = a$:

Theorem 2 (Kiguradze [6]). *Let* f *be defined for* $a < t \le b$, $|x - x_0| < r$ *and a function* $V(t,x)$ *be continuous and positive definite in* $R_0 = \{(t,x) \in \mathbb{R}^n : a < t \le b, |x| \le 2r\}$. *Assume that* $g(t,u)$ *satisfies Carathéodory conditions on any set* $\{R_c = \{(t,u) \in \mathbb{R}^2 : a \le t \le b, |u| \le c\}$, $c \in (0,\infty)$. *Suppose that* $g(t,\cdot)$ *is nondecreasing,* $g(t,0) \equiv 0$ *and the problem*

$$\frac{du}{dt} = g(t,u), \quad u(a) = 0$$

has only the trivial solution. If the conditions

$$\lim_{t\to a} V(t, x(t) - y(t)) = 0, \tag{5}$$

$$V(t, x(t) - y(t)) \le \int_a^t g(s, V(s, x(s) - y(s)) ds \tag{6}$$

hold for any two solutions $x(t)$, $y(t)$ *of* (1), *then* (1) *has at most one solution.*

3. Nonuniqueness theorems

In contradistinction with the problem of uniqueness criteria, there are only several papers dealing with problem of nonuniqueness. The necessary and sufficient conditions for the uniqueness in the scalar case was derived by T. Yosie in 1926 (see e. g. [1, pp. 81–91]). His main result is the following:

Theorem 3 (Yosie's criterion). *The scalar initial value problem has at most one solution in the interval* $[t_0, t_0 + a]$ *if and only if for every* $\varepsilon > 0$ *there exists a pair of lower- and upper- functions* $\varphi(t)$, $\psi(t)$ *with respect to the initial value problem* (1) *such that* $0 < \psi(t) - \varphi(t) < \varepsilon$ *in the interval* $(t_0, t_0 + a]$.

The first nonuniqueness criterion appeared in 1922 (see e. g. [1, p. 98]):

Theorem 4 (Tamarkine). *Let* $f(t, x)$ *be a scalar function continuous in* $R = \{(t, x) \in \mathbb{R}^2 : |t - t_0| \le a, |x - x_0| \le b\}$ *with* $(t_0, x_0) = (0, 0)$ *and for all* $(x, y) \in R$ *the condition*

$$|f(t, x) - f(t, x(t))| \ge g(|x - x(t)|)$$

holds, where $x(t)$ *is a solution of* (1), $g(u)$ *being an increasing continuous function for* $u \ge 0$, *such that* $g(0) = 0$ *and* $\int_{0+} \frac{du}{g(u)} < \infty$. *Then the initial value problem* (1) *has at least two solutions in* $[t_0 - a, t_0 + a]$.

The Tamarkine criterion was generalized by V. Lakshmikantham (1964). His nonuniqueness condition formulated for $t_0 = 0$ has a form

$$|f(t, x) - f(t, y)| \ge g(t, |x - y|), \tag{7}$$

where $g \in C(R, \mathbb{R}^+)$, $R = \{(t, u) \in \mathbb{R}^2 : 0 < t \le a, 0 \le u \le 2b\}$, $g(t, 0) \equiv 0$, $g(t, u) > 0$ for $u > 0$, and, there exists a differentiable function $u(t) \not\equiv 0$ for which

$$u'(t) = g(t, u(t)), \quad u(0) = u'_+(0) = 0.$$

Lakshmikantham's theorem was generalized by M. Samimi in 1982, however, as it was noticed by H. Stettner and Chr. Nowak, the condition (7) should be replaced by a stronger one: $f(t, x) - f(t, y) \ge g(t, x - y)$ for $x > y$. Unfortunately, the last condition cannot be fulfilled (see [9]).

The first mathematician who used Lyapunov functions to obtain nonuniqueness criterion was H. Stettner (1974). In our paper [2] a general nonuniqueness result employing Lyapunov functions for the nonsingular Cauchy problem was given. A modification of this result was presented by M. Samimi [10] in 1982. Samimi supposes the boundedness of f and uses a function $B(t)$ for the description of the behaviour of the solutions near the initial point t_0 in sense of the following Theorem 5.

In 1992, Chr. Nowak [8] attempted to remove the condition on the boundedness of f in Samimi's theorem. In the paper [3] a general nonuniqueness criterion was derived, which contains as a consequence a revised form of Nowak's nonuniqueness criterion and the most of previous known nonuniqueness criteria. The notation

$$\mathrm{D}^+V(t,x) := \limsup_{h\to 0+} \frac{V(t+h, x+hf(t,x)) - V(t,x)}{h}$$

is used and the criterion is given here in a simplified form formulated for $t_0 = a$, where $-\infty \le a < \infty$:

Theorem 5 (Kalas [3]). *Let $t_1 \in (a, A)$. Assume that $f \in C[R, \mathbb{R}^n]$, where $R = \{(t,x) \in \mathbb{R}^{n+1} : a < t < A, |x - x_0| \le b\}$, and*
(i) there exists a function $g \in C[(a,t_1] \times \mathbb{R}^+, \mathbb{R}]$ nondecreasing in the second variable and such that a certain solution $\varphi(t)$, $t \in (a, t_1]$ of

$$u' = g(t,u)$$

satisfies conditions

$$\varphi(t_1) > 0, \qquad \lim_{t\to a+} \frac{\varphi(t)}{B(t)} = 0,$$

where $B \in C[(a,t_1], \mathbb{R}]$ is positive;
(ii) $V \in C[R, \mathbb{R}^+]$ is such that

$$V(t_1, y_0) < \varphi(t_1) \qquad \text{for some} \;\; y_0 \in \mathbb{R}^n,\; |y_0 - x_0| < b, \tag{8}$$

$$V(t,x) > \varphi(t) \qquad \text{for} \;\; a < t < t_1,\; |x - x_0| = b, \tag{9}$$

$$V(t,x) \ge \Phi(t)\,\Psi(|x - z(t)|) \qquad \text{for} \;\; a < t \le t_1,\; |x - x_0| < b, \tag{10}$$

where $\Phi \in C[(a,t_1], \mathbb{R}^+]$, $\Psi \in C[[0,2b), \mathbb{R}^+]$, $z \in C[(a,t_1], \mathbb{R}^n]$ satisfy

$$\liminf_{t\to a+} \frac{\Phi(t)}{B(t)} > 0, \quad \Psi(0) = 0, \quad \Psi(u) > 0 \;\; \text{for} \;\; u \in (0, 2b) \tag{11}$$

and

$$\lim_{t\to a+} z(t) = x_0, \quad |z(t) - x_0| < b \;\; \text{for} \;\; t \in (a, t_1];$$

(iii) there exists a positive function $\varepsilon \in C[(a,t_1), \mathbb{R}^+]$ such that $V(t,x)$ satisfies locally the Lipschitz condition with respect to x for $(t,x) \in \Omega_\varphi$ and

$$\mathrm{D}^+V(t,x) \ge g(t, V(t,x)) \qquad \text{on} \;\; \Omega_\varphi$$

holds, Ω_φ being defined by

$$\Omega_\varphi = \{(t,x) \in \mathbb{R}^{n+1} : \varphi(t) < V(t,x) < \varphi(t) + \varepsilon(t), a < t < t_1, |x - x_0| < b\}. \tag{12}$$

Then the problem (1) has at least two different solutions $x(t)$ on $(a, t_1]$ such that

$$\lim_{t\to a+} \frac{V(t,x(t))}{B(t)} = 0$$

is valid.

All the mentioned nonuniqueness criteria have the disadvantage that they cannot be applied for the n-th order differential equations. In the following result formulated for $t_0 = a$, where $-\infty \leq a < \infty$, we use Lyapunov functions that need not be positive definite in x (in sense of the condition (10)), but only in some components of x and thus we need the estimations only of several components of f. Such a result is applicable to the n-order differential equation. In the result we use the projection Pr defined by $\operatorname{Pr} x = (x_{i_1}, \ldots, x_{i_l})$, where i_j $(j = 1, \ldots, l)$ are integers such that $1 \leq i_1 < \cdots < i_l \leq n$.

Theorem 6 (Kalas [4]). *Let $f \in C(R, \mathbb{R}^n)$, where $R = \{(t, x) \in \mathbb{R}^{n+1} : a < t < A, |x - x_0| \leq b\}$. Put $\mu(t) := \max_{|x-x_0|\leq b} |f(t, x)|$. Suppose that*

$$\int_{a+} \mu(t)dt < \infty$$

holds and choose $t_1 \in (a, A)$ such that

$$\int_a^{t_1} \mu(t)dt \leq b/2$$

is valid. Assume that
(i) there exists a function $g \in C[(a, t_1] \times \mathbb{R}^+, \mathbb{R}]$ nondecreasing in the second variable and such that a certain solution $\varphi(t)$, $t \in (a, t_1]$ of

$$u' = g(t, u)$$

satisfies conditions

$$\varphi(t_1) > 0, \qquad \lim_{t\to a+} \frac{\varphi(t)}{B(t)} = 0,$$

where $B \in C[(a, t_1], \mathbb{R}^+]$ is positive;
(ii) $V(t, x) \in C[R, \mathbb{R}^+]$ and there exists $y_0 \in \mathbb{R}^l$, $|y_0 - \operatorname{Pr} x_0| < b/2$, such that

$$V(t_1, y) < \varphi(t_1) \qquad \text{for } \; y \in \mathbb{R}^n, \; |y - x_0| \leq b, \; \operatorname{Pr} y = y_0,$$

and

$$V(t, x) \geq \Phi(t)\, \Psi(|\operatorname{Pr} x - z(t)|) \qquad \text{for } \; a < t \leq t_1, \; |x - x_0| < b,$$

where $\Phi \in C[(a, t_1], \mathbb{R}^+]$, $\Psi \in C[[0, 2b), \mathbb{R}^+]$, $z \in C[(a, t_1], \mathbb{R}^l]$ satisfy (11) and

$$\lim_{t\to a+} z(t) = \operatorname{Pr} x_0, \quad |z(t) - \operatorname{Pr} x_0| < b \;\; \text{for } t \in (a, t_1];$$

(iii) there exists a positive function $\varepsilon \in C[(a, t_1), \mathbb{R}^+]$ such that $V(t, x)$ satisfies locally the Lipschitz condition with respect to x for $(t, x) \in \Omega_\varphi$ and

$$\mathrm{D}^+ V(t, x) \geq g(t, V(t, x)) \qquad \text{on } \; \Omega_\varphi$$

holds, Ω_φ being defined by (12). Then the problem (1) has at least two different *solutions $x(t)$ on $(a, t_1]$ such that*

$$\lim_{t\to a+} \frac{V(t,x(t))}{B(t)} = 0$$

is valid.

Proof. For the proof see [4].

Theorem 6 is formulated for the nonsingular Cauchy problem. Recently a result which attemps to extend the last result to a singular case was published in [5]. Moreover a vector Lyapunov function instead of a scalar one is used, which allows to apply achieved results to a wider class of differential equations. For the formulation of the result we need the following notation:

$\lvert\cdot\rvert$	Hölder's 1-norm (sum of the absolute values of components);
l	fixed number from the set $\{1, \cdots, n\}$;
$i_1, i_2, \cdots, i_l$	integers $1 \le i_1 < i_2 < \cdots < i_l \le n$;
I	$:= \{i_1, i_2, \cdots, i_l\}$;
N	$:= \{1, 2, \cdots, n\}$;
$\tilde{R}^k_{a,A}$	$:=\{(t,x) \in \mathbb{R}^{k+1} : a < t < A,\ x \in \mathbb{R}^k\}$;
$\hat{R}^n_{a,A}$	$:=\{(t,x) \in \mathbb{R}^{n+1} : a < t \le A,\ x \in \mathbb{R}^n\}$;
$R^k_{\alpha,A;\varrho}$	$:=\{(t,x) \in \mathbb{R}^{k+1} : \alpha < t < A, \lvert x\rvert \le \varrho\}$;
$\mathcal{L}[\hat{R}^n_{a,A}, {\mathbb{R}^+}^k]$	class of all functions $V(t,x)\colon \hat{R}^n_{a,A} \to {\mathbb{R}^+}^k$ with following property: $V(t,\cdot)$ is uniformly continuous and if $a < \alpha < \beta \le A$, then $V(t,x(t))$ is absolutely continuous on $[\alpha,\beta]$ for any absolutely continuous function $x : [\alpha,\beta] \to \mathbb{R}^n$;
$K[\tilde{R}^k_{a,A}, \mathbb{R}^n]$	class of all mappings $\tilde{R}^k_{a,A} \to \mathbb{R}^n$ which satisfy Caratheodory conditions on $R^k_{\alpha,A;\varrho}$ for any $\alpha \in (a,A)$, $\varrho \in (0,\infty)$;
$N_0(a, A; \tau_1, \cdots, \tau_n)$	$:=\{\Lambda = (\lambda_{ij}(t))^n_{i,j=1} : \lambda_{ij} \in L[[a,A], \mathbb{R}^+]\}$ such that the system of differential inequalities $\lvert x_i'(t)\rvert \le \sum_{j=1}^n \lambda_{ij}(t)\lvert x_j(t)\rvert$, $t \in [a,A]$, $i \in N$ possesses no nontrivial solution $x(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \in AC[[a,A], \mathbb{R}^n]$ satisfying $x_i(\tau_i) = 0$ $(i = 1, 2, \cdots, n)$;
$N_I(a, A)$	$:=N_0(a, A; \tau_1, \cdots, \tau_n)$, where $\tau_i = A$ for $i \in I$ and $\tau_i = a$ for $i \in N \setminus I$.

In the theorem, the initial value problem (1) with $t_0 = a$, where $-\infty \le a < \infty$, will be considered. We shall assume, that the vector function $f = (f_1, \cdots, f_n) \in K[\tilde{R}^n_{a,A}, \mathbb{R}^n]$ is such that there are $c_i \in \mathbb{R}$ $(i \in I)$, $\Lambda = (\lambda_{ij})^n_{i,j=1} \in N_I(a,A)$, $\mu_i \in L[[a,A], \mathbb{R}^+]$ $(i \in N)$ for which

$$-f_i(t,x)\operatorname{sgn}(x_i - c_i) \le \sum_{j=1}^n \lambda_{ij}(t)\lvert x_j\rvert + \mu_i(t) \qquad (i \in I)$$

and

$$f_i(t,x)\operatorname{sgn}(x_i - x_{0i}) \le \sum_{j=1}^{n} \lambda_{ij}(t)|x_j - x_{0j}| + \mu_i(t) \qquad (i \in N \setminus I)$$

hold for $(t,x) = (t, x_1, \dots, x_n) \in \tilde{R}^n_{a,A}$.

Theorem 7 (Kalas [5]). *Assume that*

(i) there exists a function $g = (g_1, \dots, g_k) \in K[\tilde{R}^k_{a,A}, \mathbb{R}^k]$ *such that any component* $g_j(t, u_1, \dots, u_{j-1}, \cdot, u_{j+1}, \dots, u_k)$ *is nondecreasing for* $j = 1, \dots, k$ *and there is a solution* $\varphi(t) = (\varphi_1(t), \dots, \varphi_k(t))$, $t \in (a, A)$ *of*

$$u' = g(t,u)$$

satisfying

$$\varphi(t) > 0, \qquad \lim_{t\to a+} \varphi(t) = 0, \qquad \liminf_{t\to A-} \varphi(t) > 0;$$

(ii) $V(t,x) = (V_1(t,x), \dots, V_k(t,x)) \in \mathcal{L}[\hat{R}^n_{a,A}, {\mathbb{R}^+}^k]$ *and there exists* $y_0 \in \mathbb{R}^l$ *such that*

$$\sup\{V_j(A,y) : y \in \mathbb{R}^n, \operatorname{Pr} y = y_0\} < \liminf_{t\to A-} \varphi_j(t) \ \ (j = 1, \dots, k)$$

and,

$$|V(t,x)| \ge \Psi(|\operatorname{Pr} x - z(t)|) \quad \text{for } a < t < A,$$

where $\Psi \in C[\mathbb{R}^+, \mathbb{R}^+]$, $z \in C[(a,A), \mathbb{R}^l]$ *are such that*

$$\Psi(0) = 0, \qquad \Psi(u) > 0 \quad \text{for } u > 0, \qquad \lim_{t\to a+} z(t) = \operatorname{Pr} x_0;$$

(iii) there exist positive functions $\varepsilon_j \in C[(a,A), \mathbb{R}^+]$ *such that*

$$V_j'(t,x(t)) \ge g_j(t, \varphi_1(t), \dots, \varphi_{j-1}(t), V_j(t,x(t)), \varphi_{j+1}(t), \dots, \varphi_k(t))$$

holds for $j = 1, 2, \dots, k$ *and for any solution* $x(t)$ *of* (1) *a. e. on any interval* $(\alpha_1, \alpha_2) \subseteq (a, A)$ *for which*

$$V_i(t,x(t)) < \varphi_i(t) + \varepsilon_i(t) \qquad \text{on } (\alpha_1,\alpha_2), \qquad (i = 1, \dots, k), \tag{13}$$

$$V_j(t,x(t)) > \varphi_j(t) \qquad \text{on } (\alpha_1,\alpha_2). \tag{14}$$

Then the initial value problem (1) *possesses at least two different solutions* $x(t)$ *on* $[a, A]$*, either of which satisfies* $V(t, x(t)) \le \varphi(t)$ *for* $t \in (a, A)$.

As a consequence we easily obtain the result for the nonuniqueness for the n-th order differential equation (for details see [5]).

Corollary 1. *Let* $\tilde{f} \in K[\tilde{R}^n_{a,A}, \mathbb{R}]$. *Suppose* $c \in \mathbb{R}$, $\lambda, \mu \in L[[a,A], \mathbb{R}^+]$ *are such that*

$$-\tilde{f}(t, x_1, \dots, x_n)\operatorname{sgn}(x_n - c) \le \lambda(t)|x_n| + \mu(t)$$

for $(t,x) \in \tilde{\mathbb{R}}^n_{a,A}$. *Assume that*

(i) there exists a function $g \in K[\tilde{R}^1_{a,A}, \mathbb{R}]$ *such that* $g(t,\cdot)$ *is nondecreasing and there is a solution* $\varphi(t)$, $t \in (a,A)$ *of* $u' = g(t,u)$ *satisfying*

$$\varphi(t) > 0, \qquad \lim_{t\to a+} \varphi(t) = 0;$$

(ii) there are $z \in C[[a,A],\mathbb{R}]$, $\varepsilon \in C[(a,A),\mathbb{R}^+]$ *such that* z *is absolutely continuous on* $[\alpha, A]$ *for any* $\alpha \in (a,A)$, $z(a) = x_{0n}$ *and*

$$(\tilde{f}(t,x_1,\dots,x_n) - z'(t))\,\mathrm{sgn}(x_n - z(t)) \ge g(t, |x_n - z(t)|)$$

holds on $\hat{\Omega} = \{(t,x_1,\dots,x_n) \in \mathbb{R}^{n+1} : \varphi(t) < |x_n - z(t)| < \varphi(t) + \varepsilon(t), a < t < A\}$ *for almost all* $t \in (a,A)$. *Then the initial value problem*

$$v^{(n)} = \tilde{f}(t,v,v',\cdots,v^{(n-1)}),$$
$$v(a) = x_{01},\ v'(a) = x_{02},\ \cdots,\ v^{(n-1)}(a) = x_{0n}$$

is nonunique.

Finally, notice that very interesting results for nonuniqueness of a singular Cauchy-Nicolletti problem were achieved by I. Kiguradze [7]. The sufficient conditions are given in the form of one-sided inequalities for the components of the right-hand side f. The estimating expression for the j-th component f_j of f depends on t and x_j and is linear in $|x_j|$. The proofs of Theorem 6 and Theorem 7 are based on the combination of the Lyapunov function method with the modified method of I. Kiguradze [7]. We mention also the paper [9], where the differences between the nonsingular and the singular initial value problem are analyzed.

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A MODIFIED STRONG SQUEEZING PROPERTY AND THE EXISTENCE OF INERTIAL MANIFOLDS OF SEMIFLOWS

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Abstract. Sometimes so-called cone invariance and squeezing properties are used to show the existence of inertial manifolds for evolution equations. We propose and motivate a modification of these properties for semiflows. We show that the cone invariance and modified squeezing properties together with a coercivity assumption are sufficient for a general, continuous semiflow to have an inertial manifold with exponential tracking property.

Keywords. semiflow; inertial manifold; asymptotic phase; cone invariance property; squeezing property

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1. Inertial Manifolds for Semiflows

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space and let S be a semiflow on $\mathbb{X}$, i.e., let $S\colon \mathbb{R}_{\geq 0} \times \mathbb{X} \to \mathbb{X}$, $S^t x := S(t,x)$ satisfy

(S1) $(S^t)_{t\in\mathbb{R}_{\geq 0}}$ is a strongly continuous semigroup of (nonlinear) continuous operators, i.e.,

$$S^0 = I, \quad S^t S^\theta = S^{t+\theta} \text{ for all } t, \theta \geq 0,$$

and $S(\cdot, x)$ and $S^t = S(t,\cdot)$ are continuous for all $x \in \mathbb{X}$, $t \in \mathbb{R}_{\geq 0}$.

Our goal is to find a submanifold M of $\mathbb{X}$ with the following properties:

(M1) M is a finite-dimensional Lipschitz submanifold of $\mathbb{X}$;

(M2) M is *positively invariant* with respect to S, i.e.,

$$\forall u \in M\ \forall t \geq 0\colon\ S^t u \in M; \tag{1}$$

(M3) M has the *exponential tracking property*, i.e., there is an $\eta > 0$ such that, for every $x \in \mathbb{X}$, there are $x' \in M$, $c \geq 0$ with $S^t x' \in M$ and

$$\left\| S^t x - S^t x' \right\| \leq c\mathrm{e}^{-\eta t} \quad \text{for all } t \geq 0.$$

Obviously, such a manifold is a generalization of inertial manifolds for evolution equations which were first introduced and studied by P. Constantin, C. Foias, B. Nicoalenko, G.R. Sell and R. Temam [4,3,1], see also [14], and [6,12] for the exponential tracking property.

As usual, we look for M as a trivial submanifold of $\mathbb{X}$, i.e., we look for

$$M = \operatorname{graph}(m) := \{\xi + m^*(\xi)\colon \xi \in \pi_1\mathbb{X}\}$$

as the graph of function m over a finite-dimensional subspace $\mathbb{X}_1$ of $\mathbb{X}$, where π_1 is a continuous projector from $\mathbb{X}$ onto $\mathbb{X}_1$. Moreover, m shall belong to the Banach space $\mathbb{G} = \mathrm{C_b}(\pi_1\mathbb{X}, \pi_2\mathbb{X})$ of continuous, bounded functions and shall satisfy the Lipschitz inequality

$$\|m(\xi_1) - m(\xi_2)\| \leq \chi \|\xi_1 - \xi_2\| \quad \text{for all } \xi_1, \xi_2 \in \pi_1\mathbb{X}$$

with some fixed $\chi > 0$.

In Sect. 2, we introduce a modification of the cone invariance and squeezing properties (called *modified strong squeezing property*) as a natural geometric assumption on a semiflow to have an inertial manifold as graph of a bounded, globally Lipschitz function over a finite-dimensional subspace. In Sect. 3, we show that this property together with a coercivity property is actually sufficient for the existence of an inertial manifold. In the both last sections, we give a short application to evolution equations and we propose some extensions to more general results.

2. The Strong Squeezing Properties

Cone Invariance Property: If we look for $m \in \mathbb{G}$ satisfying the Lipschitz condition

$$\|m(\xi_1) - m(\xi_2)\| \leq \chi \|\xi_1 - \xi_2\| \quad \text{for all } \xi_1, \xi_2 \in \pi_1\mathbb{X}, \tag{2}$$

and if we don't have additional boundedness properties, we have to look for m in $\mathfrak{M}$, where $\mathfrak{M}$ is the set of all $m \in \mathbb{G}$ with (2). Introducing the cone

$$C_\chi := \{x \in X\colon\ \|\pi_2 x\| \leq \chi \|\pi_1 x\|\},$$

we have

$$m \in \mathfrak{M} \text{ if and only if } m \in \mathbb{G} \text{ and } \forall x \in \operatorname{graph}(m)\colon\ \operatorname{graph}(m) \in x + C_\chi. \tag{3}$$

The required positive invariance (1) and the equivalence (3) yield

$$x_1, x_2 \in \mathrm{graph}(m),\ t \geq 0 \text{ imply } S^t x_1 - S^t x_2 \in C_\chi.$$

Since we only know $m \in \mathfrak{M}$ and because of (3), we replace $x_i \in \mathrm{graph}(m)$ by $x_1 - x_2 \in C_\chi$ and get the following relation

$$\text{(CIP)} \qquad x_1 - x_2 \in C_\chi \text{ implies } S^t x_1 - S^t x_2 \in C_\chi \text{ for } t \geq 0$$

as a natural assumption for the existence of the manifold.

Since (CIP) means the invariance of the cone C_χ with respect to the difference of two positive trajectories, (CIP) is called ***cone invariance property***.

Squeezing Properties: In order to motivate the squeezing properties, we consider the following situation: We assume that S satisfies a cone invariance property (CIP) with parameter $\chi > 0$, and we assume that we have a positively invariant manifold $M = \mathrm{graph}(m)$, $m \in \mathfrak{M}$, with exponential tracking property. Concretely, we assume that for each $x_1 \in \mathbb{X} \setminus M$ there is a $\tilde{x}_1 \in M$ with

$$\text{(4)} \qquad \left\| S^t x_1 - S^t \tilde{x}_1 \right\| \leq c_1 \mathrm{dist}(x_1, M) \mathrm{e}^{-\eta t} \quad \text{for all } t \geq 0,$$

i.e., we assume that the exponential decays of the difference of the trajectory and its asymptotic phase is estimated by the initial distance of x_1 to the manifold.

We sharpen the assumptions on m by the additional assumption that m actually has a Lipschitz constant $L < \chi$.

Then there is a constant $c_2 > 0$ such that

$$\text{(5)} \qquad \forall x, y, z \in \mathbb{X} \text{ with } x - z \notin C_\chi, y - z \in C_L\colon\ \ \|x - z\| \leq c_2 \|x - y\|.$$

Let $x_1 \in \mathbb{X} \setminus M$ and $\tilde{x}_1 \in M$ with (4) and $x_1 - \tilde{x}_1 \notin C_\chi$, and let $\theta > 0$ and $x_2 \in M$ with $S^\theta x_1 - S^\theta x_2 \notin C_\chi$. Then (CIP) implies $S^t x_1 - S^t x_2 \notin C_\chi$ for $t \in [0, \theta]$. With $x = S^t x_1$, $y = S^t \tilde{x}_1$, $z = S^t x_2$ and (5), we obtain

$$\text{(6)} \qquad \left\| S^t x_1 - S^t x_2 \right\| \leq c_2 \left\| S^t x_1 - S^t \tilde{x}_1 \right\| \leq c_1 c_2 \mathrm{dist}(x_1, M) \mathrm{e}^{-\eta t}$$

for all $\theta > 0$, $t \in [0, \theta]$ and all $x_2 \in M$ with $S^\theta x_2 - S^\theta x_1 \notin C_\chi$. Since $\mathrm{dist}(x_1, M) \leq \|x_1 - x_2\|$ and $\|x_1 - x_2\| \leq \sqrt{1 + \chi^{-2}}\, \|\pi_2[x_1 - x_2]\|$, we obtain

$$\left\| S^t x_1 - S^t x_2 \right\| \leq c_3 \|\pi_2[x_1 - x_2]\| \, \mathrm{e}^{-\eta t}$$

with some $c_3 > 0$ and for all $\theta > 0$, $t \in [0, \theta]$ and all $x_1 \in \mathbb{X} \setminus M$, $x_2 \in M$ with $S^\theta x_1 - S^\theta x_2 \notin C_\chi$.

For unknown M, this leads to the assumption

$$\text{(SP)} \qquad \begin{array}{l} \text{There are } \chi_2, \eta > 0 \text{ such that } \theta > 0,\ S^\theta x_1 - S^\theta x_2 \notin C_\chi \text{ imply} \\ \|S^t x_1 - S^t x_2\| \leq \chi_2 \|\pi_2[x_1 - x_2]\| \, \mathrm{e}^{-\eta t} \text{ for all } t \in [0, \theta] \end{array}$$

called ***squeezing property***.

Let us restart with (6). Estimating $\operatorname{dist}(x_1, M) \le \|\pi_2[x_1 - x_3]\|$ with $x_3 \in M$ and $\pi_1 x_3 = \pi_1 x_1$, replacing $x_3 \in M$ by $x_3 - x_2 \in C_\chi$, and replacing $S^\theta x_1 - S^\theta x_2 \notin C_\chi$ by the sharper assumption $\pi_1 S^\theta x_1 = \pi_1 S^\theta x_2$, we find the following modification of squeezing property:

(modSP) There are $\chi_{21}, \chi_{22}, \eta > 0$ such that $\theta > 0$, $\pi_1 S^\theta x_1 = \pi_1 S^\theta x_2$ imply $\|\pi_i[S^t x_1 - S^t x_2]\| \le \chi_{2i} \|\pi_2[x_1 - x_3]\| \mathrm{e}^{-\eta t}$ for all $t \in [0, \theta]$ and all x_3 with $\pi_1 x_3 = \pi_1 x_1$ and $x_3 - x_2 \in C_\chi$

called ***modified squeezing property***.

The combination of the cone invariance property (CIP) with the squeezing property (SP) is called ***strong squeezing property***, see [11]. Analogously, the combination of the cone invariance property (CIP) with the modified squeezing property (modSP) is called ***modified strong squeezing property***.

In the next section we will see the usefullness of the modified strong squeezing property for the existence proof of an inertial manifold. Before this, we compare the strong squeezing property with the modified strong squeezing property.

Checking the proofs of cone invariance properties found in [2,5,6,10,11,14], one can see that the number χ usually is a solution of an inequality $F(\chi) > 0$, where $F\colon\ ]0, \infty[\ \to \mathbb{R}$ is a smooth function. Obviously, at least in these cases a second cone invariance property is satisfied. At least in [11, Proposition 3], such a second cone invariance property is explicitly used.

Lemma 1. *Let the cone invariance property* (CIP) *and the squeezing property* (SP) *be satisfied with the parameter $\chi > 0$. Suppose, there exists $\chi' > \chi$ such that we have a second cone invariance property with χ' instead of χ. Then the modified squeezing property* (modSP) *is satisfied with $\chi_{21} := \frac{\chi_2 \chi'}{\chi(\chi' - \chi)}$, $\chi_{22} := \frac{\chi_2 \chi'}{\chi' - \chi}$.*

Proof. Let $x_1, x_2 \in \mathbb{X}$ with $\pi_1 S^\theta x_1 = \pi_1 S^\theta x_2$ and $\pi_2 S^\theta x_1 \ne \pi_2 S^\theta x_2$. Then $S^\theta x_1 - S^\theta x_2 \notin C_\chi$ and $S^\theta x_1 - S^\theta x_2 \notin C_{\chi'}$. The cone invariance property implies $S^t x_1 - S^t x_2 \notin C_\chi$ and $S^t x_1 - S^t x_2 \notin C_{\chi'}$ for all $t \in [0, \theta]$, i.e., we have

$$\chi \left\|\pi_1[S^t x_1 - S^t x_2]\right\| \le \chi' \left\|\pi_1[S^t x_1 - S^t x_2]\right\| < \left\|\pi_2[S^t x_1 - S^t x_2]\right\| \tag{7}$$

for all $t \in [0, \theta]$. Let $x_3 \in \mathbb{X}$ with $\pi_1 x_3 = \pi_1 x_1$ and $x_3 - x_2 \in C_\chi$, i.e.,

$$\|\pi_2[x_2 - x_3]\| \le \chi \|\pi_1[x_2 - x_1]\| . \tag{8}$$

Using (7) and (8), we find $\chi' \|\pi_1[x_1 - x_2]\| \le \|\pi_2[x_1 - x_3]\| + \chi \|\pi_1[x_1 - x_2]\|$ and, hence,

$$\|\pi_1[x_1 - x_2]\| \le \frac{1}{\chi' - \chi} \|\pi_2[x_1 - x_3]\| .$$

By the squeezing property (SP) and (7), we have

$$\begin{aligned} \left\|\pi_2[S^t x_1 - S^t x_2]\right\| &\le \chi_2 \mathrm{e}^{\eta t} \left(\|\pi_2[x_1 - x_3]\| + \|\pi_2[x_3 - x_2]\|\right) \\ &\le \chi_2 \mathrm{e}^{\eta t} \left(\|\pi_2[x_1 - x_3]\| + \chi \|\pi_1[x_1 - x_2]\|\right) \\ &\le \frac{\chi_2 \chi'}{\chi' - \chi} \mathrm{e}^{\eta t} \|\pi_2[x_1 - x_3]\| , \end{aligned}$$

for all $t \in [0, \theta]$, i.e., (modSP) holds.

Thus, the strong squeezing property together with a second cone invariance property implies our modified strong squeezing property, i.e., in general, the modified strong squeezing property is the weaker assumption.

3. Construction of Inertial Manifolds

Let S be a semiflow on the Banach space $\mathbb{X}$. Let $\mathbb{X}_1$ be a finite-dimensional subspace of $\mathbb{X}$, π_1a continuous projector from $\mathbb{X}$ onto $\mathbb{X}_1$ and let $\pi_2 = I - \pi_1$. We assume that S satisfies the cone invariance property (CIP) and the modified squeezing property (modSP) with fixed $\chi > 0$. As technical assumptions we need

(S2) S satisfies the coercivity property $\|\pi_1 S^t x\| \to \infty$ as $\|\pi_1 x\| \to \infty$ in $\mathbb{X}$ for $t \geq 0$.

(S3) There is a positively invariant strip $\sum := \{x \in \mathbb{X} : \|\pi_2 x\| \leq \sigma\}$.

Theorem 1. *Under the above assumptions, there is an inertial manifold* $M = \mathrm{graph}(m)$ *with bounded* $m\colon \pi_1\mathbb{X} \to \pi_2\mathbb{X}$ *satisfying a global Lipschitz condition with constant* χ*. Moreover, for each* $x_1 \in \mathbb{X}$*, there is a* $x_2 \in M$ *with*

$$\left\|\pi_i[S^t x_1 - S^t x_2]\right\| \leq \chi_{2i} \left\|\pi_2 x_1 - m^*(\pi_1 x_1)\right\| \mathrm{e}^{-\eta t} \quad \textit{for all } t > 0.$$

Proof. We devide the proof into the following three steps:

Step 1: The Graph Transformation Mapping. We wish to construct $M = \mathrm{graph}(m^*)$ by an graph transformation mapping, i.e., m^* shall be the fixed point of suitable mappings $G^\theta\colon \mathfrak{M} \to \mathbb{G}$, $\theta > 0$, with

$$\mathrm{graph}(G^\theta m) = S^\theta \mathrm{graph}(m) \quad \text{for all } m \in \mathfrak{M}.$$

where $\mathfrak{M}$ is the set of all $m \in \mathbb{G}$ with (2) and $\mathrm{graph}(m) \subset \sum$. Concretely, we wish to define G^θ by $(G^\theta m)(\xi) := \pi_2 S^\theta x$ if $\pi_1 S^\theta x = \xi$. For it, we have to show that, for any $\xi \in \pi_1\mathbb{X}$, $\theta > 0$, $m \in \mathfrak{M}$, the boundary value problem

$$x \in \mathrm{graph}(m), \quad \pi_1 S^\theta x = \xi \tag{9}$$

has a unique solution $x = X(\theta, \xi, m)$.

Let $\theta > 0$, $\xi \in \pi_1\mathbb{X}$, $m \in \mathfrak{M}$, and x_1, x_2 with

$$\pi_1 S^\theta x_1 = \pi_1 S^\theta x_2 = \xi \quad \text{and} \quad x_2 \in \mathrm{graph}(m).$$

If we choose $x_3 := \pi_1 x_1 + m(\pi_1 x_1)$, then the modified squeezing property (modSP) implies

$$\left\|\pi_i[S^t x_1 - S^t x_2]\right\| \leq \chi_{2i} \left\|\pi_2 x_1 - m(\pi_1 x_1)\right\| \mathrm{e}^{-\eta t} \quad \text{for all } t \in [0, \theta]. \tag{10}$$

In particular, for $x_1 \in \operatorname{graph}(m)$, we have $\pi_2 x_1 = m(\pi_1 x_1)$ and (10) implies

$$\forall \theta > 0 \forall x_1, x_2 \in \operatorname{graph}(m)\colon\ \pi_1 S^\theta x_1 = \pi_1 S^\theta x_2 \Longrightarrow x_1 = x_2, \tag{11}$$

i.e., for each $\theta > 0$, $m \in \mathfrak{M}$, $\xi \in \pi_1\mathbb{X}$ there is at most one $x \in \operatorname{graph}(m)$ with $\pi_1 S^\theta x = \xi$.

Let $\theta > 0$, $m \in \mathfrak{M}$ be fixed and let $H\colon \pi_1\mathbb{X} \to\colon \pi_1\mathbb{X}$ be defined by $H(\zeta) := \pi_1 S^\theta(\zeta + m(\zeta))$.

By the continuity of S^θ, H is continuous with inverse H^{-1} given by $H^{-1}(\xi) = \pi_1 X(\theta, \xi, m)$ on $H\pi_1\mathbb{X}$. In order to show $H\pi_1\mathbb{X} = \pi_1\mathbb{X}$, we wish to show the continuity of H^{-1}. Suppose, there is a $\xi \in \pi_1\mathbb{X}$ such that H^{-1} is not continuous at ξ. Then there are $\varepsilon > 0$ and a sequence $(\xi_k)_{k\in\mathbb{N}}$ in $\pi_1\mathbb{X}$ such that $\xi_k \to \xi$ as $k \to \infty$ and

$$\|\zeta - \zeta_k\| \geq \varepsilon \quad \text{for all } k \in \mathbb{N} \tag{12}$$

where $\zeta := X(\theta, \xi, m)$, $\zeta_k := X(\theta, \xi_k, m)$.

First we suppose that there is a subsequence of $(\zeta_k)_{k\in\mathbb{N}}$, denoted for shortness again by $(\zeta_k)_{k\in\mathbb{N}}$, with $\|\zeta_k\| \to \infty$ as $k \to \infty$. Then the coercivity property (S2) implies $\left\|\pi_1 S^\theta(\zeta_k + m(\zeta_k))\right\| \to \infty$ in contradiction to $S^\theta(\zeta_k + m(\zeta_k)) \to S^\theta(\zeta + m(\zeta))$.

Remains the boundedness of $(\zeta_k)_{k\in\mathbb{N}}$. Since $\pi_1\mathbb{X}$ is finite-dimensional space $\pi_1\mathbb{X}$, there is a convergent subsequence, denoted for shortness again by $(\zeta_k)_{k\in\mathbb{N}}$, with a limit $\zeta_\infty \in \pi_1\mathbb{X}$. By the continuity of S^θ, we have $S^\theta(\zeta_\infty + m(\zeta_\infty)) = S^\theta(\zeta + m(\zeta))$, and hence $\zeta = \zeta_\infty$ in contrast to (12) and (11).

Therefore, H and H^{-1} are continuous. Because of (S2), we have $\|H(\xi)\| \to \infty$ for $\|\xi\| \to \infty$. Thus, H is a homeomorphism from $\pi_1\mathbb{X}$ onto $\pi_1\mathbb{X}$ and hence we have $H\pi_1\mathbb{X} = \pi_1\mathbb{X}$. Therefore, for each $\theta > 0$, $m \in \mathfrak{M}$, $\xi \in \pi_1\mathbb{X}$, we have a unique solution $X(\theta, \xi, m)$ of (9), and we can define the graph transformation mappings G^θ by

$$\left(G^\theta m\right)(\xi) = \pi_2 S^\theta X(\theta, \xi, m) \quad \text{for } \theta > 0, m \in \mathfrak{M}, \xi \in \pi_1\mathbb{X}.$$

Step 2: Fixed-Points of the Graph Transformation Mapping. Let $\theta > 0$, $m \in \mathfrak{M}$, $\xi_1, \xi_2 \in \pi_1\mathbb{X}$ be arbitrary. By (S3) we have $\operatorname{graph}(G^\theta m) \subset \sum$. By the cone invariance property (CIP), we have

$$\begin{aligned}\left\|\left(G^\theta m\right)(\xi_1) - \left(G^\theta m\right)(\xi_2)\right\| &\leq \chi \left\|\pi_1[S^\theta X(\theta, \xi_1, m) - S^\theta X(\theta, \xi_2, m)]\right\| \\ &= \chi \|\xi_1 - \xi_2\|,\end{aligned}$$

i.e., G^θ maps $\mathfrak{M}$ into itself for each $\theta > 0$.

Now let $\theta > 0$, $\xi \in \pi_1\mathbb{X}$, $m_1, m_2 \in \mathfrak{M}$, and x_1, x_2 with

$$\pi_1 S^\theta x_1 = \pi_1 S^\theta x_2 = \xi \quad \text{and} \quad x_i \in \operatorname{graph}(m_i).$$

If we choose $x_3 := \pi_1 x_1 + m_2(\pi_1 x_1)$, then $x_3 - x_2 \in C_\chi$ and (modSP) imply

$$\left\|\pi_2[S^\theta x_1 - S^\theta x_2]\right\| \leq \chi_{22} \|m_1(\pi_1 x_1) - m_2(\pi_1 x_1)\| \, \mathrm{e}^{-\eta\theta} \quad \text{for } \theta \geq 0.$$

Thus

$$\left\|\left(G^\theta m_1\right)(\xi)-\left(G^\theta m_2\right)(\xi)\right\| \leq \chi_{22}\mathrm{e}^{-\eta\theta}\left\|m_1(\pi_1 X(\theta,\xi,m_1))-m_2(\pi_1 X(\theta,\xi,m_1))\right\|,$$

i.e.,

$$\left\|G^\theta m_1 - G^\theta m_2\right\|_{\mathbb{G}} \leq \kappa(\theta)\left\|m_1 - m_2\right\|_{\mathbb{G}} \quad \text{for all } \theta > 0 \text{ and } m_1, m_2 \in \mathfrak{M},$$

where $\kappa(\theta) := \chi_{22}\mathrm{e}^{-\eta\theta}$. Since $\eta > 0$, there is a $\theta_0 > 0$ with $\kappa(\theta) < 1$ for $\theta \geq \theta_0$. Thus, for $\theta \geq \theta_0$, G^θ is a contractive self-mapping on the closed subset $\mathfrak{M}$ of the Banach space $\mathbb{G}$. Hence, for each $\theta \geq \theta_0$, there is a unique fixed-point $m^{(\theta)}$ of G^θ in $\mathfrak{M}$.

Let $p \in \mathbb{N}_{>0}$. Then $m^{(\theta)}$ is a fixed-point of $G^{p\theta}$ and hence $m^{(p\theta)} = m^{(\theta)}$ for $\theta \geq \theta_0$ and $p \in \mathbb{N}_{>0}$. Let $q \in \mathbb{N}_{>0}$. Because of

$$G^\theta\left(G^{\frac{1}{q}\theta}m^{(\theta)}\right) = G^{\frac{1}{q}\theta}\left(G^\theta m^{(\theta)}\right) = G^{\frac{1}{q}\theta}m^{(\theta)}$$

and the uniqueness of the fixed-point $m^{(\theta)}$ of G^θ, $m^{(\theta)}$ is the unique fixed-point of $G^{\frac{1}{q}\theta}$ for $\theta \geq \theta_0$ and each $q \in \mathbb{N}_{>0}$. Thus, for each $\theta > 0$, there is a unique fixed-point $m^{(\theta)}$ of G^θ and we have $m^{(\frac{p}{q}\theta)} = m^{(\theta)}$ for $\theta > 0$ and all $p, q \in \mathbb{N}_{>0}$. Hence,

$$S^{\frac{p}{q}\theta_0}x \in \operatorname{graph}(m^{(\theta_0)}) \quad \text{for } u \in \operatorname{graph}(m^{(\theta_0)}) \text{ and } m, n \in \mathbb{N}_{>0}$$

and the continuity of $t \mapsto S^t u$ yields

$$S^\theta x \in \operatorname{graph}(m^{(\theta_0)}) \quad \text{for } \theta > 0 \text{ and } x \in \operatorname{graph}(m^{(\theta_0)}).$$

Thus, $m^* := m^{(\theta_0)} = m^{(\theta)}$ for all $\theta > 0$, and $M = \operatorname{graph}(m^*)$ is positively invariant with respect to S.

Step 3: Existence of Asymptotic Phases. Let $x_1 \in \mathbb{X}$ and let $(t_k)_{k\in\mathbb{N}}$ be a monotonously increasing sequence of real numbers t_k with $t_k \to \infty$ for $k \to \infty$. Further, let $\zeta_k := \pi_1 X(t_k, S^{t_k}x_1, m^*)$. By (10), we have

$$\left\|\pi_i[S^t x_1 - S^t X(t_k, S^{t_k}x_1, m^*)]\right\| \leq \chi_{2i}\left\|\pi_2 x_1 - m^*(\pi_1 x_1)\right\|\mathrm{e}^{-\eta t} \quad \text{for all } t \in [0, t_k].$$

In particular, we find $\|\pi_1 x_1 - \zeta_k\| \leq \chi_{21}\|\pi_2 x_1 - m^*(\pi_1 x_1)\|$. If $\pi_1\mathbb{X}$ is finite dimensional, then the bounded and closed set

$$\{\zeta \in \pi_1\mathbb{X}\colon\ \|\pi_1 x_1 - \zeta\| \leq \chi_{21}\|\pi_2 x_1 - m^*(\pi_1 x_1)\|\}$$

is compact. Thus, there is subsequence of $(t_k)_{k\in\mathbb{N}}$, denoted again by $(t_k)_{k\in\mathbb{N}}$, such that $(\zeta_k)_{k\in\mathbb{N}}$ is converging to some $\zeta^* \in \pi_1\mathbb{X}$. Let $x_2 := \zeta^* + m^*(\zeta^*)$. Then

$$\begin{aligned}
&\|\pi_i[S^t x_1 - S^t x_2]\| \\
&\quad \leq \|\pi_i[S^t x_1 - S^t X(t_k, S^{t_k}x_1, m^*)]\| + \|\pi_i[S^t X(t_k, S^{t_k}x_1, m^*) - S^t x_2]\| \\
&\quad \leq \chi_{2i}\|\pi_2 x_1 - m^*(\pi_1 x_1)\|\mathrm{e}^{-\eta t} + \|\pi_i[S^t(\zeta_k + m^*(\zeta_k) - S^t(\zeta^* + m^*(\zeta^*)]\|
\end{aligned}$$

for all $\theta > 0$, $t \in [0, \theta]$ and all $k \in \mathbb{N}_{>0}$ with $t_k \geq \theta$. By the continuity of m^* and S, and because of $\zeta_k \to \zeta^*$, the second term can be made arbitrary small on $[0, \theta]$ choosing k large enough. Therefore,

$$\left\|\pi_i[S^t x_1 - S^t x_2]\right\| \leq \chi_{2i} \left\|\pi_2 x_1 - m^*(\pi_1 x_1)\right\| \mathrm{e}^{-\eta t} \quad \text{for all } \theta > 0, t \in [0, \theta],$$

i.e., $t \mapsto S^t x_2$ is an asymptotic phase of $t \mapsto S^t x_1$ in M.

4. Application to Evolution Equations

We consider the evolution equation

$$\dot{u} + Au = f(u) \tag{13}$$

with selfadjoint, positive definite densly defined linear operator A in the separable Hilbert space $(\mathbb{H}, |\cdot|_0)$. Further let $f \in \mathrm{C_b}(D(A^\alpha), \mathbb{H})$ satisfy the Lipschitz inequality

$$|f(u) - f(u')|_0 \leq L|u - u'|_\alpha \quad \text{for all } u, u' \in D(A),$$

where $\alpha \in [0, \frac{1}{2}]$. Let π_1 be the orthogonal projection from $\mathbb{H}$ onto the N-dimensional subspace of $\mathbb{H}$ spanned by the N eigenvectors belonging to the first N eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ of A, counted with their multiplicity.

Then (13) generates a semiflow S on $\mathbb{X} = D(A^\alpha)$ satisfying (S1), cf. [7,9,8] for (S1). The coercivity property (S2) follows from the variation of constant formula, the boundedness of f and $|\pi_1 \mathrm{e}^{-At} u| \geq C\mathrm{e}^{-\lambda_N t}|\pi_1 u|$. Studying the quadratic form $Q_\chi(u) = |\pi_2 u|_\alpha^2 - \chi^2 |\pi_1 u|_\alpha^2$ along the difference of solutions of (13), in [8] was shown, that there is a $\chi > 0$ with

$$\frac{d}{dt} Q_\chi(S^t u_1 - S^t u_2) \leq \Lambda(\chi) Q_\chi(S^t u_1 - S^t u_2) \text{ and } \Lambda(\chi) < 0 \tag{14}$$

if the spectral gap condition

$$\lambda_{N+1} - \lambda_N > cL\left(\lambda_N^\alpha + \lambda_{N+1}^\alpha\right) \tag{15}$$

holds with $c = 1$. Romanov showed in [13], that the spectral gap condition (15) is sharp in the following sense: For each $c \in [0, 1[$, there are two-dimensional evolution equations (13) in $\mathbb{X} = \mathbb{R}^2$ without inertial manifold (i.e., here instable manifolds) but satisfying (15).

In particular, we may choose $\chi = \chi_0 := \sqrt{\lambda_N^\alpha \lambda_{N+1}^{-\alpha}}$. Moreover, in [8] was shown that (14) implies the modified strong squeezing property (CIP), (modSP) of S. In particular, for $\chi = \chi_0$ we may choose $\eta := -\lambda_{N+1} + L\lambda_{N+1}^\alpha$, $\chi_{21} := \frac{1}{\chi_0 - \underline{\chi}}$, $\chi_{22} := \frac{\chi_0}{\chi_0 - \underline{\chi}}$, where $\underline{\chi} < \overline{\chi}$ are the positive solutions of

$$\left(\lambda_{N+1} - \lambda_N\right)^2 \chi^2 = L^2\left(\chi^2 + 1\right)\left(\lambda_N^{2\alpha} + \chi^2 \lambda_{N+1}^{2\alpha}\right).$$

5. Extensions

Let $\mathbb{X}$ be densely imbedded in the Banach space $\mathbb{Y}$. If the cone invariance and modified squeezing property are required only with respect to the weaker norm $\|\cdot\|_{\mathbb{Y}}$, we need an additional smoothing property of S in the form that there is a function $c_0\colon\ ]0,\infty[\ \to\ ]0,\infty[$ with $\|S^t u\|_{\mathbb{X}} \le c_0(t)\,\|u\|_{\mathbb{Y}}$ for $u \in \mathbb{X}$ and $t > 0$. This approach allows $\alpha \in [0,1[$ for the evolution equation (13) if $\mathbb{Y} = D(A^\nu)$ with $\nu \in [0, \min\{\alpha, \frac{1}{2}\}]$, see [8].

Another approach consists in the construction of a manifold $M = \operatorname{graph}(m)$ with bounded domain $D(m) \subset \mathbb{X}_1$ as an overflowing invariant manifold, see [8]. For it we need some overflowing and inflowing properties of the semiflow on the boundary of a subset V of $\mathbb{X}$ in which the manifold shall be constructed. Then the technical assumption (S2) can be removed, since the needed bijectivity of the corresponding mapping H can be shown by the homotopy theorem. For the evolution equation (13), this allows to replace the global boundedness and Lipschitz assumptions on f by corresponding assumptions on V.

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PROPERTY A OF THE $(n+1)^{th}$ ORDER DIFFERENTIAL EQUATION $\left[\frac{1}{r_1(t)}\ \left(x^{(n)}(t)+p(t)x(t)\right)\ \right]' = f(t, x(t), \cdots, x^{(n)}(t))$.

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Abstract. The aim of this contribution is to study properties of solutions of the $n+1^{th}$-order differential equation of the form

$$\left[\frac{1}{r_1(t)}\ \left(x^{(n)}(t)+p(t)x(t)\right)\ \right]' = f(t, x(t), \cdots, x^{(n)}(t)) . \tag{1}$$

where $n \geq 2$ is a natural number. A new approach using "submersivity" of a solution of an equation is presented, by means of it a sufficient condition for the property A is proved. This approach can be also used to prove necessary condition for the property A.

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1. Preliminaries

The main goal of this paper is to study certain properties of solutions of the differential equations, which are very appropriate for exploring the Property A. In this paper we consider only the proper solutions of the equations.

A solution $u(.) \in \mathbb{C}^n[T_0, \infty)$ is called oscillatory at $+\infty$ if it is proper, and there exists a sequence of numbers $\{t_k\}, k \in N$ such that $t_k \in [T_0, \infty)$, $u(t_k) = 0$, $k \in N$ and $\lim\limits_{t\to\infty} t_k = +\infty$ hold.

A solution $u(.)$ is called non-oscillatory proper (briefly non-oscillatory) if it is proper and there exists a number $\bar{t} \in R^+$ such that $u(t) \neq 0$ for $t \geq \bar{t}$.
Briefly, we can say that the proper solution is said to be oscillatory, if it has a sequence of zeros converging to $+\infty$, otherwise is said to be non-oscillatory.

We will say that an equation has the Property A, if each proper solution of this equation is oscillatory when n is even and is either oscillatory or satisfies the condition

$$\lim_{t\to\infty} u^{(i)}(t) = 0\,, \qquad \text{monotonically}, \qquad i = 0, 1, \ldots, n-1\,,$$

when n is odd.

2. "Submersivity" of a solution of the equation $y^{(n)}(t) + \alpha_1(t)\ y^{(n-1)}(t) + ... + \alpha_n(t)\ y(t) + p(t)\ y(t) = r(t)$.

One can describe "submersivity" as the ability of the function not to overcome a certain level ε for a certain time interval $[t_0, t_0 + \delta]$. The function having these properties behaves as follows: from a certain $t > t_0$, it dives under a certain level of ε and keeps being under this level maximally during a time interval δ.

Let us find a criterion of "submersivity" as the simplest possible conditions to be imposed on the equations, usually to the left-hand side of the equations.

This property is of major importance for exploring the questions about the oscillating and non-oscillating properties of a solution, and it can be directly used to prove necessary and sufficient conditions for property A.

Similar problems were posed and solved by I. T. Kiguradze [5] for the equation of the form $u^{(n)} + u^{(n-2)} = f(t, u, u', \ldots, u^{(n-1)})$. In his paper for the first time was considered the case of oscillatory left-hand side operator. The results given in it fill this gap to some extent.

Consequently the knowledge about situation in the oscillatory cases was studied in a few papers for the third order diff. equation. The result of this kind was presented e.g. by Cecchi, Došlá, Marini [1], Greguš, Graef [2], Greguš, Gera, Graef, [3,4].

Similar properties of solutions were investigated by several authors, namely for the equation of the form $u^{(n)} = f(t, u, u', \ldots, u^{(n-1)})$. Many results also have been obtained for the equation of the type $u^{(n)} + \sum p_k(t) u^{(k)} = f(t, u, \ldots, u^{(n-1)})$, with a disconjugated left-hand side operator.

Our aim is a little different: We consider here *the case on the left-hand side kernel operator of the equations (1) can be oscillatory.* The new what we bring to this problem was directly the assumption on the oscillatory left-hand side kernel operator.

"Submersivity" properties can help us to explore the questions of oscillation of solutions in the case of the oscillatory left-hand side operator. Similar theorem, as follows, appeared in [5], but only for the case $\alpha_i(t) \equiv 0$, $p(t) \equiv 1$.

Theorem 1. *Let $n \geq 2$ and let the functions $\{\alpha_i(.)\}_{i=1}^{n}$, $p(.)$, $r(.) \in \mathbb{C}[T_0, \infty)$ satisfy the assumptions*

$$\text{(2)} \qquad (i) \quad \text{for all } i \in \{1, \cdots, n\}, \qquad \lim_{t\to\infty} \alpha_i(t) = 0\,,$$

(ii) there exist constants $r_{max},\ r_{min} > 0$ such that

$$\text{(3)} \qquad |\, p(t)\,| < r_{max} \quad \text{and} \quad r_{min} \leq r(t) \leq r_{max} \qquad \text{for all } \ t \in [T_0, \infty).$$

Then for each $\delta_0 > 0$ and each $p_0 \in (0,1)$, there exist $T \geq T_0$ and $\varepsilon > 0$ with the following property:

If $y(.) \in \mathbb{C}^n[T_0, \infty)$ is a non-negative solution of the differential equation

$$\text{(4)} \qquad y^{(n)}(t) + \alpha_1(t)\, y^{(n-1)}(t) + \ldots + \alpha_n(t)\, y(t) + p(t)\, y(t) = r(t)\,,$$

then for all $t_0 > T$ and for all $\delta > \delta_0$

$$\text{(5)} \qquad \mu\left([t_0, t_0 + \delta_0] \cap y^{-1}[0, \varepsilon]\right) \leq p_0\ \delta\,,$$

where μ denotes the Lebesgue measure of sets.

Proof. Let the functions $\{\alpha_i(.)\}_{i=1}^{n}$, $p(.), r(.)$ and constants r_{max}, r_{min}, δ_0, p_0 satisfied the conditions (2) and (3). Let m be the least natural number satisfying the inequality $r_{min} < 2^{2+n(n+1)}\,(2^m - 1)$. Let

$$q_{min} = \frac{r_{min}}{2^m} \qquad \text{and} \qquad q_{max} = r_{max} + \frac{r_{min}}{2^m}\,(2^m - 1)\,.$$

Moreover, put

$$\text{(6)} \qquad \alpha_0 = \frac{2.2^{n\,(n+1)}}{q_{min}}\,, \qquad \varepsilon_{max} = \left(\min\{p_0\,\delta_0\ ,\ \frac{r_{min}}{2\,r_{max}}\}\right)^n\,.$$

Let $p_0 \in (0,1)$ be an arbitrary, but fixed number. Lemma 5.1 from [6] ensures the existence of a constant $P_1 > 0$ such that: If $z(.) \in C^n[0,1]$ is a solution of the differential equation

$$\text{(7)} \qquad z^{(n)}(t) + p_1(t)\, z^{(n-1)}(t) + \cdots + p_n(t)\, z(t) = \alpha \cdot q(t)\,,$$

with the property

$$\text{(8)} \qquad 0 \leq z(t) \leq 1 \qquad \text{for all } \ t \in [0,1]\,,$$

where the functions $p_i(.),\ q(.)$ satisfy conditions

$$\text{(9)} \qquad 0 < q_{min} \leq q(t) \leq q_{max}\,, \qquad P(t) = \sum_{i=1}^{n} |\, p_i(t)\,| \leq P_1 \qquad \forall t \in [0,1]\,,$$

then for the constant α from the equation (7) we have $\alpha \leq \frac{2.2^{n(n+1)}}{q_{min}}$.

Theorem 5.2 from [6] guaranties that for arbitrary constants q_{min}, q_{max}, α_{max}, such that $\alpha_{max} > 0$, $0 < q_{min} \leq q_{max}$, and $p \in (0,1)$, there exist constants $P_2 > 0$ and $\varepsilon_2 \in (0,1)$, with the following property:

If $z(.) \in C^n[0,1]$ is a solution of the differential equation (7) such that (8) and $z(0) = 1$, where the constant α and functions $p_i(.),\ q(.) \in C[0,1]$ satisfy condition (9) and

$$0 < \alpha \leq \alpha_{max}\,, \qquad P(t) = \sum_{i=1}^{n} \mid p_i(t) \mid \leq P_2 \qquad \forall t \in [0,1]\,,$$

then $\mu(z^{-1}[0,\varepsilon_2]) \leq p$.

First we will find the required $T > T_0$ and $\varepsilon > 0$. Set $\varepsilon = \varepsilon_2\ \frac{\varepsilon_{max}}{\alpha_0}$ and choose T_1 sufficiently large such that

$$\sum_{i=1}^{n} |\alpha_i(t)| \leq \min\{P_1, P_2\}\,, \qquad \forall t > T_1\,. \tag{10}$$

Then the required T can be defined by $T = \max\{T_1, T_0\}$. It is enough to prove that ε and T chosen in this way ensure the validity of Theorem 1.

Let $y \in \mathbb{C}^n[T,\infty)$ be a solution of the equation (4), such that $y(t) \geq 0$ for all $t \in [T,\infty)$. For $t_0 > T$ and $\delta > \delta_0$ define the set M by $M = (t_0, t_0+\delta) \cap y^{-1}(-1, \frac{\varepsilon_{max}}{\alpha_0})$. The set M is open, so it can be expressed as at most countable union of the disjoint open intervals $(t_i, t_i+\delta_i)$, i. e. , $M = \bigcup_{i=1}^{l} (t_i, t_i+\delta_i)\,, 1 \leq l \leq \infty$. Note, if M is an empty set, the assertion of Theorem 1 holds.

Let us take one of these intervals $(t_j, t_j+\delta_j)$, $j \in \{1,\ldots,l\}$ and set $s = t_j$ and $\tilde{\delta} = \min\{1, \delta_j\}$. Applying the transformation

$$z(t) = \frac{\alpha_0}{\varepsilon_{max}}\ y(s+\tilde{\delta}t)\,, \tag{11}$$

the equation (4) can be changed into the form

$$z^{(n)}(t) + p_1(t)\ z^{(n-1)}(t) + \cdots + p_n(t)\ z(t) = \alpha \cdot q(t)\,,$$

where

$$\begin{aligned} z^{(k)}(t) &= \tfrac{\alpha_0}{\varepsilon_{max}}\ \tilde{\delta}^k\ y^{(k)}(s+\tilde{\delta}t)\,, & p_k(t) &= \tilde{\delta}^k\ \alpha_k(s+\tilde{\delta}t)\,, \\ q(t) &= r(s+\tilde{\delta}t) - p(s+\tilde{\delta}t)\ \ y(s+\tilde{\delta}t)\,, & \alpha &= \frac{\alpha_0}{\varepsilon_{max}}\ \ \tilde{\delta}^n\,. \end{aligned} \tag{12}$$

It is easy to verify the fulfilling of the assumptions of Lemma 5.1 from [6] and Theorem 5.1 from [6] by the functions $z(.), p_i(.), q(.)$ and the constant α.

So, summarizing, we have $0 < q_{min} \leq q(t) \leq q_{max}$. Next, it is clear that $0 \leq z(t) \leq 1$, for all $t \in [0,\tilde{\delta}]$. According to (10), and the definitions of $\tilde{\delta}$ and $p_i(.)$ we have $\sum_{i=1}^{n} |p_i(t)| \leq P_1$. Hence Lemma 5.1 from [6], whose assumptions are fulfilled, ensures that the constant $\alpha > 0$, given by (12) satisfies the inequality $\alpha \leq \frac{2.2^{n(n+1)}}{q_{min}}$. This inequality implies

$$\frac{\alpha_0}{\varepsilon_{max}}\,\tilde{\delta}^n \leq \frac{2.2^{n(n+1)}}{q_{min}}$$

which using (6) gives

$$\frac{\tilde{\delta}^n}{\varepsilon_{max}} \leq 1\,.$$

Due to (6) we have

$$\tilde{\delta}^n \leq \left(\min\{p_0\,\delta_0\,, \frac{r_{min}}{2\,r_{max}}\}\right)^n, \qquad \text{or} \qquad \tilde{\delta} \leq \left(\min\{p_0\,\delta_0\,, \frac{r_{min}}{2\,r_{max}}\}\right),$$

which means that

$$\tilde{\delta} \leq p_0\;\delta_0 \qquad \text{and} \qquad \tilde{\delta} \leq \frac{r_{min}}{2\,r_{max}} \leq \frac{1}{2}\,.$$

Clearly, for all intervals $[t_i, t_i+\delta_i],\ i \geq 2$, the assumptions of Theorem 5.1 from [6] are fulfilled. The property $y(t_i) = \frac{\varepsilon_{max}}{\alpha_0}$ implies $z(t_i) = 1$. Due to Theorem 5.1 from [6] there exists a positive constant ε_2 such that $\mu\left(z^{-1}[0, \varepsilon_2]\right) \leq p_0$.
From (11) we can see that

$$\frac{1}{\delta_i}\,\mu\left([t_i, t_i+\delta_i] \cap y^{-1}[0, \varepsilon_2\,\frac{\varepsilon_{max}}{\alpha_0}]\right) \leq p_0\,. \tag{13}$$

This clearly forces

$$\mu\left([t_i, t_i+\delta_i] \cap y^{-1}[0, \varepsilon]\right) \leq p_0\;\delta_i \tag{14}$$

on all intervals $[t_i, t_i+\delta_i]$, for $i \geq 2$.

It remains to prove the validity of the estimation (14) on the interval $[t_1, t_1+\delta_1]$.

If, $y(t_1) = y(t_0) = \frac{\varepsilon_{max}}{\alpha_0}$, then the estimation (14) is evidently true.

If, $y(t_1) < \frac{\varepsilon_{max}}{\alpha_0}$, then using the backward transformation $x(t) = y(t_1+\delta_1-t)$ we obtain $x(t_1) = y(\delta_1) = \frac{\varepsilon_{max}}{\alpha_0}$. Let $z(t) = \frac{\alpha_0}{\varepsilon_{max}}\;x(t_1+\delta_1 t)$. Then by similar arguments we can obtain the estimation $\mu\left(z^{-1}[0, \varepsilon_2]\right) \leq p_0$ also on the interval $[t_1, t_1+\delta_1]$. Thus (compare with (13))

$$\frac{1}{\delta_1}\mu\left([t_1, t_1+\delta_1] \cap x^{-1}[0, \varepsilon_2\,\frac{\varepsilon_{max}}{\alpha_0}]\right) \leq p_0$$

and further

$$\mu\left([t_1, t_1+\delta_1] \cap x^{-1}[0, \varepsilon]\right) = \mu\left([t_1, t_1+\delta_1] \cap y^{-1}[0, \varepsilon]\right) \leq p_0\delta_1\,. \tag{15}$$

Recall that $\varepsilon = \varepsilon_2 \, \frac{\varepsilon_{max}}{\alpha_0}$, and $\varepsilon_2 \in (0,1)$. From (14) and (15) we can conclude that

$$\begin{aligned}
&\mu\left([t_0, t_0+\delta_0] \cap y^{-1}[0,\varepsilon]\right) \leq \mu\left(M \cap y^{-1}[0,\varepsilon]\right) = \\
&= \mu\left([\bigcup_{i=1}^{l}(t_i, t_i+\delta_i)] \cap y^{-1}[0,\varepsilon]\right) \leq \sum_{\substack{1<i\leq l \\ t_i>t_0}} \mu\left([t_i, t_i+\delta_i] \cap y^{-1}[0,\varepsilon]\right) + p_0\,\delta_1 \leq \\
&\leq p_0\,\delta_1 + \sum_{\substack{1<i\leq l \\ t_i>t_0}} p_0\;\delta_i \leq p_0 \left[\delta_1 + \sum_{1<i\leq l} \mu\left(t_i, t_i+\delta_i\right)\right] = p_0\;\mu(M) \leq p_0\;\delta\,,
\end{aligned}$$

which is the required conclusion. □

3. Formulation of the Problem

The aim of this paper is to study properties of solutions of the nonlinear $n+1^{th}$ - order differential equation of the form

$$\left[\frac{1}{r_1(t)}\;\left(x^{(n)}(t) + p(t)x(t)\right)\;\right]' = f(t, x(t), \cdots, x^{(n)}(t))\,. \tag{16}$$

where $n \geq 2$ is a natural number.

Let M_1 and M_2 be constants such that $0 < M_1 \leq M_2$.

Let the functions $p(.)$, $r_1(.)$ and $f = f(t, x_0, \ldots, x_n)$ satisfy conditions:

– Let the function $r_1(.) \in \mathbb{C}^1[0,\infty)$ and have the property

$$0 < M_1 \leq r_1(t) \leq M_2\,, \qquad \forall t \geq \tilde{T} > 0\,, \tag{17}$$

– Let the function $p(.) \in \mathbb{C}^1[0,\infty)$ have the property

$$0 < M_1 \leq p(t) \leq M_2\,, \qquad \forall t \geq \tilde{T} > 0\,, \tag{18}$$

– Let the function $f = f(t, x_0, x_1, \ldots, x_n)$ be continuous on $R^+ \times R^{n+1}$ and has the sign property

$$f(t, x_0, x_1, \ldots, x_n)\, x_0 \leq 0\,. \tag{19}$$

Moreover there exist functions $p_0(.)$, and $\omega(.) \in \mathbb{C}[0,\infty)$ such that

$$f(t, x_0, x_1, \ldots, x_n)\; sign\;(x_0) \leq -p_0(t)\,\omega(|x_0|)\,, \tag{20}$$

where the functions $p_0(.)$, and $\omega(.)$ have the properties

$$\omega : \quad [0,\infty) \to [0,\infty) \text{ is non-decreasing function}\,, \tag{21}$$
$$\omega(0) = 0\,, \tag{22}$$
$$\omega(s) > 0\,, \qquad \forall s > 0\,, \tag{23}$$

$$p_0 : \quad [0,\infty) \to [0,\infty) \tag{24}$$
$$p_0(t) \not\equiv 0 \text{ on any subinterval of } [0,\infty)\,,$$
$$\text{and the function } p_0(t) \text{ is strongly non-integrable}$$
$$\text{on the interval } [1,\infty)\,. \tag{25}$$

Definition 1. We call a function $f(.)$ **strongly non-integrable**, if $f(.)$ is non-negative and locally integrable function on an interval $[T,\infty)$ and if there exist the constants $\delta_0 > 0$ and $p_0 \in (0,1)$ with the property:

For each set $M \in \mathcal{B}(\mathcal{R})$ such that $M \subset [T,\infty)$ and

$$\mu\left(M \cap [t_0, t_0+\delta]\right) \geq (1-p_0)\,\delta \qquad \forall t_0 \geq T\,,\ \forall \delta \geq \delta_0 \tag{26}$$

the function $f(.)$ satisfies

$$\int\limits_M f\ d\mu = +\infty\,. \tag{27}$$

Remark 1. It is easy find a function, which is strongly non-integrable:
e.g. let $f(.) \in \mathbb{C}[T,\infty)\,, f(t) > s > 0\,, \forall t \in [T,\infty)$, where s be an positive constant. $f(.)$ is a non-integrable function and in the sense of the Definition 1 is strongly non-integrable, too.

Remark 2. Non all non-integrable function are strongly non-integrable:
e.g. let $f(.) \in \mathbb{C}[T,\infty)$, be defined by

$$f(t) = \begin{cases} 1/t^2\,, & t \in [2n, 2n+1] \\ 1\,, & t \in (2n+1, 2n+2) \quad \text{for all } n \in N\,. \end{cases}$$

If we take $M = \cup_{n\in N}\ [2n, 2n+1]$ then M satisfy (26) e.g. for $p_0 = 1/4\,, \delta_0 = 2$ we get $\int\limits_M f\ d\mu < +\infty$.

4. “Submersivity ” of a solution of the equation $\left[\frac{1}{r_1(t)}\ \left(x^{(n)}(t) + p(t)x(t)\right)\ \right]' = f(t, x(t), \cdots, x^{(n)}(t))\,.$

In this section we will examine “submersivity ” of a solution of the equation

$$\left[\frac{1}{r_1(t)}\ \left(x^{(n)}(t) + p(t)x(t)\right)\ \right]' = f(t, x(t), \cdots, x^{(n)}(t))\,.$$

We will compare the properties of the solution of (4) with the properties of solutions of our equation. Our aim was to find conditions of " submersivity" for the solution of the equation $y^{(n)}(t) + \alpha_1(t)\, y^{(n-1)}(t) + ... + \alpha_n(t)\, y(t) + p(t)\, y(t) = r(t)$.

We will see that the similar conditions of "submersivity" can be found for the solutions of the equation (16). The conclusions obtained in the next theorem, we will use for proving the main theorem of this contribution.

Let us define on some interval $[\tilde{T}, \infty)$, for a function $x(.) \in \mathbb{C}^{n+1}\,[T, \infty)$ the function

$$\alpha_1(t) = \frac{1}{r_1(t)}\ \left[x^{(n)}(t) + p(t)\, x(t)\right]\ , \tag{28}$$

Theorem 2. *Let functions $p(.)$, $r_1(.) \in \mathbb{C}^1[T_0, \infty)$ satisfy the assumptions (17) and (18) on the interval $[T_0, \infty)$. Let the function $f = f(t, x_0, x_1, \dots, x_n)$ be continuous on $R^+ \times R^{n+1}$ and satisfy properties (19)–(25). Moreover let $0 < \lim\limits_{t\to\infty} \alpha_1(t)$.*

Then for each $\delta_0 > 0$ and each $p_0 \in (0, 1)$, there exist $T > T_0$ and $\varepsilon > 0$ with the following property:

If $x(.) \in \mathbb{C}^{n+1}[T_0, \infty)$ is a non-negative solution of the differential equation (16), then for all $t_0 > T$ and for all $\delta > \delta_0$ we have

$$\mu\left([t_0, t_0 + \delta] \cap \left[\frac{x}{\alpha_1}\right]^{-1} [0, \varepsilon]\right) \le p_0\, \delta\,, \tag{29}$$

where μ denotes the Lebesgue measure of sets.

Proof. Let the function $\alpha_1(t)$ be defined by (28). Due to sign property (19) we get

$$\alpha_1{}'(t) = f(t, x(t), x'(t), \dots, x^{(n)}(t)) \le 0\,.$$

Since $0 < \lim\limits_{t\to\infty} \alpha_1(t)$ and $\alpha_1{}'(t) \le 0$, it follows that $0 < \lim\limits_{t\to\infty} \alpha_1(t) < \infty$. Further according to (28) we have

$$x^{(n)}(t) + p(t)\, x(t) = \alpha_1(t)\, r_1(t)\,. \tag{30}$$

The equation (30) formally can be written in the form (4), where $\alpha_i(t) \equiv 0$ for all $i \in \{1, \dots, k\}$ and $r(.) = \alpha_1(.)\, r_1(.)$. Hence, by Theorem 1 whose assumptions are satisfied on some $[T', \infty)\,, T' \ge T_0$, for all constants $\delta_0 > 0$ and $p_0 \in (0, 1)$ there exist $T \ge T' \ge T_0$ and $\varepsilon_1 > 0$ such that for all $t_0 > T$ and for all $\delta > \delta_0$ we get $\mu\left([t_0, t_0 + \delta_0] \cap x^{-1}[0, \varepsilon_1]\right) \le p_0\ \delta$. If we take $\varepsilon = \varepsilon_1 / \sup\limits_{t\ge T} \alpha_1(t)$, then it holds

$$\mu\left([t_0, t_0 + \delta] \cap \left[\frac{x}{\alpha_1}\right]^{-1} [0, \varepsilon]\right) \le p_0\, \delta\,,$$

which proves the claim of our theorem. □

5. The Main Theorem

Theorem 3. *Let $x(.) \in \mathbb{C}^{n+1}[0,\infty)$ be a solution of the differential equation of the form (16), where $n \geq 2$ is natural s number.*

Let the functions $p(.)$ and $r_1(.)$ satisfy the conditions (18) and (17) respectively.

Let further the right-hand side of (16) satisfy (19) - (25).

Then $x(.)$ is either oscillatory solution of the diff. equation (16), or there exists some function $\alpha(t)$, with the property

$$\alpha(t) \geq 0\,, \qquad \forall t \in [T_0,\infty), T_0 \geq 0\,, \qquad \textit{and} \ \lim_{t\to\infty} \alpha(t) = 0\,,$$

and $x(.)$ solves the diff. equation

$$x^{(n)}(t) + p(t)\ x(t) = \alpha(t)\ sign\ x(t)\,,$$

on some neighbourhood of infinity.

Proof. Let $x(.)$ be a proper non-oscillatory solution of the diff. eq. (16).

It is sufficient to study a non-negative non-oscillatory solution $x(t)$ on an interval $[\tilde{T},\infty)$. Otherwise if $x(t) \leq 0$ in $[\tilde{T},\infty)$, then the function

$$y(t) = -x(t)\,, \qquad t \in [\tilde{T},\infty)\,,$$

satisfies the differential equation

$$\left[\frac{1}{r_1(t)}\ \Big(y^{(n)}(t) + p(t)y(t)\Big)\ \right]' = f_1(t, y(t), y'(t), \cdots, y^{(n)}(t))\,, \tag{31}$$

where the function $f_1(t, x_0, x_1, \ldots, x_n) = -f(t, -x_0, -x_1, \ldots, -x_n)$ has all properties of the function f, i.e. f_1 is continuous on $R^+ \times R^{n+k}$ and satisfies the conditions (19) – (25).

Thus all properties of non-negative solutions of the equation (31) can be transformed to the similar ones of the non-positive solution of (16).

Hence in this theorem we will consider only non-negative solution and the statement will be true also for the non-positive solutions.

Let the function $\alpha_1(t)$ be define by (28). We conclude from the sign property (19) of the equation (16) on interval $[\tilde{T},\infty)$ that

$$\alpha'_1(t) = f(t, x(t), x'(t), \ldots, x^{(n)}(t)) \leq 0 \qquad \text{for}\ \ x(t) \geq 0\,,$$

hence that

$$\alpha'_1(t) \leq 0\,, \qquad \forall t \in [\tilde{T},\infty)\,. \tag{32}$$

By (32), it is obvious that $\alpha_1(t)$ is non-increasing function on interval $[\tilde{T}, \infty)$. Thus there exists $T', \ T' \geq \tilde{T}$ such that the function $\alpha_1(t)$ does not change its sign on interval $[T', \infty)$.

We can certainly assume the existence of such $T_0\,, \ T_0 \geq T'$ with the property

$$sign\ (\alpha_1(t)) \ = \ \text{constant}\,, \qquad \forall t \in [T_0, \infty)\,. \tag{33}$$

The proof will be divided into three cases.

(A) $\lim\limits_{t\to\infty} \alpha_1(t) < 0\,,$

(B) $\lim\limits_{t\to\infty} \alpha_1(t) > 0\,,$

(C) $\lim\limits_{t\to\infty} \alpha_1(t) = 0\,.$

5.1. (A) $\lim\limits_{t\to\infty} \alpha_1(t) < 0$

If $\lim\limits_{t\to\infty} \alpha_1(t) < 0$, then there exists $T_1\,, \ T_1 \geq T_0$ such that for all $t \geq T_1$, $\alpha_1(t) \leq -\varepsilon < 0$, where $\varepsilon > 0$. By the definition of $\alpha_1(t)$ and the constants $M_1\,, M_2$ we get $x^{(n)}(t) + p(t)x(t) = \alpha_1(t)\ r_1(t) \leq -\varepsilon\ M_1 < 0\,.$

As $x(t) \geq 0$, and $p(t) > 0$ on $[T_1, \infty)$ we have $x^{(n)}(t) \leq -\varepsilon\ M_1 < 0\,.$ Let $T_1 \leq t_1 \leq t$ and $\tau \in [t_1, t]$. By integration we come to the inequality

$$x^{(n-1)}(t) \leq x^{(n-1)}(t_1) - \varepsilon\ M_1\ (t - t_1)\,.$$

In the limit case, if $t \to \infty$ we obtain

$$\lim_{t\to\infty} x^{(n-1)}(t) \leq x^{(n-1)}(t_1) - \varepsilon\ M_1 \lim_{t\to\infty}(t - t_1) = -\infty\,,$$

which contradicts $x(t) \geq 0$.

5.2. (B) $\lim\limits_{t\to\infty} \alpha_1(t) > 0$

Consider the functions $p_0(.)$, and $\omega(.) \in \mathbb{C}[0, \infty)$, for which (20) – (25) hold.
By Theorem 2 for all $\delta_0 > 0$, $p_0 \in (0, 1)$, there exist $T \geq T_0$ and $\varepsilon > 0$ with the property: If $x(.) \in \mathbb{C}^{n+1}[T_0, \infty)$ is a non-negative solution of the differential equation (16), then for all $t_0 > T$ and for all $\delta > \delta_0$ we have

$$\mu\left([t_0, t_0 + \delta] \cap \left[\frac{x}{\alpha_1}\right]^{-1} [0, \varepsilon]\right) \leq p_0\ \delta\,, \tag{34}$$

where μ denotes the Lebesgue measure of sets.

Let the constants ε and T be given by Theorem 2. Let us denote by $\mathcal{M}$ the set

$$\mathcal{M} = \{t\, : t \geq T,\ x(t) \geq \varepsilon\ \alpha_1(t)\}\,. \tag{35}$$

The set $\mathcal{M}$ has the property (26) from Definition 1, Theorem 2, Definition 1 and the assumptions (24), (25) yield $\int\limits_{\mathcal{M}} p_0(t)\ d\mu = +\infty$.

Due to (33) we have $\alpha_1(t) \geq 0$ for all $t \geq T \geq T_0$. Choose arbitrary t_0, t such that $T \leq t_0 \leq t$. We have

$$0 \leq \alpha_1(t) = \alpha_1(t_0) + \int\limits_{t_0}^{t} \alpha_1{}'(s)\ ds = \alpha_1(t_0) - \int\limits_{t_0}^{t} |f(s, x(s), \cdots, x^{(n)}(s))|\ ds\,.$$

and hence $|\alpha_1(t_0)| \geq \int\limits_{t_0}^{t} |f(s, x(s), \cdots, x^{(n)}(s))| ds$, which for $t \to \infty$ implies, $|\alpha_1(t_0)| \geq \int\limits_{t_0}^{\infty} |f(s, x(s), \cdots, x^{(n)}(s))| ds$. Putting $t = t_0$ we obtain

$$|\alpha_1(t)| \geq \int\limits_{t}^{\infty} |f(s, x(s), \cdots, x^{(n)}(s))|\ ds\,, \qquad \forall t \geq t_0 \geq T\,.$$

If we use the previous result, for all $t \geq T$ we get

$$|\alpha_1(t)| \geq \int\limits_{t}^{\infty} |f(s, x(s), \cdots, x^{(n)}(s))|\ ds \overset{(20)}{\geq} \int\limits_{t}^{\infty} p_0(s)\ \omega(|x(s)|)\ ds\,.$$

Define the function $n : [T, \infty) \to \{0, 1\}$ as follows

$$n(s) = \begin{cases} 1 & s \in \mathcal{M} \\ 0 & s \notin \mathcal{M}\,, \end{cases}$$

where the set $\mathcal{M}$ is given by (35).

Since the functions $\alpha_1(t)$ do not change their signs on the interval $[T_0, \infty)$ and $\alpha'_1(t) \leq 0$, the function $\alpha_1(t)$ is non-increasing function on interval $[T_0, \infty)$ with the property $\lim\limits_{t\to\infty} \alpha_1(t) = \tilde{\alpha}_1 > 0$. Put $\varepsilon_1 = \frac{\tilde{\alpha}_1}{\alpha_1(T_0)}$. We thus get

$$\alpha_1(\tau) \geq \tilde{\alpha}_1 = \varepsilon_1\ \alpha_1(T_0) \geq \varepsilon_1\ \alpha_1(t)\ , \qquad \forall \tau \geq t \geq T_0\,.$$

Hence

$$\alpha_1(\tau) \geq \varepsilon_1\ \alpha_1(t)\,, \qquad \forall \tau \geq t \geq T\,. \tag{36}$$

Since the function $\omega(.)$ is non-decreasing non-negative function and using the estimation (36), we obtain on the set $\mathcal{M}$

$$\omega(|x(s)|) = \omega(x(s)) \geq \omega(\varepsilon\ \alpha_1(s))\,.$$

According to the above definition of the function $n(.)$

$$\omega(x(s)) \geq n(s)\ \omega(\varepsilon\ \alpha_1(s)) \geq n(s)\ \omega(\varepsilon\ \varepsilon_1 \alpha_1(t))\,, \qquad \forall s \geq t \geq T\,,$$

be valid. Further, the function $\omega(.)$ satisfies (22), (23) and due to (36) we get

$$\alpha_1(t) \geq \int_t^\infty p_0(s)\ n(s)\ \omega(\varepsilon\ \varepsilon_1\ \alpha_1(t_0))\ ds\,, \qquad \forall\ t \geq t_0 \geq T\,,$$

and hence

$$\infty > \frac{\alpha_1(t_0)}{\omega(\varepsilon\ \varepsilon_1\ \alpha_1(t_0))} \geq \int_{t_0}^\infty p_0(s)\ n(s)\ ds = \int_{\mathcal{M}\cap[t_0,\infty)} p_0(s)\ ds = +\infty\,.$$

which is impossible.

The cases (A) and (B) led to contradiction. Therefore the case (C), holds.

5.3. (C) $\lim_{t\to\infty} \alpha_1(t) = 0$

Due to sign property (32) we get $\alpha'_1(t) \leq 0$ for all $t \geq T_0$ and $\lim_{t\to\infty} \alpha_1(t) = 0$ and hence

$$\alpha_1(t) \geq 0\,, \qquad \forall t \geq T_2 \geq T_0\,.$$

If we take $\alpha(t) = r_1(t)\ \alpha_1(t)$, then we have $\lim_{t\to\infty} \alpha(t) = 0$ and $\alpha(t) \geq 0$, for all $t \geq T_2 \geq T_0$.

From the above it follows that, either $x(.)$ is oscillatory solution, or $x(.)$ is proper non-oscillatory solution on some interval $[T_0, \infty)$ and then there exists the function $\alpha(t)\,,\ \alpha(t) \geq 0$ for all $t \in [T_2, \infty)$, with the property $\lim_{t\to\infty} \alpha(t) = 0$, such that $x(.)$ will be a solution of equation

$$x^{(n)}(t) + p(t)x(t) = \alpha(t)\ sgn\ x(t)\,,$$

and the proof is complete. □

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STURM-LIOUVILLE DIFFERENCE EQUATIONS AND BANDED MATRICES

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Abstract. In this paper we consider *discrete* Sturm-Liouville eigenvalue problems of the form

$$L(y)_k := \sum_{\mu=0}^{n} (-\Delta)^\mu \{r_\mu(k) \Delta^\mu y_{k+1-\mu}\} = \lambda \rho(k) y_{k+1}$$

$$\textit{for} \;\; 0 \le k \le N-n \;\; \textit{with} \;\; y_{1-n} = \cdots = y_0 = y_{N+2-n} = \cdots = y_{N+1} = 0,$$

where N and n are integers with $1 \le n \le N$ and with the assumptions that $r_n(k) \ne 0$, $\rho(k) > 0$ for all k. These problems correspond to eigenvalue problems for symmetric, banded matrices $\mathcal{A} \in \mathbb{R}^{(N+1-n)\times(N+1-n)}$ with band-width $2n+1$. We present the following results: - a formula for the chracteristic polynomial of $\mathcal{A}$, which yields a *recursion* for its calculation - an *oscillation theorem*, which generalizes Sturm's well-known theorem on Sturmian chains, and - an inversion formula, which shows that *every* symmetric, banded matrix corresponds uniquely to a Sturm-Liouville eigenvalue problem of the above form.

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1. Introduction

We consider *discrete* Sturm-Liouville eigenvalue problems (with eigenvalue parameter λ) of the form

$$L(y)_k := \sum_{\mu=0}^{n} (-\Delta)^{\mu} \{ r_{\mu}(k) \Delta^{\mu} y_{k+1-\mu} \} = \lambda \rho(k) y_{k+1} \tag{1}$$

for $0 \leq k \leq N - n$, where $\Delta y_k = y_{k+1} - y_k$, and with the boundary conditions

$$y_{1-n} = \cdots = y_0 = y_{N+2-n} = \cdots = y_{N+1} = 0, \tag{2}$$

where N and n are fixed integers with $1 \leq n \leq N$ and where we always assume that

$$r_n(k) \neq 0 \quad for \;\; all \;\; k. \tag{3}$$

These problems correspond to eigenvalue problems for symmetric, banded matrices $\mathcal{A}$ of size $(N + 1 - n) \times (N + 1 - n)$ with band-width $2n + 1$. In particular, $\mathcal{A}$ is *tridiagonal* in the case $n = 1$.

In this paper we essentially formulate and discuss our results while detailed proofs will be given in a forthcoming paper. The following theorems will be presented:

– a formula for the characteristic polynomial of $\mathcal{A}$ (Theorem 1). This result yields also a *recursion* for its calculation. In the case $n = 1$ we obtain the well-known algorithm, which is commonly used in numerical analysis to handle eigenvalue problems for tridiagonal matrices (cf. [[4], pp. 305; [8], pp. 134; [9], pp. 299]).
– an *oscillation theorem* (Theorem 2). This result generalizes Sturm's well-known theorem on Sturmian chains (cf. e.g. [[4], Theorem 8.5-1 or [8], Sätze 4.8 and 4.9]).
– an *inversion formula* (Theorem 3). This identity can be used to calculate the matrix $\mathcal{A}$ when the discrete Sturm-Liouville operator from equation (1) is given and vice versa. Hence, *every* symmetric, banded matrix with bandwidth $2n+1$ corresponds uniquely to such a Sturm-Liouville operator.

Our method and most of our results have continuous counterparts along the lines of the book [6] (cf. also [7]).

2. Discrete Sturm-Liouville equations and associated Hamiltonian systems

In this section we give the connection between discrete Sturm-Liouville equations and Hamiltonian difference systems (cf. [[1], Proposition 5]), and we introduce the important notions of *conjoined bases* and *focal points* of it (cf. [[1], Definitions 1 and 3]). Moreover, the *multiplicity* of focal points is defined according to [3]. It will turn out that these multiplicities always equal one for Hamiltonian systems, which we treat here, i.e. which originate from Sturm-Liouville equations.

Lemma 1. *A vector* $y = (y_k)_{1-n}^{N+1} \in \mathbb{R}^{N+1-n}$ *solves the Sturm-Liouville difference equation* (1) *for* $0 \le k \le N - n$ *if and only if* (x, u) *solves the Hamiltonian difference system*

$$\Delta x_k = A x_{k+1} + B_k u_k\,,\ \Delta u_k = (C_k - \lambda \tilde{C}_k) x_{k+1} - A^T u_k \tag{4}$$

for $0 \le k \le N$, *where we use the following* **notation:**

A, B_k, C_k, $\tilde{C}_k$ *are* $n \times n$*-matrices defined by*

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}\,,\ B_k = \frac{1}{r_n(k)} B \quad \textit{with}\ \ B = \textit{diag}\,(0, \ldots, 0, 1)\,,$$

$$C_k = \textit{diag}\,(r_0(k), \ldots, r_{n-1}(k))\,,\ \tilde{C}_k = \rho(k) \tilde{C} \quad \textit{with}\ \ \tilde{C} = \textit{diag}\,(1, 0, \ldots, 0)\,,$$

for $0 \le k \le N$, *and* $x_k = (x_k^{(\nu)})_{\nu=0}^{n-1}$, $u_k = (u_k^{(\nu)})_{\nu=0}^{n-1} \in \mathbb{R}^n$ *are defined by*

$$x_k^{(\nu)} = \Delta^\nu y_{k-\nu}\,,\ u_k^{(\nu)} = \sum_{\mu=\nu+1}^{n} (-\Delta)^{\mu-\nu-1} \{ r_\mu(k) \Delta^\mu y_{k+1-\mu} \}$$

for $0 \le \nu \le n-1\,, 0 \le k \le N+1$ *with suitably chosen* $y_{N+2}, \ldots, y_{N+n+1}$ *(which are used for* $u_{N+2-n}, \ldots, u_{N+1}$*).*

Definition 1. Assume that (3) holds.

(i) A pair $(X, U) = (X_k, U_k)_{k=0}^{N+1}$ is called a conjoined basis of (4), if the real $n \times n$-matrices X_k, U_k solve (4) for $0 \le k \le N$, and if
$$X_0^T U_0 = U_0^T X_0 \ \text{ and } \ \operatorname{rank}(X_0^T, U_0^T) = n \quad \text{holds.}$$

(ii) Suppose that (X, U) is a conjoined basis of (4) and let $0 \le k \le N$. We say that X has no focal point in the interval $(k, k+1]$ if
$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k \ \text{ and } \ D_k := X_k X_{k+1}^\dagger \tilde{A} B_k \ge 0 \quad \text{holds,}$$
where $\tilde{A} := (I - A)^{-1}$. Moreover, if $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ and $D_k \not\ge 0$, then $\operatorname{ind} D_k$ is called the multiplicity of the focal point of X in the interval $(k, k+1)$.

Remark 1.

(i) For a matrix M we denote by $\operatorname{Ker} M$ the kernel of M, $\operatorname{ind} M$ denotes the index of M, i.e., the number of negative eigenvalues of M, provided M is symmetric (and real), and $M^\dagger$ denotes the Moore-Penrose inverse of M. Moreover, $M \ge 0$ means that M is symmetric (and real) and non-negative definite. Observe that D_k is symmetric, if $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ (cf. [[1], Proposition 1]).

(ii) For our Sturm-Liouville difference equations the multiplicity of focal points, which we defined only in case $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$, always equals 1, because $\operatorname{rank} D_k \le \operatorname{rank} B = 1$.

3. Associated quadratic functionals and banded matrices

For $y = (y_k)_{k=1}^{N+1-n} \in \mathbb{R}^{N+1-n}$ we define a quadratic functional $\mathcal{F}$, which corresponds to the Sturm-Liouville operator $L(y)$ from equation (1), by

$$\mathcal{F}(y) := \sum_{k=0}^{N} \sum_{\mu=0}^{n} r_\mu(k)(\Delta^\mu y_{k+1-\mu})^2 \,,$$

where we assume (2), i.e., $y_{1-n} = \cdots = y_0 = y_{N+2-n} = \cdots = y_{N+1} = 0$.

Lemma 2. *The following formulas hold.*

(i) $\mathcal{F}(y) = y^T \mathcal{A} y$, *where* $\mathcal{A} \in \mathbb{R}^{(N+1-n)\times(N+1-n)}$ *is a symmetric, banded matrix with band-width* $2n+1$, *which is defined by*

$$a_{k+1,k+1+t} = (-1)^t \sum_{\mu=t}^{n} \sum_{\nu=t}^{\mu} \binom{\mu}{\nu} \binom{\mu}{\nu - t} r_\mu(k+\nu)$$

for $0 \le t \le n$ *and* $0 \le k \le N-n-t$.

(ii) $(\mathcal{A}y)_{k+1} = L(y)_k$ *for* $0 \le k \le N-n$ *with* $L(y)_k$ *given by* (1).

Observe that $\mathcal{A}$ is a tridiagonal $N \times N$-matrix in the case $n = 1$. In the sequel we use the **notation**:

$\mathcal{A}_{N+1} = \mathcal{A} \in \mathbb{R}^{(N+1-n)\times(N+1-n)}$ is the symmetric, banded matrix as defined in Lemma 2, and $\mathcal{A}_k \in \mathbb{R}^{(k-n)\times(k-n)}$ is defined correspondingly for $n+1 \le k \le N+1$. Moreover, let $\mathcal{A}(\lambda) := \mathcal{A} - \lambda \mathcal{D}$ with
$\mathcal{D} := \operatorname{diag}(\rho(0), \ldots, \rho(N-1))$, and as before, $\mathcal{A}_k(\lambda)$ is defined accordingly.

The following statement follows directly from Lemma 2.

Corollary 1. *The discrete Sturm-Liouville eigenvalue problem* (1) *and* (2) *from Section 1 is equivalent with the algebraic eigenvalue problem (matrix pencil)*

$$\mathcal{A}y = \lambda \mathcal{D} y \quad \text{or} \quad \mathcal{A}(\lambda) y = 0.$$

4. Results

We assume throughout that (X, U) is the so-called *principal solution* of (4), i.e., $X = X_k(\lambda)$, $U_k = U_k(\lambda)$ satisfy (4) with

$$X_0 \equiv 0, \; U_0 \equiv I \,. \tag{5}$$

Moreover, as in the previous sections, $y = (y_k)_{k=1}^{N+1-n} \in \mathbb{R}^{N+1-n}$ satisfies (2), i.e., $y_{1-n} = \cdots = y_0 = y_{N+2-n} = \cdots = y_{N+1} = 0$, and

$$\mathcal{F}(y) = \sum_{k=0}^{N} \sum_{\mu=0}^{n} r_\mu(k)(\Delta^\mu y_{k+1-\mu})^2 \,, \;\; D_k = X_k X_{k+1}^\dagger \tilde{A} B_k \,(= D_k(\lambda)).$$

First, we cite some auxiliary results mainly from [1].

4.1. Auxiliary results

Lemma 3. *The following assertions hold, provided* (3) *and* (5) *are fulfilled.*

(i) $X_0, \ldots, X_n$ *are independent of* λ.
(ii) $\det X_k = 0\,,\ D_k = 0\,,\ Ker\, X_{k+1} \subset Ker\, X_k$ *for* $k = 0, \ldots, n-1$.
(iii) $\det X_n = \{r_n(0) \cdots r_n(n-1)\}^{-1} \neq 0\,.$
(iv) $\det X_k(\lambda) \neq 0$ *for* $n \le k \le N+1$, *if* λ *is sufficiently small, provided* $\rho(k) > 0$ *for* $0 \le k \le N-n$.
(v) $D_k(\lambda) = \dfrac{1}{r_n(k)} \dfrac{\det X_k(\lambda)}{\det X_{k+1}(\lambda)} B\,,$ *provided* $\det X_{k+1}(\lambda) \neq 0$, *for* $n \le k \le N$.

Proof. The assertions (i) and (iii) are derived in a forthcoming paper. The assertion (ii) is contained in [[1], Proposition 6], and (iv) follows from [[1], Satz 9], because

$$\mathcal{F}(y) - \lambda \sum_{k=0}^{N-n} \rho(k) y_{k+1}^2 > 0 \ \ for \quad \lambda \le \lambda_0,$$

if $y \neq 0$ and $\rho(k) > 0$ for $0 \le k \le N-n$. Finally, the assertion (v) is shown in [[2], Lemma 4.1].

Observe that $X_k(\lambda)$, $U_k(\lambda)$ are matrix-polynomials in λ, so that $D_k(\lambda)$ is a rational function of λ as follows from Lemma 3 (v). Hence, if $\rho(k) > 0$ for all k, then $\det X_k(\lambda) \neq 0$ for $n \le k \le N+1$ and all $\lambda \in \mathbb{R} \setminus \mathcal{N}$ with a *finite* set $\mathcal{N}$. The next result follows from [[1], Proposition 1] and Lemma 3.

Lemma 4. *(Picone's identity) Suppose* (2), (3), *and* (5), *and assume that* $\det X_k(\lambda) \neq 0$ *for* $n \le k \le N+1$. *Then*

$$\mathcal{F}(y) - \lambda \sum_{k=0}^{N-n} \rho(k) y_{k+1}^2 = \sum_{k=n}^{N} z_k^T D_k z_k,$$

where $z_k = u_k - U_k(\lambda) X_k^{-1}(\lambda) x_k$ *with* x_k, u_k *as in Lemma 1.*

The next statement with the notation of Section 3 follows immediately from Lemma 3 and Lemma 4.

Corollary 2. *Under the assumptions of Lemma 4*

$$y^T(\mathcal{A}_{N+1} - \lambda \mathcal{D}) y = \sum_{k=n}^{N} r_n(k) \frac{\det X_{k+1}(\lambda)}{\det X_k(\lambda)} w_{k+1-n}^2,$$

where $w_\nu = y_\nu + \sum_{\mu=\nu+1}^{\nu+n} \alpha_\mu y_\mu$ *with suitable coefficients* $\alpha_\mu = \alpha_\mu(\nu, \lambda)$. *Hence,* $w = Ty$ *with* $T = \begin{pmatrix} 1 & \star & \star \\ \vdots & \ddots & \star \\ 0 & \cdots & 1 \end{pmatrix}$, *so that* $\det T = 1$.

4.2. Main results

First, the Lemmas 3 and 4 with Crollary 2 yield our first result, which states a formula for the characteristic polynomial of $\mathcal{A}$ and its recursive calculation.

Theorem 1. *(Recursion) Assume* (3), (5), *and suppose that*

$$\rho(k) > 0 \quad \text{for } \ 0 \le k \le N - n \tag{6}$$

holds. Then, with the notation of Section 3,

$$\det(\mathcal{A} - \lambda\,\mathcal{D}) = r_n(0)\cdots r_n(N)\det X_{N+1}(\lambda) \tag{7}$$

for all $\lambda \in \mathbb{R}$. *Moreover, by* (4) *and* (5), $X_{N+1}(\lambda)$ *is given by the recursion*

$$X_{k+1} = \tilde{A}(X_k + B_k U_k)\,,\; U_{k+1} = (C_k - \lambda\tilde{C}_k)X_{k+1} + (I - A^T)U_k$$

for all $0 \le k \le N$ *with* $X_0 = 0$, $U_0 = I$.

Proof. By Lemma 3 and Lemma 4 we have that

$$\begin{aligned}\det\mathcal{A}(\lambda) =& r_n(n)\frac{\det X_{n+1}(\lambda)}{\det X_n(\lambda)}\cdots r_n(N)\frac{\det X_{N+1}(\lambda)}{\det X_N(\lambda)}\\ =& r_n(0)\cdots r_n(N)\det X_{N+1}(\lambda).\end{aligned}$$

Next, the general oscillation theorem for Hamiltonian systems from reference [3] implies a corresponding result here.

Theorem 2. *(Oscillation) Under the assumptions of Theorem 1 let* $\lambda \in \mathbb{R}$ *with* $\det X_k(\lambda) \ne 0$ *for* $n \le k \le N+1$. *Then, the number of eigenvalues (including multiplicities) of the eigenvalue problem* (1), (2) *from Section 1, which are less than* λ, *equals the number of focal points of* $X(\lambda)$ *in the interval* $(0, N+1]$.

Remark 2. Observe first, that the multiplicity of an eigenvalue λ is given by the rank of the kernel of $X_{N+1}(\lambda)$. Hence, it is an integer in $\{1, \ldots, n\}$. Moreover, by Remark 1, the focal points of $X(\lambda)$ are all simple, i.e., of multiplicity one, and their number in $(0, N+1]$ equals the number of the elements of the set

$$\{k :\; n \le k \le N \ \text{ with } \ r_n(k)\frac{\det X_{k+1}(\lambda)}{\det X_k(\lambda)} < 0\}.$$

The next corollary is just another formulation of Theorem 2. It generalizes the well-known theorem of Sturm on “Sturmian chains”(cf. [[4], Theorem 8.5-1 and [8], Sätze 4.8 and 4.9). Moreover, it yields the Poincaré separation theorem for banded matrices (cf. [[5], 4.3.16 Corollary]).

Corollary 3. *Under the assumptions of Theorem 2 and the previous notation define polynomials* $f_k(t)$ *by*

$$f_k(t) := \det\mathcal{A}_k(t) \quad \text{for } \ n+1 \le k \le N+1 \quad \text{and} \quad f_n(t) \equiv 1. \tag{8}$$

Then the number of zeros of $f_{N+1}(t)$ *(including multiplicities), which are less than* λ, *equals the number of sign changes of* $\{f_k(\lambda)\}$ *for* $n \le k \le N+1$, *i.e.,* $\{f_k(\lambda)\}$ *is a “Sturmian chain”.*

Proof. The assertion follows from Theorem 1 and Theorem 2, because

$$f_k(\lambda) = r_n(0)\cdots r_n(k-1)\det X_k(\lambda)$$

for $n \le k \le N+1$, so that

$$\frac{f_{k+1}(\lambda)}{f_k(\lambda)} = r_n(k)\frac{\det X_{k+1}(\lambda)}{\det X_k(\lambda)} .$$

Finally, we have the following inversion formula, where the "easy" part is the assertion (i) of Lemma 2, while the main formula will be proved in detail via generating functions in a forthcoming paper as already mentioned in the introduction.

Theorem 3. *(Inversion) The following inversion formulas hold:*

$$r_\mu(k+\mu) = \tag{9}$$

$$(-1)^\mu \sum_{s=\mu}^{n} \left\{ \binom{s}{\mu} a_{k+1,k+1+s} + \sum_{l=1}^{s-\mu} \frac{s}{l}\binom{\mu+l-1}{l-1}\binom{s-l-1}{s-\mu-l} a_{k+1-l,k+1-l+s} \right\},$$

for $0 \le \mu \le n$ *and all* k, *if and only if the* $a_{\mu\nu}$ *are given by*

$$a_{k+1,k+1+t} = (-1)^t \sum_{\mu=t}^{n}\sum_{\nu=t}^{\mu} \binom{\mu}{\nu}\binom{\mu}{\nu-t} r_\mu(k+\nu) \tag{10}$$

for $0 \le t \le n$ *and all* k.

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SADDLE CONNECTIONS IN PLANAR SYSTEMS

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Abstract. The class of planar autonomous systems with a small parameter-dependent perturbation is considered. We derive a sufficient condition for existence of a saddle connection in such system.

AMS Subject Classification. 34C37

Keywords. Saddle connection, stable and unstable manifolds, small perturbation, Hamiltonian system.

1. Introduction

We consider systems of the form

$$\dot{x} = f(x) + \varepsilon g(x, \alpha), \quad x \in R^2, \quad \varepsilon, \alpha \in R, \tag{1}$$

where f, g are C^r, $r \geq 2$, and bounded on bounded sets, ε is a small parameter. Such systems are viewed as planar systems with a small perturbation which depends on a real parameter α. If we assume that unperturbed system (for $\varepsilon = 0$) possesses a saddle connection, then a natural question arises whether there are values of a parameter α for which a perturbed system possesses a saddle connection. There are many results related to similar questions, see for instance [1] for the problem of existence of periodic orbits in a perturbed system, or [4, §4.4], where the impact of a small time-dependent periodic perturbation on homoclinic orbit in Hamiltonian systems is studied. The paper [3] explores existence and number of periodic and homoclinic orbits, but only for a particular Hamiltonian system (whirling pendulum equation) with a special perturbation (a friction). None of the results in mentioned (and other) works has been directly applicable to our problem. To solve it, we follow a geometrical point of view as it is presented in [2].

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2. Assumptions and background material

We will assume that for $\varepsilon = 0$ (1) has two saddle points p_1 and p_2, which are connected by heteroclinic trajectory Γ. (The reasoning in the case of a saddle connected to itself by a homoclinic loop is very similar). More precisely, one branch, say Γ^u, of the global unstable manifold W^u of p_1 coincides with one branch, say Γ^s of the global stable manifold W^s of p_2, and they form a saddle connection Γ (see Fig. 1a).

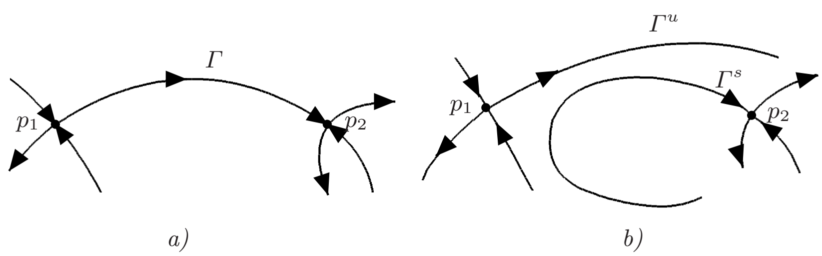

Fig. 1. The phase portrait of $\dot{x} = f(x) + \varepsilon g(x, \alpha)$ for a) $\varepsilon = 0$, b) $\varepsilon \neq 0$.

This situation is not resistent to perturbations – in general, any perturbation will break the saddle connection, although the local phase portraits will not change under a small perturbation (see Fig. 1b). Particularly, the following facts are well-known for (1) with $\varepsilon \neq 0$ (for details we refer the reader to [2, §4.5] and the references given there):

F1 For each ε sufficiently small, (1) has two unique saddles $p_1^\varepsilon = p_1^\varepsilon + \mathcal{O}(\varepsilon)$, $p_2^\varepsilon = p_1^\varepsilon + \mathcal{O}(\varepsilon)$. This is a straightforward application of the implicit function theorem, since Jacobi matrices $Df(p_1)$, $Df(p_2)$ are invertible (they have nonzero real eigenvalues).

F2 Perturbed local stable and unstable manifolds of the saddles p_1^ε, p_2^ε‘are C^r-close to unperturbed local stable and unstable manifolds of the saddles p_1, p_2. This fact follows from invariant manifold theory.

F3 If we denote by $\gamma(t)$ a solution of the unperturbed system lying in Γ, by $\gamma^u(t)$ and $\gamma^s(t)$ solutions of the perturbed system lying in Γ_ε^u and Γ_ε^s (branches of W_ε^u and W_ε^s corresponding to Γ^u and Γ^s), the following expressions holds, with uniform validity in the indicated intervals:

$$\begin{aligned} \gamma^s(t) &= \gamma(t) + \varepsilon\gamma_1^s(t) + \mathcal{O}(\varepsilon^2), \quad t \in [0, \infty), \\ \gamma^u(t) &= \gamma(t) + \varepsilon\gamma_1^u(t) + \mathcal{O}(\varepsilon^2), \quad t \in (-\infty, 0]. \end{aligned} \tag{2}$$

Here $\gamma_1^s(t)$ and $\gamma_1^u(t)$ are solutions of the first variational equations

$$\dot{\gamma}_1^{s,u}(t) = Df(\gamma(t))\gamma_1^{s,u}(t) + g(\gamma(t), \alpha). \tag{3}$$

This fact represents both local and global dynamics – near a saddle point (infinite time interval) it is governed by exponential attraction and repulsion, while away from a saddle (finite time interval) the closeness of solutions may be derived thanks to Gronwall's inequality.

In what follows, we will look for values of parameter α for which the saddle connection persists. The main idea is to measure, in some sense, the distance between perturbed branches Γ_ε^u and Γ_ε^s of the global manifolds W_ε^u and W_ε^s.

3. The distance function

Let $p \in \Gamma$ be a nonsingular point ($f(p) \neq 0$), and $p^u \in \Gamma_\varepsilon^u$, $p^s \in \Gamma_\varepsilon^s$ are lying on the normal $f^{\perp(p)}$ to Γ at p (Fig. 2). Then we define the oriented distance between Γ_ε^u and Γ_ε^s at the point p as

$$d(\varepsilon, \alpha) = \frac{f(p) \wedge (p^u - p^s)}{|f(p)|},$$

where $a \wedge b = a^\perp \cdot b$ is the wedge product.

We denote $\gamma(t)$, $\gamma^s(t)$ and $\gamma^u(t)$ solutions lying in Γ, Γ_ε^s and Γ_ε^u for which

$$\gamma(0) = p, \quad \gamma^s(0) = p^s, \quad \gamma^u(0) = p^u. \tag{4}$$

Using (2) and (4), we can write

$$d(\varepsilon, \alpha) = \varepsilon \frac{f(\gamma(0)) \wedge (\gamma_1^u(0) - \gamma_1^s(0))}{|f(\gamma(0))|} + \mathcal{O}(\varepsilon^2).$$

Now we define the time dependent distance function

$$\Delta(t) = f(\gamma(t)) \wedge (\gamma_1^u(t) - \gamma_1^s(t))$$

which may be written as $\Delta(t) = \Delta^u(t) - \Delta^s(t)$ with $\Delta^{s,u}(t) = f(\gamma(t)) \wedge \gamma_1^{s,u}(t)$. Note that

$$d(\varepsilon, \alpha) = \varepsilon \frac{\Delta(0)}{|f(\gamma(0))|} + \mathcal{O}(\varepsilon^2).$$

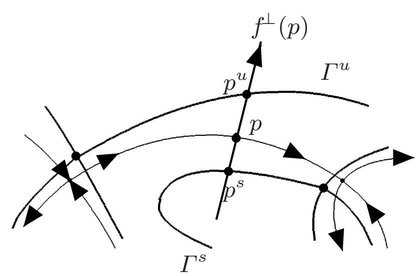

Fig. 2. Definition of the distance function.

The derivative of $\varDelta^{s,u}(t)$ with respect to time is

$$\dot{\varDelta}^{s,u}(t) = Df(\gamma(t))\dot{\gamma}(t) \wedge \gamma_1^{s,u}(t) + f(\gamma(t)) \wedge \dot{\gamma}_1^{s,u}(t).$$

Using (3) and the fact that $\dot{\gamma}(t) = f(\gamma(t))$, we obtain, after some matrix calculations,

$$\dot{\varDelta}^{s,u}(t) = \mathrm{Tr}\,(Df(\gamma(t)))\,\varDelta^{s,u}(t) + f(\gamma(t)) \wedge g(\gamma(t), \alpha).$$

Integrating the last equation from 0 to ∞ for $\varDelta^s$ and from $-\infty$ to 0 for $\varDelta^u$ yields

$$\varDelta^s(\infty) - \varDelta^s(0) = \int\limits_0^\infty f(\gamma(t)) \wedge g(\gamma(t), \alpha) e^{-\int_0^t \mathrm{Tr}\,(Df(\gamma(s)))\,\mathrm{d}s}\,\mathrm{d}t,$$

$$\varDelta^u(0) - \varDelta^u(-\infty) = \int\limits_{-\infty}^0 f(\gamma(t)) \wedge g(\gamma(t), \alpha) e^{\int_t^0 \mathrm{Tr}\,(Df(\gamma(s)))\,\mathrm{d}s}\,\mathrm{d}t.$$

Since

$$\varDelta^s(\infty) = \lim_{t\to\infty} f(\gamma(t)) \wedge \gamma_1^s(t),$$

where $\gamma_1^s(t)$ is bounded and $\lim_{t\to\infty} f(\gamma(t)) = f(p_2) = 0$, we have $\varDelta^s(\infty) = 0$. Similarly $\varDelta^u(-\infty) = 0$. Then

$$\varDelta^s(0) = \int\limits_0^\infty f(\gamma(t)) \wedge g(\gamma(t), \alpha) e^{-\int_0^t \mathrm{Tr}\,(Df(\gamma(s)))\,\mathrm{d}s}\,\mathrm{d}t.$$

In the case when the unperturbed system is Hamiltonian, i.e. $f = \left(\dfrac{\partial H}{\partial x_2}, -\dfrac{\partial H}{\partial x_1}\right)$ for some differentiable function $H(x_1, x_2)$, we have $\mathrm{Tr}\,(Df) \equiv 0$, and

$$\varDelta(0) = \int_{-\infty}^{\infty} f(\gamma(t)) \wedge g(\gamma(t), \alpha) dt,$$

which is the homoclinic Melnikov function [2, p. 187].

In the next, we will use more suitable notation $\varDelta(0) = M(\alpha)$, which takes into account the fact that $\varDelta(0)$ depends on α. Thus

$$d(\varepsilon, \alpha) = \varepsilon \frac{M(\alpha)}{|f(p)|} + \mathcal{O}(\varepsilon^2). \tag{5}$$

Now we are ready to state and prove the main result:

Theorem 1. *Let there exist α_0 such that $M(\alpha_0) = 0$, $M'(\alpha_0) \neq 0$. Then for each ε sufficiently small there exists $\alpha(\varepsilon) = \alpha_0 + \mathcal{O}(\varepsilon)$ such that the perturbed system*

$$\dot{x} = f(x) + \varepsilon g(x, \alpha(\varepsilon))$$

possesses a saddle connection, which is C^r-close to the saddle connection of the unperturbed system.

Proof. We rewrite (5) in the form $d(\varepsilon,\alpha)=\varepsilon\overline{d}(\varepsilon,\alpha)$, where

$$\overline{d}(\varepsilon,\alpha)=\frac{M(\alpha)}{|f(p)|}+\mathcal{O}(\varepsilon).$$

Then, for $\varepsilon\neq 0$, d vanishes if and only if $\overline{d}$ vanishes. For α_0 with indicated properties we obtain

$$\overline{d}(0,\alpha_0)=0,\qquad \frac{\partial\overline{d}}{\partial\alpha}(0,\alpha_0)\neq 0.$$

The implicit function theorem ensures the existence of a smooth curve of points $(\varepsilon,\alpha(\varepsilon))$ passing throw $(0,\alpha(0))$, $\alpha(0)=\alpha_0$, with a property

$$\overline{d}(\varepsilon,\alpha(\varepsilon))=0.$$

It means that the oriented distance between Γ_ε^u and Γ_ε^s at the point p is zero, which implies, thanks to the uniqueness theorem, that they coincide, forming a saddle connection. The C^r-closeness is ensured by F3.

4. Example

We will seek parameter α_0 for which there exists a smooth curve of parameters $\alpha(\varepsilon)$ with the property: the planar system

$$\begin{aligned}\dot{x}&=y\\ \dot{y}&=-\sin x+\varepsilon y(\cos x+\alpha(\varepsilon))\end{aligned}\tag{6}$$

has a saddle connection that is C^r-close to the upper saddle connection of the planar pendulum equation, i.e. the system

$$\begin{aligned}\dot{x}&=y\\ \dot{y}&=-\sin x.\end{aligned}\tag{7}$$

To obtain the value of α_0, we will compute $M(\alpha)$ for (6). First, we recall that the planar pendulum equation (7) is a Hamiltonian system with the energy

$$H(x,y)=\frac{y^2}{2}-\cos x+1.$$

Saddles $-\pi$, π are connected by two heteroclinic orbits

$$y=\pm\sqrt{2(\cos x+1)}$$

(upper and lower saddle connections) corresponding to the energy level $h=2$. Then

$$M(\alpha)=\int_{-\infty}^{\infty}y^2(t)(\cos x(t)+\alpha)\mathrm{d}t.$$

Using the fact that $y\mathrm{d}t = \mathrm{d}x$, and trigonometrical identity $\cos x + 1 = 2\cos^2 \dfrac{x}{2}$, we obtain that along the upper saddle connection

$$M(\alpha) = \int_{-\pi}^{\pi} y(\cos x + \alpha)\mathrm{d}x = 8(\alpha + \frac{1}{3}).$$

Consequently, if we denote $\alpha_0 = -\dfrac{1}{3}$, then

$$M(\alpha_0) = 0, \qquad M'(\alpha_0) \neq 0.$$

By Theorem 1, for each ε sufficiently small there exists $\alpha(\varepsilon) = -\frac{1}{3} + \mathcal{O}(\varepsilon)$ such that (6) has an upper saddle connection. Moreover, from the definition of $d(\varepsilon, \alpha)$ we can deduce that for $\alpha > \alpha(\varepsilon)$ the unstable manifold of $[-\pi, 0]$ is lying above the stable manifold of $[\pi, 0]$, and reversely for $\alpha < \alpha(\varepsilon)$ (see Fig. 3, where the situation is depicted for two values of ε). The similar result may be obtained for the lower saddle connection.

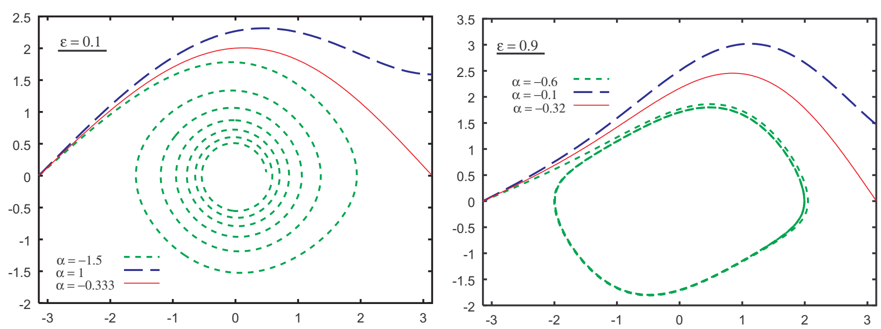

Fig. 3. Phase portraits of (6).

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COMPARISON THEOREMS FOR HALF–LINEAR SECOND ORDER DIFFERENCE EQUATIONS

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Abstract. In the paper new comparison theorems for half–linear difference equation

$$\Delta\Big(R_k\Phi(\Delta z_k)\Big) + C_k\Phi(z_{k+1}) = 0$$

are derived. The main tool is variational technique developed for half–linear difference equations in Řehák [5].

AMS Subject Classification. 39A10

Keywords. half–linear difference equation, comparison theorem

1. Introduction

The well–known result from the calculus of variations states, that there is an equivalence between disconjugacy of second order ordinary differential equation

$$(r(t)y')' + c(t)y = 0$$

on the interval (a, b) and nonnegativity of quadratic functional

$$\int_a^b \Big(r(t)\eta'^2(t) - c(t)\eta^2(t)\Big)\mathrm{d}t$$

defined on the class of functions $\eta \in W_0^{1,2}(a, b)$. This classical result has been later extended in various directions. The generalizations include $n-$dimensional problem with general boundary conditions [1], singular functional [2], p-degree

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functionals [3,4] and also discrete p-degree functionals [5]. The relationship between the functional and the corresponding equation is frequently used in the comparison and oscillation theory of differential equations, see. e.g. [6].

In this paper we will study the second order half–linear difference equation

$$L[z_k] = \Delta\Big(R_k\Phi(\Delta z_k)\Big) + C_k\Phi(z_{k+1}) = 0 \tag{E}$$

for $k \in [0, n]$ and the corresponding discrete scalar p-degree functional

$$J(x) = A|x_0|^p + \sum_{k=0}^{n}\Big(R_k|\Delta x_k|^p - C_k|x_{k+1}|^p\Big). \tag{J}$$

The relationship between Eq. (E) and (J) will be used further in the proof of comparison theorems, which compare Eq. (E) with another half–linear discrete differential equation

$$l[y_k] = \Delta\Big(r_k\Phi(\Delta y_k)\Big) + c_k\Phi(y_{k+1}) = 0. \tag{e}$$

Remark that, unless stated explicitly, under the interval $[m, n]$ we actually mean the discrete set $\{m, m + 1, \dots n\}$. Similarly under the term function we actually mean the sequence.

The following lemma presents our main tool – Picone-type identity for half–linear difference equations. It is a simplified version of the Picone identity published for Eq. (E) in Řehák [5].

Lemma 1 ([5]). *If $L[z_k] = 0$ for $k \in [0, n]$ and $z_k \neq 0$ for $k \in [0, n + 1]$, then for $k \in [0, n]$*

$$\Delta\bigg\{-|x_k|^p R_k\frac{\Phi(\Delta z_k)}{\Phi(z_k)}\bigg\} = C_k|x_{k+1}|^p - R_k|\Delta x_k|^p + \frac{R_k z_k}{z_{k+1}}G_k(x, z), \tag{P}$$

where

$$G_k(x, z) = \frac{z_{k+1}}{z_k}|\Delta x_k|^p - \frac{z_{k+1}\Phi(\Delta z_k)}{z_k\Phi(z_{k+1})}|x_{k+1}|^p + \frac{z_{k+1}\Phi(\Delta z_k)}{z_k\Phi(z_k)}|x_k|^p$$

holds. The function $G_k(\cdot, \cdot)$ satisfies

$$G_k(x, z) \geq 0 \tag{1}$$

with equality if and only if $\Delta x_k = x_k\frac{\Delta z_k}{z_k}$, i.e. if and only if $x_{k+1} = x_k\frac{z_{k+1}}{z_k}$.

Lemma 2. *If $x_{k+1} = x_k\frac{z_{k+1}}{z_k}$ for $k \in [0, n]$, then $x_k = \frac{x_0}{z_0}z_k$ for $k \in [0, n + 1]$.*

Proof. By induction.

2. Main results

In connection with Eq. (E) we will study also the first order Riccati-type difference equation

$$\text{(R)} \qquad \Delta w_k + C_k + w_k\Big(1 - \frac{R_k}{\Phi(\Phi^{-1}(R_k)+\Phi^{-1}(w_k))}\Big) = 0,$$

where $\Phi^{-1}(\cdot)$ denotes the inverse function to the function $\Phi(\cdot)$.

The relationship between functional (J) and Eqs. (E), (R) has been studied in [5]. Here it is proved the equivalence between disconjugacy of (E), existence of solution of (R) and positive definiteness of (J) on the class of functions satisfying $x_0 = 0 = x_{n+1}$.

The difference between these results and the results from this paper lies in another type of boundary conditions for the function x. The fact that we use another types of boundary conditions causes that we obtain information about solution of Eq. (E) given by another initial condition, than in [5].

First let us recall the definition of generalized zero, which is known to be the convenient substitution for zeros of the continuous function.

Definition 1. An interval $(m, m+1]$ is said to contain a *generalized zero* of a solution z_k of Eq. (E) if $z_m \neq 0$ and $R_m z_m z_{m+1} \leq 0$.

The following theorem establishes the relationship between the half–linear equation, Riccati equation and the $p-$degree functional. Results of this type are sometimes referred as Reid's Roundabout–type theorem.

Theorem 1. *The following statements are equivalent:*

(i) The solution z_k of Eq. (E) given by $R_0\Phi\Big(\frac{\Delta z_0}{z_0}\Big) = A$ satisfies $R_k z_k z_{k+1} > 0$ for $k \in [0, n]$.
(ii) Equation (R) has a solution on $[0, n]$ such that $w_0 = A$ and $R_k + w_k > 0$ on $[0, n]$.
(iii) Functional (J) is positive definite on the class of functions defined on $[0, n+1]$ satisfying $x_{n+1} = 0$.

Proof. "(i)$\Longleftrightarrow$(ii)" If z_k is the solution of (E) satisfying $R_k z_k z_{k+1} > 0$ for $k \in [0, n]$, then the function $w_k = R_k\Phi\Big(\frac{\Delta z_k}{z_k}\Big)$ is well-defined on $[0, n+1]$ and satisfies (R) and $R_k + w_k > 0$ on $[0, n]$, which follows from [5].

Conversely, if w_k is a solution of Eq. (R) satisfying $R_k + w_k > 0$, then $z_{k+1} = z_k\Big(1 + \Phi^{-1}\big(\frac{w_k}{r_k}\big)\Big)$ defines solution of Eq. (E) satisfying $R_k z_k z_{k+1} > 0$ for $k \in [0, n]$.

In addition, $R_0\Phi\Big(\frac{\Delta z_0}{z_0}\Big) = A$ is equivalent to $w_0 = A$.

"(i)$\Longrightarrow$(iii)" Let x be defined on $[0, n+1]$ and $x_{n+1} = 0$. Summation of Picone identity (P) for $k \in [0, n]$ gives

$$J(x) = A|x_0|^p + \sum_{k=0}^{n}\Big[\Delta\Big(|x_k|^p R_k \Phi\Big(\frac{\Delta z_k}{z_k}\Big)\Big) + \frac{R_k z_k z_{k+1}}{z_{k+1}^2} G_k(x,z)\Big] \geq$$
$$A|x_0|^p + |x_{n+1}|^p R_{n+1}\Phi\Big(\frac{\Delta z_{n+1}}{z_{n+1}}\Big) - |x_0|^p R_0 \Phi\Big(\frac{\Delta z_0}{z_0}\Big) = 0$$

and the functional is positive semidefinite.

The equality holds throughout only if $G_k(x,z) = 0$ for $k \in [0,n]$. From here it follows $x_{k+1} = x_k \frac{z_{k+1}}{z_k}$ for $k \in [0,n]$, or equivalently $x_k = z_k \frac{x_0}{z_0}$ for $k = [0, n+1]$. In view of the fact $x_{n+1} = 0 \neq z_{n+1}$, it holds $x_0 = 0$ and $x \equiv 0$. Hence $J(x) = 0$ only if $x \equiv 0$ and the functional is positive definite.

"(iii)$\Longrightarrow$(i)" Suppose, by contradiction, that the functional is positive definite and for the solution z of Eq. (E) given (uniquely up to the constant multiple) by the condition $R_0\Phi\Big(\frac{\Delta z_0}{z_0}\Big) = A$ there exists $N \in [0,n]$ such that

$$\begin{aligned} R_k z_k z_{k+1} &> 0 \quad \text{ for } 0 \leq k < N \\ R_N z_N z_{N+1} &\leq 0. \end{aligned}$$

Denote

$$x_k = \begin{cases} z_k & k \in [0,N] \\ 0 & k \in [N+1, n+1]. \end{cases}$$

Since $z_0 \neq 0$, clearly $x \not\equiv 0$. Suppose $N \geq 1$. From the definition of the function x it follows $L[x_k] = 0$ for $k \in [0, N-2]$. Summation by parts gives

$$\begin{aligned} J(x) &= A|x_0|^p + \sum_{k=0}^{n}\Big[R_k|\Delta x_k|^p - C_k|x_{k+1}|^p\Big] \\ &= A|x_0|^p + \big[x_k R_k \Phi(\Delta x_k)\big]_{k=0}^{n+1} - \sum_{k=0}^{n} x_{k+1}L[x_k] \\ &= -\sum_{k=0}^{N} x_{k+1}L[x_k] = -x_N L[x_{N-1}] \\ &= -z_N\Big[\Delta(R_{N-1}\Phi(\Delta x_{N-1})) + C_{N-1}\Phi(z_N)\Big] \\ &= z_N\Big[R_{N-1}\Phi(\Delta z_{N-1}) - R_N\Phi(\Delta x_N) + \Delta(R_{N-1}\Phi(\Delta z_{N-1}))\Big] \\ &= z_N R_N \Phi(\Delta z_N) + z_N R_N \Phi(z_N), \end{aligned}$$

since $\Delta x_N = -z_N$ and $\Delta x_{N-1} = \Delta z_{N-1}$. Hence $J(x) = R_N|z_N|^p\Big[\Phi\big(\frac{\Delta z_N}{z_N}\big) + 1\Big]$. Now $z_{N+1} \neq 0$. Really, if $z_{N+1} = 0$ would hold, then $J(x) = 0$, a contradiction. Hence

$$J(x) = \frac{R_N z_N z_{N+1}}{z_{N+1}^2}|z_N|^p\Big[\frac{z_{N+1}}{z_N}\Phi\big(\frac{z_{N+1}}{z_N} - 1\big) + \frac{z_{N+1}}{z_N}\Big].$$

In view of the fact $R_N z_N z_{N+1} \leq 0$ and with respect to inequality $\alpha\Phi(\alpha-1)+\alpha \geq 0$ we obtain $J(x) \leq 0$, a contradiction. This contradiction ends the proof.

Corollary 1 (Leighton type comparison theorem). *Let y_k be solution of Eq. (e), such that $y_{n+1} = 0 \neq y_0$. Denote $a = r_0 \Phi\big(\frac{\Delta y_0}{y_0}\big)$. Let A be such that*

$$V(y) := (A-a)|y_0|^p + \sum_{k=0}^{n} \Big[(R_k - r_k)|\Delta y_k|^p - (C_k - c_k)|y_{k+1}|^p \Big] \le 0.$$

Then the solution of Eq. (E) given by $R_0 \Phi\big(\frac{\Delta z_0}{z_0}\big) = A$ has a generalized zero on $[0, n+1]$, i.e., there exists $i \in [0, n]$ such that $R_i z_i z_{i+1} \le 0$ holds.

Proof. Define the functional $j(x) = \alpha |x_0|^p + \sum_{k=0}^{n} r_k |\Delta x_k|^p - c_k |x_{k+1}|^p$. Using summation by parts we obtain $j(y) = 0$ and hence $J(y) = J(y) - j(y) = V(y) \le 0$. Since $y \not\equiv 0$, the statement follows from Theorem 1.

An immediate consequence is the following

Corollary 2. *Let y_k be solution of Eq. (e), such that $y_{n+1} = 0 \neq y_0$. Denote $a = r_0 \Phi\big(\frac{\Delta y_0}{y_0}\big)$. Let $A < a$, $R_k \le r_k$ on $[0, n]$ and $c_k \le C_k$ on $[0, n-1]$. Then the solution of Eq. (E) given by $R_0 \Phi\big(\frac{\Delta z_0}{z_0}\big) = A$ has a generalized zero on $[0, n+1]$, i.e., there exists $i \in [0, n]$ such that $R_i z_i z_{i+1} \le 0$ holds.*

Corollary 3. *Let y_k be solution of Eq. (e), such that $y_{n+1} = 0 \neq y_0$. Denote $a = r_0 \Phi\big(\frac{\Delta y_0}{y_0}\big)$. Let A be such that*

$$\mathcal{V}(y) := (A - \frac{R_0}{r_0} a)|y_0|^p - \sum_{k=0}^{n} \Big\{ \Delta\Big(\frac{R_k}{r_k}\Big) r_k \Phi(\Delta y_k) y_{k+1} + (C_k - \frac{R_{k+1}}{r_{k+1}} c_{k+1}) |y_{k+1}|^p \Big\} \le 0.$$

Then the solution of Eq. (E) given by $R_0 \Phi\big(\frac{\Delta z_0}{z_0}\big) = A$ has a generalized zero on $[0, n+1]$, i.e., there exists $i \in [0, n]$ such that $R_i z_i z_{i+1} \le 0$ holds.

Proof. Let y_k be solution of (e) on $[0, n]$ satisfying $y_{n+1} = 0 \neq y_0$. Then

$$\begin{aligned} L[y_k] &= \Delta\big(R_k \Phi(\Delta y_k)\big) + C_k \Phi\big(y_{k+1}\big) = \Delta\Big(\frac{R_k}{r_k} r_k \Phi(\Delta y_k)\Big) + C_k \Phi(y_{k+1}) \\ &= \Delta\Big(\frac{R_k}{r_k}\Big) r_k \Phi(\Delta y_k) + \frac{R_{k+1}}{r_{k+1}} \Delta\Big(r_k \Phi(\Delta y_k)\Big) + C_k \Phi(y_{k+1}) \\ &= \Delta\Big(\frac{R_k}{r_k}\Big) r_k \Phi(\Delta y_k) + \Phi(y_{k+1}) \Big[C_k - \frac{R_{k+1}}{r_{k+1}} c_k \Big]. \end{aligned} \tag{2}$$

Since the integration by parts shows that

$$\begin{aligned} \sum_{k=0}^{n} \Delta\Big(\frac{R_k}{r_k}\Big) r_k \Phi(\Delta y_k) y_{k+1} &= \Big[R_k \Phi(\Delta y_k) y_{k+1} \Big]_{n=0}^{n+1} - \sum_{k=0}^{n} \frac{R_{k+1}}{r_{k+1}} \Delta\Big(r_k \Phi(\Delta y_k) y_{k+1}\Big) \\ &= R_{n+1} y_{n+2} \Phi(\Delta y_{n+1}) - R_0 y_1 \Phi(\Delta y_0) \\ &\quad - \sum_{k=0}^{n} \frac{R_{k+1}}{r_{k+1}} \Big[\Delta\Big(r_k \Phi(\Delta y_k)\Big) y_{k+1} + \Delta y_{k+1} r_{k+1} \Phi(\Delta y_{k+1}) \Big] \end{aligned}$$

$$
\begin{aligned}
&= R_{n+1}y_{n+2}\Phi(\Delta y_{n+1}) - R_0 y_1 \Phi(\Delta y_0) \\
&\quad - \sum_{k=0}^{n}\Big[-\frac{R_{k+1}}{r_{k+1}}c_k|y_{k+1}|^p + R_{k+1}|\Delta y_{k+1}|^p\Big] \\
&= R_{n+1}y_{n+2}\Phi(\Delta y_{n+1}) - R_0 y_1 \Phi(\Delta y_0) \\
&\quad - \sum_{k=0}^{n}\Big[-\frac{R_{k+1}}{r_{k+1}}c_k|y_{k+1}|^p + R_k|\Delta y_k|^p\Big] - R_{n+1}|\Delta y_{n+1}|^p + R_0|\Delta y_0|^p \\
&= R_{n+1}y_{n+1}\Phi(\Delta y_{n+1}) - R_0 y_0 \Phi(\Delta y_0) \\
&\quad - \sum_{k=0}^{n}\Big[R_k|\Delta y_k|^p - \frac{R_{k+1}}{r_{k+1}}c_k|y_{k+1}|^p\Big]
\end{aligned}
$$

the following relation holds

$$\sum_{k=0}^{n} y_{k+1}L[y_k] = R_{n+1}\Phi(\Delta y_{n+1})y_{n+1} - R_0\Phi(\Delta y_0)y_0 - \sum_{k=0}^{m}\Big[R_k|\Delta y_k|^p - C_k|y_{k+1}|^p\Big].$$

Then in view of (2) and $y_{n+1} = 0$, clearly

$$J(y) = |y_0|^p\Big[A - R_0\Phi\Big(\frac{\Delta y_0}{y_0}\Big)\Big] - \sum_{n=0}^{n} y_{k+1}L[y_k] = \mathcal{V}(y)$$

and the statement follows from Theorem 1.

3. Open problems

In the oscillation theory of discrete differential equations the concept of generalized zeros is used. This is caused by the fact that the sequence R_k is allowed to attain also negative values. However in the boundary conditions of the functional (J) "exact" zeros are used. It could be interesting to remove this disharmonicity and to find out, whether the concept of generalized zeros in boundary conditions would produce some fruitful extension of discrete variational technique.

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THE l^p TRICHOTOMY FOR DIFFERENCE SYSTEMS AND APPLICATIONS

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Abstract. The notion of l^p trichotomy for a linear difference system is here considered as extension of exponential trichotomy and l^p dichotomy. The main properties are analyzed and necessary and sufficient conditions for the existence are given. The asymptotic behavior of solutions of a quasi-linear system $x(n+1) = A(n)x(n) + f(n, x(n))$ is studied under the assumption that the associated linear system possesses a l^p trichotomy.

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1. Introduction

Consider the nonlinear difference system in $\mathbb{R}^m$

$$y(n+1) = A(n)y(n) + f(n, y(n)), \qquad n \in \mathbb{Z} \tag{1}$$

where $A(n)$ is a $m \times m$ invertible matrix for every $n \in \mathbb{Z}$ and f is a continuous function from $\mathbb{Z} \times \mathbb{R}^m$ into $\mathbb{R}^m$. Our aim is to study the existence of bounded solutions of (1) having zero limit as $n \to \pm\infty$, under the assumption that the solutions of the associated linear (homogeneous) system

$$x(n+1) = A(n)x(n), \qquad n \in \mathbb{Z} \tag{2}$$

are not all bounded on $\mathbb{Z}$.

In the continuous case the study of the existence on the whole real line of zero convergent solutions as $t \to \pm\infty$ of a linear differential system often has been accomplished by introducing suitable assumptions on the asymptotic behavior of a fundamental matrix. For instance in [15] the notion of S-S trichotomy is introduced and is employed to study the existence of invariant splittings for linear differential systems. Later a stronger notion of trichotomy, namely exponential trichotomy, was introduced in [8], still in the continuous case. These notions were extended afterwards to the discrete case and during the last years many authors dealt with exponential or ordinary trichotomy of difference systems, giving necessary and sufficient conditions for the existence, proving the roughness and applying these results to nonlinear difference systems, see for instance [2], [9], [11]. We refer the reader to [7] for the basic theory of dichotomies and to [1] for the extension to difference equations.

Here, in section 2, l^p trichotomy for a linear system (2) will be introduced and the main asymptotic properties of the solutions of this system will be analyzed. We point out that l^p trichotomy can be considered as an extension to the l^p spaces of exponential trichotomy, as well as l^p dichotomy is an extension of exponential dichotomy [18].

In section 3 the boundary value problem

$$\begin{cases} y(n+1) = A(n)y(n) + f(n, y(n)), \qquad n \in \mathbb{Z} \\ y(+\infty) = 0, \quad y(-\infty) = 0 \end{cases} \tag{3}$$

will be considered, assuming that the associated linear system has a l^p trichotomy and using a topological approach based on Schauder-Tychonoff fixed point theorem.

The results obtained extend some of the results in [10], [12]–[14], [18] and improve some of those in [9], [2], [11]. A comparison will be made throughout the paper.

2. l^p TRICHOTOMY FOR LINEAR DIFFERENCE SYSTEMS

Let $X(n)$ be a fundamental matrix of (2). We recall the definitions of l^p, exponential and ordinary dichotomy for reader's convenience.

Definition 1 ([18], [10], [12]–[14]). System (2) is said to have a l^p *dichotomy on* $\mathbb{Z}^+ = \{0, 1, 2, \cdots\}$, $1 \le p < \infty$, if there exist a projection P^+ and a constant $K^+ > 0$ such that for every $n \in \mathbb{Z}^+$

$$\begin{gathered} \left[\sum_{s=-1}^{n-1} |X(n)P^+X^{-1}(s+1)|^p \right]^{1/p} < K^+ \\ \left[\sum_{s=n-1}^{\infty} |X(n)(I-P^+)X^{-1}(s+1)|^p \right]^{1/p} < K^+. \end{gathered} \tag{4}$$

Analogously system (2) has a l^p *dichotomy on* $\mathbb{Z}^- = \{0, -1, -2, \cdots\}$, $1 \le p < \infty$, if there exist a projection P^- and a constant $K^- > 0$ such that

$$\begin{aligned} &\left[\sum_{s=-\infty}^{n-1} |X(n)P^-X^{-1}(s+1)|^p\right]^{1/p} < K^- \\ &\left[\sum_{s=n-1}^{-1} |X(n)(I-P^-)X^{-1}(s+1)|^p\right]^{1/p} < K^-. \end{aligned} \tag{5}$$

System (2) has an *exponential dichotomy on* $\mathbb{Z}^+$ if there exist a projection P_0 and constants $M > 0$, $0 < \beta < 1$ such that

$$|X(n)P_0X^{-1}(s)| < M\beta^{n-s}, \quad 0 \le s \le n$$

$$|X(n)(I-P_0)X^{-1}(s)| < M\beta^{s-n}, \quad 0 \le n \le s.$$

The exponential dichotomy on $\mathbb{Z}^-$ is defined in a similar way. If the above two inequalities hold with $\beta = 1$, then system (2) has an *ordinary dichotomy on* $\mathbb{Z}^+$. Clearly ordinary dichotomy is equivalent to l^∞ dichotomy.

The above mentioned notions of dichotomy can be regarded as kinds of conditional stability in future for the linear system (2). In particular system (2) is uniformly stable (in future) if and only if it has an ordinary dichotomy on $\mathbb{Z}^+$ with projection the identity operator, it is asymptotically uniformly stable (in future) if and only if it has an exponential dichotomy on $\mathbb{Z}^+$ with projection the identity operator, it is l^p stable (in future) if and only if it has a l^p dichotomy on $\mathbb{Z}^+$ with projection the identity operator ([17], see also [4], [5]).

If one is interested in the asymptotic behavior of the solutions of (2) both in the future and in the past, then it may be useful to generalize the above kinds of dichotomies. For instance in [9], [11] the exponential trichotomy is considered as a generalization of exponential dichotomy on $\mathbb{Z}$ and it is employed to study the asymptotic behavior in the future and in the past of the solutions of perturbed difference systems. Analogously it is possible to generalize the l^p dichotomy on $\mathbb{Z}$ in the following way:

Definition 2. System (2) is said to have a l^p trichotomy on $\mathbb{Z}$ with $1 \le p < \infty$, if there exist three mutually orthogonal projections P_1, P_2, P_3, with $P_1+P_2+P_3 = I$, and a constant $K > 0$, such that

$$\begin{aligned} &\left[\sum_{s=-\infty}^{n-1} |X(n)P_1X^{-1}(s+1)|^p\right]^{1/p} < K \\ &\left[\sum_{s=n-1}^{\infty} |X(n)P_2X^{-1}(s+1)|^p\right]^{1/p} < K \\ &\left[\sum_{s=-1}^{n-1} |X(n)P_3X^{-1}(s+1)|^p\right]^{1/p} < K \qquad \text{for } n \ge 0 \end{aligned} \tag{6}$$

$$\left[\sum_{s=n-1}^{-1} |X(n)P_3X^{-1}(s+1)|^p\right]^{1/p} < K \qquad \text{for } n \le 0.$$

It is worth to remark that the l^p trichotomy is a property that does not depend on the fixed fundamental matrix. Indeed, if $Y(n)$ is another fundamental matrix of (2), then there exists a nonsingular matrix C such that $X(n) = Y(n)C$ and $|X(n)P_jX^{-1}(s+1)| = |Y(n)CP_jC^{-1}Y^{-1}(s+1)|$. Only the projections depend on the fixed fundamental matrix.

From Definition 2 l^p trichotomy on $\mathbb{Z}$ implies l^p dichotomy on $\mathbb{Z}^+$ and on $\mathbb{Z}^-$, and l^p dichotomy on $\mathbb{Z}$ implies a trivial l^p trichotomy, with the projection $P_3 = 0$. In particular the following holds:

Proposition 1. *The following statements are equivalent:*

i) *System (2) has a l^p trichotomy on $\mathbb{Z}$, with projections P_1, P_2, P_3.*
ii) *There exist two projections P, Q, such that $PQ = QP$, $P + Q - PQ = I$ and a positive constant N, such that*

$$(7)\qquad \begin{aligned}
&\left[\sum_{s=-1}^{n-1} |X(n)PX^{-1}(s+1)|^p\right]^{1/p} < N, \qquad n \ge 0\\
&\left[\sum_{s=n-1}^{\infty} |X(n)(I-P)X^{-1}(s+1)|^p\right]^{1/p} < N\\
&\left[\sum_{s=n-1}^{-1} |X(n)QX^{-1}(s+1)|^p\right]^{1/p} < N, \qquad n \le 0\\
&\left[\sum_{s=-\infty}^{n-1} |X(n)(I-Q)X^{-1}(s+1)|^p\right]^{1/p} < N.
\end{aligned}$$

iii) *System (2) has a l^p dichotomy on $\mathbb{Z}^+$ with projection P^+ and a l^p dichotomy on $\mathbb{Z}^-$ with projection P^-, such that $P^+P^- = P^-P^+ = P^-$. In addition the second inequality in (4) and the first one in (5) hold for every $n \in \mathbb{Z}$.*

Proof. i) $\Longrightarrow$ ii). Let $P = I - P_2$ and $Q = I - P_1$. It is trivial to check that $PQ = P_3 = QP$ and $P + Q - PQ = I$. The second and the fourth inequalities in (7) are immediately verified. With regard to the first one in (7) we have for $n \ge 0$

$$\begin{aligned}
\sum_{s=-1}^{n-1} |X(n)PX^{-1}(s+1)|^p &= \sum_{s=-1}^{n-1} |X(n)(P_1+P_3)X^{-1}(s+1)|^p\\
&\le 2^{p-1} \sum_{s=-1}^{n-1} \Big(|X(n)P_1X^{-1}(s+1)|^p + |X(n)P_3X^{-1}(s+1)|^p\Big)\\
&< 2^pK^p.
\end{aligned}$$

Similarly the last one in (7) can be proved
ii) $\Longrightarrow$ iii). Let $P^+ = P$ and $P^- = (I - Q)$. Then system (2) has a l^p dichotomy on $\mathbb{Z}^+$ with projection P^+, and a l^p dichotomy on $\mathbb{Z}^-$ with projection P^-. Further $P^+P^- = P(I-Q) = I - Q = P^- = P^-P^+$ and both the second inequality in (4) and the first one in (5) hold for every $n \in \mathbb{Z}$.
iii) $\Longrightarrow$ i). Let $P_1 = P^-$, $P_2 = I - P^+$, $P_3 = P^+ - P^- = P^+(I - P^-) = (I - P^-)P^+$. Then clearly $P_1 + P_2 + P_3 = I$ and $P_iP_j = 0$ if $i \neq j$. The proof of the inequalities (6) is quite immediate; for the last two inequalities it is sufficient to observe that $P_3 = (I - P^-)P^+$ for the first one, and that $P_3 = P^+(I - P^-)$ for the second one. □

The equivalence between conditions (i), (iii) in Proposition 1 gives the following:

Corollary 1. *System (2) has a l^p trichotomy if and only if the following two conditions are satisfied:*

a) system (2) has a l^p dichotomy both on $\mathbb{Z}^+$ and on $\mathbb{Z}^-$;
b) every solution is the sum of two solutions, one bounded on $\mathbb{Z}^+$ and the other bounded on $\mathbb{Z}^-$.

Proposition 1 permits us to give a complete description of the asymptotic behavior of the solutions of (2), both in the future and in the past. More precisely we have

Theorem 1. *If system (2) has a l^p trichotomy, $1 \leq p < \infty$, with projections P_1, P_2, P_3 corresponding to the fundamental matrix $X(n)$ s.t. $X(0) = I$, then the m-dimensional space S of all the solutions of (2) can be written as direct sum*

$$S = B_k^+ \oplus B_r^- \oplus B_{m-k-r}^\pm$$

where

B_k^+ *is the k-dimensional subspace of solutions x such that $x(0) = \eta \in Range(P_1)$, where $k = Rank(P_1)$. If $x \in B_k^+$ then $x(+\infty) = 0$ and x is unbounded for $n \to -\infty$.*

B_r^- *is the r-dimensional subspace of solutions x such that $x(0) = \nu \in Range(P_2)$, where $r = Rank(P_2)$. If $x \in B_r^-$ then $x(-\infty) = 0$ and x is unbounded for $n \to +\infty$.*

$B_{m-k-r}^\pm$ *is the subspace of solutions x such that $x(0) = \mu \in Range(P_3)$, where $m - k - r = Rank(P_3)$. If $x \in B_{m-k-r}^\pm$ then $x(\pm\infty) = 0$.*

In particular a solution of (2) is bounded for all $n \in \mathbb{Z}$ if and only if it has zero limit as $n \to \pm\infty$.

Proof. If system (2) has a l^p trichotomy on $\mathbb{Z}$, with projections P_1, P_2, P_3, from Proposition 1, (2) has also a l^p dichotomy on $\mathbb{Z}^+$ with projection $P^+ = I - P_2$ and a l^p dichotomy on $\mathbb{Z}^-$ with projection $P^- = P_1$. In addition $P^+P^- = P^-P^+ = P^-$ and also $(I - P^-)(I - P^+) = (I - P^+)(I - P^-) = I - P^+$.

Let x be a solution of (2). Then $x(n) = X(n)P_1x(0)+X(n)P_2x(0)+X(n)P_3x(0)$. From ([16], [18]) and using the fact that (2) has a l^p dichotomy on $\mathbb{Z}^+$ and on $\mathbb{Z}^-$ we obtain

$$\begin{aligned}
\lim_{n\to+\infty}|X(n)P_1x(0)| &= \lim_{n\to+\infty}|X(n)P^-x(0)| = \lim_{n\to+\infty}|X(n)P^+P^-x(0)| = 0\\
\lim_{n\to-\infty}|X(n)P_1x(0)| &= \lim_{n\to-\infty}|X(n)P^-x(0)| = +\infty \quad \text{if } P_1x(0)\neq 0\\
\lim_{n\to+\infty}|X(n)P_2x(0)| &= \lim_{n\to+\infty}|X(n)(I-P^+)x(0)| = +\infty \quad \text{if } P_2x(0)\neq 0\\
\lim_{n\to-\infty}|X(n)P_2x(0)| &= \lim_{n\to-\infty}|X(n)(I-P^+)x(0)|\\
&= \lim_{n\to-\infty}|X(n)(I-P^-)(I-P^+)x(0)| = 0\\
\lim_{n\to+\infty}|X(n)P_3x(0)| &= \lim_{n\to+\infty}|X(n)P^+(I-P^-)x(0)| = 0\\
\lim_{n\to-\infty}|X(n)P_3x(0)| &= \lim_{n\to-\infty}|X(n)(I-P^-)P^+x(0)| = 0.
\end{aligned}$$

This ends the proof of the first part of the proposition. To prove the second assertion it is sufficient to observe that necessarily $P_1x(0) = P_2x(0) = 0$ in order to have a solution of (2) bounded on all $\mathbb{Z}$. □

As l^p trichotomy is more general than exponential trichotomy, the previous results extend the correspondent ones in [9], [2], [11]. Further the notion of trichotomy allows to consider the behavior of the solutions of (2) on the whole set $\mathbb{Z}$, therefore the results in Theorem 1 imply the corresponding ones in [18].

Remark 1. It is also possible to give an estimate of the rate of convergence towards zero of the various terms, see [18], [16].

3. Applications to nonlinear boundary value problems

Suppose that (2) has a l^p trichotomy and consider the associated nonlinear system (1). The following holds:

Proposition 2. *Assume:*

i) system (2) has a l^p trichotomy, $1\leq p<\infty$, with projections P_1, P_2, P_3 associated with the fundamental matrix $X(n)$ s.t. $X(0) = I$;
ii) there exists a function $g : \mathbb{Z}\times\mathbb{R}^+ \mapsto \mathbb{R}^+$, continuous with respect to the second variable $\forall n \in \mathbb{Z}$ and such that

$$|f(n,c)| \leq g(n,|c|), \quad n\in\mathbb{Z}, c\in\mathbb{R}^m \tag{8}$$

$$\max_{v\in[0,r]} g(n,v) = g_r(n) \in l^q, \quad r\in\mathbb{R}^+,\ 1/p+1/q=1,\ (p=1, q=\infty). \tag{9}$$

Then every bounded solution of (1) *is solution of*

$$
\begin{aligned}
y(n) = & X(n)P_3 y(0) + \sum_{s=-\infty}^{n-1} X(n)P_1 X^{-1}(s+1)f(s,y(s)) \\
& - \sum_{s=n}^{+\infty} X(n)P_2 X^{-1}(s+1)f(s,y(s)) + \sum_{s=0}^{n-1} X(n)P_3 X^{-1}(s+1)f(s,y(s)) \\
& - \sum_{s=n}^{-1} X(n)P_3 X^{-1}(s+1)f(s,y(s))
\end{aligned} \tag{10}
$$

(with the convention $\sum_{s=a}^{b} g(s) = 0$ *if* $a > b$*) and vice versa.*

Proof. The assertion is an easy consequence of the variation of constants formula. We only sketch the proof.

Let u be a bounded solution of (1). From i) and ii) we get

$$\left| \sum_{s=n}^{+\infty} X(n)P_2 X^{-1}(s+1)f(s,u(s)) \right| \le K \|g_{\|u\|_\infty}\|_q, \quad 1 < q \le \infty$$

where K is the trichotomy constant (see Definition 2). Then for $n \ge 0$ we can write

$$
\begin{aligned}
u(n) = & X(n)P_1 u(0) + X(n)P_2 u(0) + X(n)P_3 u(0) \\
& + \sum_{s=0}^{n-1} X(n)(P_1+P_3)X^{-1}(s+1)f(s,u(s)) \\
& + \sum_{s=0}^{+\infty} X(n)P_2 X^{-1}(s+1)f(s,u(s)) - \sum_{s=n}^{+\infty} X(n)P_2 X^{-1}(s+1)f(s,u(s)).
\end{aligned}
$$

The sequence

$$\left\{ \sum_{s=0}^{n-1} X(n)(P_1+P_3)X^{-1}(s+1)f(s,u(s)) \right\}$$

is bounded by the constant $2K\|g_{\|u\|_\infty}\|_q$, $1 < q \le \infty$. As $\lim_{n\to+\infty} |X(n)P_j u(0)| = 0$, $j = 1, 3$ (see Theorem 1) and u is bounded, the sequence

$$\left\{ X(n)P_2 \left[u(0) + \sum_{s=0}^{+\infty} X^{-1}(s+1)f(s,u(s)) \right] \right\}$$

is bounded too. From Theorem 1 it follows that

$$P_2 \left[u(0) + \sum_{s=0}^{+\infty} X^{-1}(s+1)f(s,u(s)) \right] = 0. \tag{11}$$

Now to show that u satisfies (10) it is sufficient to prove that

$$P_1\left[u(0)-\sum_{s=-\infty}^{-1} X^{-1}(s+1)f(s,u(s))\right]=0. \tag{12}$$

Since u is a solution of (1), for $n\leq -1$ we have

$$\begin{aligned}
u(n) =&X(n)P_1u(0)+X(n)P_2u(0)+X(n)P_3u(0)\\
&-\sum_{s=-\infty}^{-1} X(n)P_1X^{-1}(s+1)f(s,u(s))+\sum_{s=-\infty}^{n-1} X(n)P_1X^{-1}(s+1)f(s,u(s))\\
&-\sum_{s=n}^{-1} X(n)(P_2+P_3)X^{-1}(s+1)f(s,u(s)).
\end{aligned}$$

Following an argument similar to that above given and taking into account that u is bounded, we obtain (12), and so u satisfies (10) for $n\geq 0$. Starting from the variation of constants formula for $n\leq -1$ and taking into account (11) and (12) we obtain that u satisfies (10) for $n\leq -1$ too.

Vice versa let u be a bounded solution of (10). A standard calculation shows that u satisfies (1). □

Denote $l_0^\infty=\{u\in l^\infty:\ \lim_{n\to\pm\infty} u(n)=0\}$. From the above proposition we have

Corollary 2. *Assume conditions i) and ii) of Proposition 2 hold, with $1\leq p<\infty$. Assume also for $p=1$ ($q=\infty$)*

iii) $g(n,|c|)\leq \gamma|c|+\lambda(n)$, for every $n\in\mathbb{Z}$, $c\in\mathbb{R}^m$, where $\gamma>0$, $2K\gamma<1$ and $\lambda\in l_0^\infty$.

Then every bounded solution of (1) belongs to l_0^∞.

Proof. Let u be a bounded solution of (1). From Proposition 2 u is solution of (10). Let $1<p<\infty$ and $n\geq n_1>0$, n_1 fixed; from (11) we get

$$\begin{aligned}
&|u(n)|\leq |X(n)(P_1+P_3)|\left\{|u(0)|+\sum_{s=0}^{n_1-1}|X^{-1}(s+1)f(s,u(s))|\right\}\\
&+\sum_{s=n}^{+\infty}|X(n)P_2X^{-1}(s+1)f(s,u(s))|+\sum_{s=n_1}^{n}|X(n)(P_1+P_3)X^{-1}(s+1)f(s,u(s))|\\
&\leq |X(n)(P_1+P_3)|\left\{|u(0)|+\sum_{s=0}^{n_1-1}|X^{-1}(s+1)f(s,u(s))|\right\}\\
&+3K\left(\sum_{s=n_1}^{+\infty}(g_{\|u\|_\infty}(s))^q\right)^{1/q}
\end{aligned}$$

Choosing n_1 sufficiently large, in view of Theorem 1 we obtain $\lim_{n\to+\infty} u(n) = 0$. The assertion $\lim_{n\to-\infty} u(n) = 0$ can be proved in a similar way taking into account (12).

When $p = 1$ the proof comes using similar arguments to those in [6] (Th. 8 p. 68 and Th. 10 p. 7) with slight modifications; see also [18], Prop. 3.2. □

Remark 2. When $p = 1$ conditions i) and ii) in Proposition 2 are not sufficient to assure that every bounded solution of (1) belongs to l_0^∞. It is possible to find conditions different from iii) in Corollary 2 that, together with conditions i) and ii), assure the decaying of all the bounded solutions towards zero; for instance this happens by assuming

iv) $g_\alpha \in l_0^\infty$ *for every* $\alpha > 0$.

Finally consider the boundary value problem (3). The method here used for solving (3) is to reduce it to a fixed point problem in the Fréchet space $\mathcal{X}$ of all the sequences from $\mathbb{Z}$ into $\mathbb{R}^m$

$$\mathcal{X} := \{q : \mathbb{Z} \mapsto \mathbb{R}^m\}$$

and then to apply the Schauder-Tychonoff fixed point theorem.

Theorem 2 (Existence). *Let* $\xi \in Range(P_3)$ *be fixed. If conditions i) and ii) in Proposition 2 and, for* $p = 1$, *also condition iii) in Corollary 2 hold, and if in addition*

v) there exists a constant $\beta > 0$ *such that*

$$\sup_{n\in\mathbb{Z}} |X(n)\xi| + 3K\|g_\beta\|_q \le \beta,$$

then the boundary value problem

$$\begin{cases} y(n+1) = A(n)y(n) + f(n, y(n)), & n \in \mathbb{Z} \\ y(+\infty) = 0, \quad y(-\infty) = 0 \\ y(0) = \xi \end{cases} \tag{13}$$

has at least a solution.

Proof. Let $\Omega := \{q \in \mathcal{X} : q \in l_0^\infty, q(0) = \xi, \|q\|_\infty \le \beta\}$. Clearly Ω is a nonempty, closed, convex and bounded subset of $\mathcal{X}$. Consider the operator $F : \Omega \mapsto \mathcal{X}$ defined by (see the right end side of (10))

$$\begin{aligned}(Fq)(n) =& X(n)\xi + \sum_{s=-\infty}^{n-1} X(n)P_1X^{-1}(s+1)f(s, q(s)) \\ & - \sum_{s=n}^{+\infty} X(n)P_2X^{-1}(s+1)f(s, q(s)) + \sum_{s=0}^{n-1} X(n)P_3X^{-1}(s+1)f(s, q(s)) \\ & - \sum_{s=n}^{-1} X(n)P_3X^{-1}(s+1)f(s, q(s))\end{aligned}$$

(with the convention $\sum_{s=a}^{b} g(s) = 0$ if $a > b$).

Let $1 < p < \infty$. Assumptions i) and ii) in Proposition 2 assure that this operator is well defined, being $\Omega \subset l^\infty$. Let us show that $F(\Omega) \subseteq \Omega$. For every $q \in \Omega$, taking into account assumption v), we have

$$|(Fq)(n)| \leq \sup_{n\in\mathbb{Z}} |X(n)\xi| + 3K\|g_\beta\|_q \leq \beta.$$

Moreover from Proposition 2 and Corollary 2 it follows $(Fq)(n) \to 0$ as $n \to \pm\infty$ and $(Fq)(0) = \xi$, for every $q \in \Omega$. Thus $F(\Omega) \subseteq \Omega$. This also implies that $F(\Omega)$ is a relatively compact subset of $\mathcal{X}$, because in such a Fréchet space a subset is relatively compact if and only if it is bounded. Finally F is a continuous operator in Ω: let $\{q_k\}_{k\in\mathbb{N}}$ a sequence in Ω such that $q_k \to \bar{q}$ in $\mathcal{X}$, and consider the sequence $\{Fq_k\}_{k\in\mathbb{N}}$. We have

$$|(Fq_k)(n) - (F\bar{q})(n)| \leq 3K\left(\sum_{s=-\infty}^{+\infty} |f(s, q_k(s)) - f(s, \bar{q}(s))|^q\right)^{1/q}.$$

Note that $|f(s, q_k(s)) - f(s, \bar{q}(s))| \leq 2g_\beta(s) \in l^q$, for every $k \in n$. The continuity of f with respect to the second argument and the fact that the convergence in $\mathcal{X}$ implies the pointwise convergence, allow us to apply the dominated convergence theorem (see [3] for the formulation in the space $\mathcal{X}$). Thus $Fq_k \to F\bar{q}$ in $\mathcal{X}$ being the convergence of $(Fq_k)(n) - (F\bar{q})(n)$ towards zero uniform with respect to n. By the Schauder-Tychonoff fixed point theorem the operator F has at least a fixed point y in Ω and Proposition 2 assures that y is a solution of problem (13).

The case $p = 1$ can be treated by means of similar arguments. □

Remark 3. If $\xi \in Range(P_3)$ then $\sup_{n\in\mathbb{Z}} |X(n)\xi| = \max_{n\in\mathbb{Z}} |X(n)\xi| < \infty$. Indeed $\lim_{n\to\pm\infty} |X(n)\xi| = 0$.

Remark 4. Assumption v) in Theorem 2 is trivially satisfied if $\sup_{\alpha>0} \|g_\alpha\| < \infty$.

It is worth to remark that the choice of the Fréchet space $\mathcal{X}$ makes the proof of the compactness of $F(\Omega)$ quite immediate, while the proof of the continuity of F is not more difficult than working in a Banach space like l^∞.

The results of this section extend those in [18] and generalize those in [4], because the nonlinear discrete boundary value problem is completely solved. They also generalize some of the results in [2], [9], [11] because exponential trichotomy implies l^p trichotomy.

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ON THE RESONANCE PROBLEM FOR THE 4$^{\text{th}}$ ORDER ORDINARY DIFFERENTIAL EQUATIONS, FUČÍK'S SPECTRUM

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Abstract. We consider the boundary value problems for the fourth order nonlinear differential equation $u^{\mathrm{IV}} = f(x,u)$ together with three different boundary conditions (the Dirichlet, the periodic and the Navier boundary conditions). We discuss the existence results for these boundary value problems at resonance. Our results contain the Landesman–Lazer type conditions. We also show some numerical results concerning *Fučík's spectrum* for the boundary value problems for the differential equation $u^{\mathrm{IV}} = \mu u^+ - \nu u^-$, where $u^+ = \max\{u,0\}$ and $u^- = \max\{-u,0\}$, together with our three boundary conditions.

AMS Subject Classification. 34B15, 34L16, 65L15

Keywords. Fučík's spectrum, Landesman-Lazer type condition

1. Introduction

In this paper, we introduce some results concerning the boundary value problems for a fourth order differential equation. These results are the main results of the diploma thesis [4] that consists of three parts. The first part deals with the regularity problem of weak solutions, the second one describes *Fučík's spectrum* and the third one concerns the existence of at least one weak solution of our boundary value problems at resonance. This paper covers only the second and the third parts of the thesis [4].

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2. Fučík's spectrum

In this section, we investigate *Fučík's spectrum* of the boundary value problems for a fourth order differential equation. Let us consider a differential operator $L : D(L) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$, where Ω is a bounded domain with a smooth boundary. We define its *Fučík's spectrum* as the following set

$$A_{-1}(L) = \{(\mu,\nu) \in \mathbb{R}^2 \mid Lu = \mu u^+ - \nu u^- \text{ has a nontrivial solution}\},$$

where $u^+ = \max\{u,0\}$ and $u^- = \max\{-u,0\}$ are the positive and the negative parts of the function u. Let us denote the spectrum of L by

$$\sigma(L) = \{\lambda \in \mathbb{R} \mid Lu = \lambda u \text{ has a nontrivial solution}\}.$$

Then we have $\{(\lambda,\lambda) \in \mathbb{R}^2 \mid \lambda \in \sigma(L)\} \subseteq A_{-1}(L)$ and therefore we can regard *Fučík's spectrum* $A_{-1}(L)$ as a generalization of the spectrum $\sigma(L)$.

In our case, the differential operator L is defined by

$$Lu(x) = \frac{d^4u}{dx^4} \quad \text{for all } u \in D(L).$$

So, the main goal of our investigation will be the boundary value problems for the fourth order differential equation

$$u^{\mathrm{IV}} = \mu u^+ - \nu u^- \tag{1}$$

together with different type of boundary conditions. The knowledge of *Fučík's spectrum* is essential for studying various mathematical models, especially models with jumping nonlinearities (see e.g. [5] for some concrete applications).

Fučík's spectrum of the boundary value problems for the second order differential equation

$$u'' + \mu u^+ - \nu u^- = 0$$

together with the periodic or the Dirichlet boundary conditions is well known and can be described analytically by some explicit formulas (see [2]). But in the case of the boundary value problems for the fourth order differential equation (1), the situation is absolutely different and much more complicated. First of all, concerning these boundary value problems, we cannot describe corresponding *Fučík's spectrum* by some analytic explicit formulas, and only some kinds of its qualitative properties are known (see the papers [3], [1]). Note that in the recent paper [1], the asymptotic behavior of *Fučík's spectrum* is also studied.

2.1. The periodic boundary value problem

Let us consider the periodic boundary value problem of the form

$$\begin{cases} u^{\mathrm{IV}}(x) = \lambda u(x), \quad x \in [0,2\pi], \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi). \end{cases} \tag{2}$$

The eigenvalues of this boundary value problem (2) form the sequence

$$\lambda_k = k^4, \qquad k = 0, 1, 2, 3, \dots \tag{3}$$

The eigenvalues λ_k, $k = 1, 2, 3, \dots$ are of multiplicity 2 and two linearly independent orthogonal eigenfunctions correspond to each of them. We denote these orthogonal eigenfunctions by $v_{k,1}$ and $v_{k,2}$. They are of the form

$$v_0(x) = 1, \qquad v_{k,1}(x) = \sin kx, \quad v_{k,2}(x) = \cos kx, \qquad k = 1, 2, 3, \dots \tag{4}$$

2.1.1. Fučík's spectrum Let us consider the periodic boundary value problem

$$\begin{cases} u^{\mathrm{IV}}(x) = a^4 u^+(x) - b^4 u^-(x), \quad x \in [0, 2\pi], \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi). \end{cases} \tag{5}$$

For further considerations, let us consider the direct periodic extension to the whole real line $\mathbb{R}$ of each solution u of this boundary value problem (5). Let us denote $\varphi \in](3/4)\pi, \pi[$ the smallest positive root of the equation $\tan x + \tanh x = 0$. Further, let us define the auxiliary functions f and g by the formulas

$$f(x) = \frac{\cosh x \cos x}{\cosh x \sin x + \sinh x \cos x}, \quad g(x) = \frac{\cosh x \sin x - \sinh x \cos x}{\cosh x \sin x + \sinh x \cos x} \tag{6}$$

for $x \in]0, \varphi[$. The following theorem, which is proved in the paper [3] (some corrections of the analytical bounds for the spectrum is given in [4]), provides the description of the first branch of *Fučík's spectrum*.

Theorem 1. *The set S_1 of all pairs $(a, b) \in]0, +\infty[^2$ such that there exists a nontrivial 2π-periodic solution of the boundary value problem (5), which is composed of two semi-waves, is a curve $(a, b(a))$, where $b(a)$ is a decreasing C^∞-function defined in $]\varphi/\pi, +\infty[$ with $\lim_{a\to+\infty} b(a) = \varphi/\pi$.*

The curve S_1 is symmetric with respect to the straight line $b = a$ and fulfils $S_1 \subset G_1$, where G_1 is the set of all pairs $(a, b) \in]\varphi/\pi, +\infty[^2$ such that

$$\text{for } b \ge a,\ \left[\alpha(a,b) \ge \frac{\pi}{2},\ \xi(a,b) \ge 0\right] \vee \left[\alpha(a,b) < \frac{\pi}{2},\ \xi(a,b) \ge 0 \ge \psi(a,b)\right], \tag{7}$$

$$\text{for } b \le a,\ \left[\beta(a,b) \ge \frac{\pi}{2},\ \psi(a,b) \ge 0\right] \vee \left[\beta(a,b) < \frac{\pi}{2},\ \psi(a,b) \ge 0 \ge \xi(a,b)\right], \tag{8}$$

where

$$\alpha(a,b) = b\pi\left(1 - \frac{1}{2a}\right),\ \beta(a,b) = a\pi\left(1 - \frac{1}{2b}\right),$$

$$\xi(a,b) = \left(\frac{b}{a}\right)^2 - g\left(\pi a\left(1 - \frac{1}{2b}\right)\right),\ \psi(a,b) = \left(\frac{a}{b}\right)^2 - g\left(\pi b\left(1 - \frac{1}{2a}\right)\right).$$

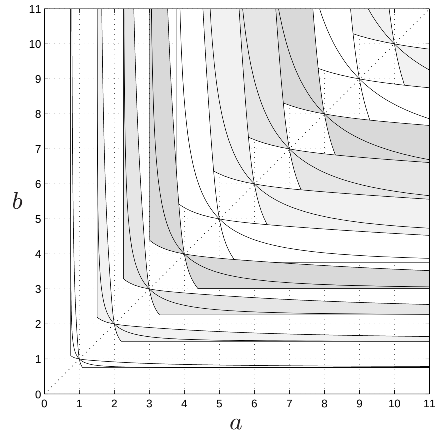

Fig. 1: The correct bounds (7), (8), (9).

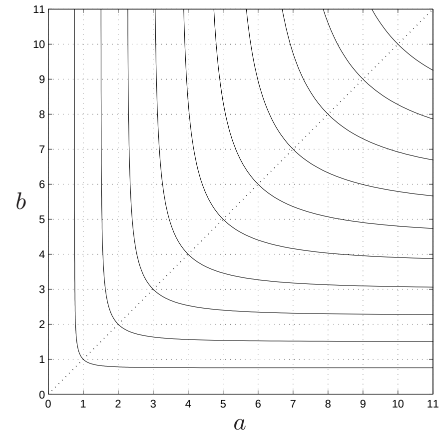

Fig. 2: *Fučík's spectrum* for the BVP (5).

The analytical bounds (7) and (8) are shown in the Figure 1. By virtue of previous Theorem 1, we can summarize the actual knowledge of *Fučík's spectrum* for the periodic boundary value problem (5) into the following items (see also [3]):

1. The set S of all pairs $(a,b) \in]0,+\infty[^2$, for which there exists a nontrivial 2π-periodic solution of the boundary value problem (5), is the countable set $\{S_k, k \in \mathbb{N}\}$ of C^∞-curves, where $S_k = \{(a,b) \in]0,+\infty[^2, (a/k, b/k) \in S_1\}$ for $k = 2,3,\ldots$, the description of the curve S_1 is given by Theorem 1.
2. The inclusion $S_k \subset G_k$ holds for all $k \in \mathbb{N}$, where

$$G_k = \{(a,b) \in]0,+\infty[^2, (a/k, b/k) \in G_1\}. \tag{9}$$

   The set G_1 is defined in Theorem 1. Thus we obtain

$$S \subset \bigcup_{k=1}^{+\infty} G_k.$$

3. For the pair $(a,b) \in S_k$, the corresponding 2π-periodic nontrivial solutions of the boundary value problem (5) have exactly $2k$ semi-waves in an interval of the length 2π. This solution is unique if the translation in the direction of the x-axes and positive multiples are not considered.

Then *Fučík's spectrum* for the periodic boundary value problem (5) is the set

$$A_{-1} = \{(a^4, b^4) \in \mathbb{R}^2 \mid (a,b) \in S\} \cup \{S_0^x, S_0^y\},$$

where S_0^x (or S_0^y, respectively) is just x-axes (y-axes, respectively). The corresponding nontrivial solutions of the boundary value problems (5) for the pairs $(a,b) \in S_0^x$ (S_0^y) are arbitrary constants $c < 0$ ($c > 0$).

2.1.2. The description of the algorithm The algorithm how to generate the points of *Fučík's spectrum* A_{-1} with some specific accuracy is in details described in [4]. It is obvious from the previous considerations that if we are able to generate the points of the set S_1 that determine the first branch of *Fučík's spectrum*, then we are able to generate automatically the other branches of *Fučík's spectrum*. It can be shown (see [3]) that the set S_1 is described by the system of two nonlinear equations

$$\begin{aligned} af(ar) + bf(b(\pi - r)) &= 0, \\ a^2 g(ar) - b^2 g(b(\pi - r)) &= 0. \end{aligned} \tag{10}$$

The principle of the algorithm is such that for the chosen fixed $r \in (\pi/2, \pi)$ we compute the parameters a and b of the system (10) numerically with some accuracy. This provides the approximation of one pair $(a, b) \in S_1$. For the complete description of the algorithm see thesis [4].

2.2. The Navier boundary value problem

Let us consider the boundary value problem of the form

$$\begin{cases} u^{\mathrm{IV}}(x) = \lambda u(x), \quad x \in [0, \pi], \\ u(0) = u''(0) = u(\pi) = u''(\pi) = 0. \end{cases} \tag{11}$$

The eigenvalues of this boundary value problem (11) and the corresponding eigenfunctions are

$$\lambda_k = k^4, \qquad v_k(x) = \sin kx, \qquad k = 1, 2, 3, \dots.$$

2.2.1. Fučík's spectrum Let us consider the boundary value problem

$$\begin{cases} u^{\mathrm{IV}}(x) = a^4 u^+(x) - b^4 u^-(x), \quad x \in [0, \pi], \\ u(0) = u''(0) = u(\pi) = u''(\pi) = 0. \end{cases} \tag{12}$$

Fučík's spectrum of this boundary value problem (12) is the set

$$A_{-1} = \{(a^4, b^4) \in \mathbb{R}^2 \mid (a, b) \in S\},$$

where S is the system of continuous curves $S = \{S_i^+, S_i^-, i \in \mathbb{N}\}$ with the following properties (see [3]):

1. Let $(a, b) \in S_i^+$ (S_i^-), then the solution u of the boundary value problem (12) is the solution of the initial value problem

$$\begin{cases} u^{\mathrm{IV}}(x) = a^4 u^+(x) - b^4 u^-(x), \quad x \in [0, +\infty[, \\ u(0) = 0, \quad u'(0) = \alpha, \quad u''(0) = 0, \quad u'''(0) = t, \end{cases} \tag{13}$$

   with $\alpha > 0$ $(\alpha < 0)$ and with some $t \in \mathbb{R}$. This solution u is uniquely determined by the choice of the constant α and has exactly $(i + 1)$ zeros in the interval $[0, \pi]$.

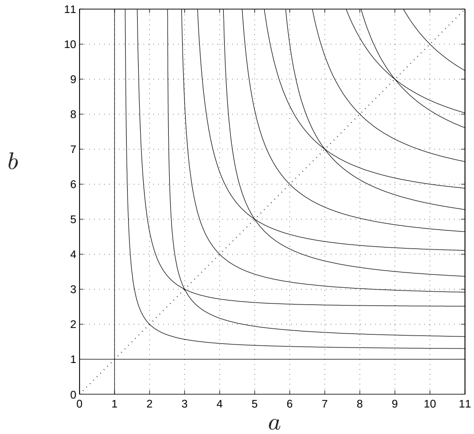

Fig. 3. *Fučík's spectrum* for the Navier BVP (12).

2. The curves S_i^+ and S_i^- are mutually symmetric with respect to the straight line $a = b$. If i is even, then $S_i^+ = S_i^-$.
3. For all $i \in \mathbb{N}$, $(S_i^+ \cup S_i^-) \cap (S_{i+1}^+ \cup S_{i+1}^-) = \emptyset$ holds.

2.2.2. The description of the algorithm We will try to explain the main idea of the algorithm for generating *Fučík's spectrum* for the easiest case. This means, we consider the second branch S_2^+ that merges in the curve S_2^-, which follows from the properties of the spectrum that we mentioned in the previous Section 2.2.1. If we restrict our attention only to the second branch S_2^+, then we know that the corresponding solutions u of the boundary value problem (12) will have exactly 3 zeros in the interval $[0, \pi]$. Further, we know that the curve S_2^+ is passing through the point $(\sqrt[4]{\lambda_2}, \sqrt[4]{\lambda_2}) = (2, 2)$ and the corresponding nontrivial solution of the boundary value problem (12) is then $v_2(x) = \sin 2x$. Due to the symmetry of *Fučík's spectrum* with respect to the straight line $a = b$, we can concentrate hereafter only on the case $a \geq b$.

We will try to find the inspiration in the classical shooting methods, which are based on a transformation of a boundary value problem into a sequence of some initial value problems. Our attention will be therefore concentrated on the initial value problem (13). There are four parameters a, b, α and t in the initial value problem (13). We will try to determine these parameters in such a way that the corresponding solution u of the problem (13) will be the solution of the boundary value problem (12) and in the interval $[0, \pi]$ will have exactly 3 zeros. If u is the solution of the boundary value problem (12), then an arbitrary positive multiple of u is also its solution. This fact can be expressed just by the parameter α. Let us therefore choose an arbitrary, but fixed value of the parameter α such that $\alpha > 0$, because we are studying the curve S_2^+. Our goal is now to find the corresponding values of the parameters b and t (for the chosen parameter

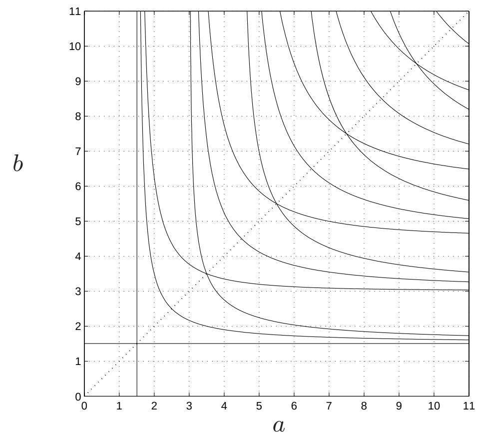

Fig. 4. *Fučík's spectrum* for the Dirichlet BVP (17).

a) such that the point (a, b) will be the point of the curve S_2^+. For more details see the second part of the thesis [4], where the complete form of the algorithm can be also found.

2.3. The Dirichlet boundary value problem

Let us consider the eigenvalue problem for the Dirichlet boundary value problem of the form

$$\begin{cases} u^{\mathrm{IV}}(x) = \lambda u(x), \quad x \in [0, \pi], \\ u(0) = u'(0) = u(\pi) = u'(\pi) = 0. \end{cases} \tag{14}$$

The eigenvalues λ_k of this boundary value problem (14) are given by

$$\lambda_k = \varphi_k^4, \quad \text{where} \quad \cos\varphi_k\pi \cosh\varphi_k\pi = 1, \quad \varphi_k \neq 0, \qquad k = 1, 2, 3, \dots \tag{15}$$

and the corresponding eigenfunctions are

$$\begin{aligned} v_k(x) = {} & [\cosh\varphi_k\pi - \cos\varphi_k\pi][\sinh\varphi_k x - \sin\varphi_k x] - \\ & -[\sinh\varphi_k\pi - \sin\varphi_k\pi][\cosh\varphi_k x - \cos\varphi_k x]. \end{aligned} \tag{16}$$

2.3.1. Fučík's spectrum Let us consider the boundary value problem

$$\begin{cases} u^{\mathrm{IV}}(x) = a^4 u^+(x) - b^4 u^-(x), \quad x \in [0, \pi], \\ u(0) = u'(0) = u(\pi) = u'(\pi) = 0. \end{cases} \tag{17}$$

Fučík's spectrum of the Dirichlet boundary value problem (17) has similar properties as *Fučík's spectrum* of the previous Navier boundary value problem (12). This can be also observed if we compare the Figures 3 and 4. The algorithm for generating *Fučík's spectrum* of the Dirichlet boundary value problem (17) is analogous to the algorithm for the previous problem (12) (see [4]).

2.4. The implementation of the algorithms

The algorithms stated in this paper can be easily modified for the problems with other boundary conditions. In general, it is possible to say that for realization of the algorithms for generating *Fučík's spectrum* it is necessary to perform the individual steps of the computations with relatively high accuracy; the higher accuracy, the higher number of the branches of *Fučík's spectrum* we would like to generate.

The mentioned algorithms for generating *Fučík's spectrum* of our three boundary value problems (the periodic boundary value problem (5), the Navier boundary value problem (12) and the Dirichlet boundary value problem (17)) were implemented in Fortran 77 on the parallel computer cluster Lyra.

Due to the required higher accuracy, for the computations generating the higher branches of *Fučík's spectrum*, the mentioned algorithms were implemented also in the system Mathematica 3.0. The algorithms were included into the system of procedures for modelling of bifurcations (MBx). For more results of our numerical experiments visit the internet site

`http://cam.zcu.cz/members/necesal/index.cz.shtml`.

3. Existence results

Let us consider the boundary value problems for the fourth order nonlinear differential equation

$$u^{\mathrm{IV}} = f(x, u)$$

together with three different boundary conditions (the Dirichlet, the periodic and the Navier boundary conditions). In this section, we discuss the existence results for these boundary value problems at resonance. Our results rely on the Landesman–Lazer type conditions.

3.1. The Dirichlet boundary value problem

Let us consider the boundary value problem of the form

$$\begin{cases} u^{\mathrm{IV}}(x) - \lambda_m u(x) + g(x, u(x)) = f(x), \quad x \in [0, \pi], \\ u(0) = u'(0) = u(\pi) = u'(\pi) = 0, \end{cases} \tag{18}$$

where $g : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is the *Carathéodory function*, the right hand side $f \in \mathrm{L}^1(0, \pi)$, λ_m is the eigenvalue of the boundary value problem (14) (see the relation (15)).

Henceforth we will assume that the function $g = g(x, s)$ satisfies the following *growth condition*. Let us suppose that there exist the function $p \in \mathrm{L}^1(0, \pi)$ and the constant $C > 0$ such that the inequality

$$|g(x, s)| \le p(x) + C|s| \tag{19}$$

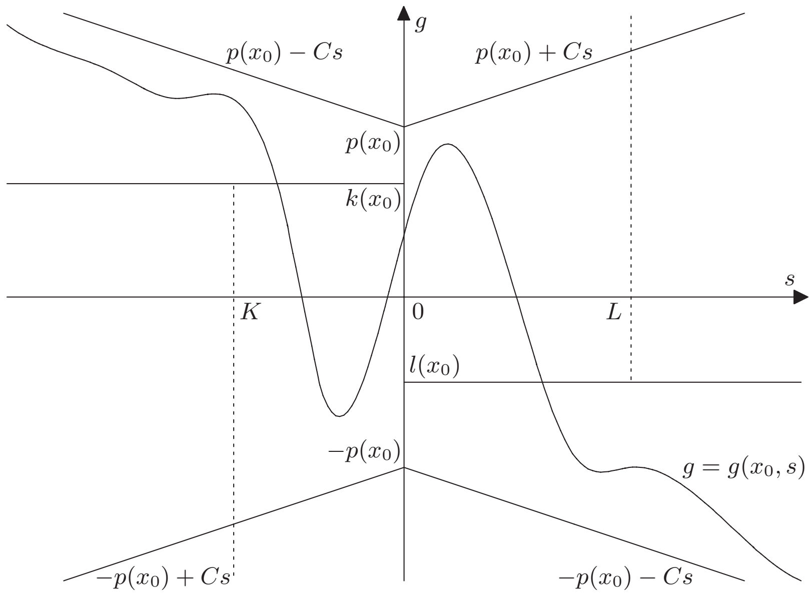

Fig. 5. The illustration of the conditions (19), (20) and (21) for fixed $x_0 \in [0, \pi]$.

holds for all $s \in \mathbb{R}$ and for a. a. $x \in [0, \pi]$. Moreover, let us suppose that there exist the functions $k, l \in \mathrm{L}^1(0, \pi)$ and the constants $K, L \in \mathbb{R}$, $K < 0 < L$, such that

$$g(x, s) \geq k(x) \quad \text{for all } s \leq K \text{ and for a. a. } x \in [0, \pi], \tag{20}$$

$$g(x, s) \leq l(x) \quad \text{for all } s \geq L \text{ and for a. a. } x \in [0, \pi]. \tag{21}$$

Let us denote $\mathrm{H} = \mathrm{W}_0^{2,2}(0, \pi)$ the Sobolev space on the interval $]0, \pi[$ with the inner product and the norm

$$(u, v) = \int_0^\pi u''(x) v''(x)\, dx \quad \text{and} \quad \|u\| = \sqrt{(u, u)}, \quad \text{respectively.}$$

We say that u is the *weak solution* of the boundary value problem (18), if $u \in \mathrm{H}$ and the integral identity

$$\int_0^\pi u''(x) v''(x)\, dx - \lambda_m \int_0^\pi u(x) v(x)\, dx + \int_0^\pi g(x, u(x)) v(x)\, dx = \int_0^\pi f(x) v(x)\, dx$$

holds for all $v \in \mathrm{H}$.

Theorem 2 (Sublinear growth). *Let us suppose that the function $g = g(x, s)$ satisfies all assumptions stated above and, moreover,*

$$\lim_{s \to \pm\infty} \frac{g(x, s)}{s} = 0 \tag{22}$$

uniformly for a. a. $x \in [0, \pi]$. *Denote*

$$g^{+\infty}(x) = \limsup_{s \to +\infty} g(x, s), \qquad g_{-\infty}(x) = \liminf_{s \to -\infty} g(x, s).$$

Then the boundary value problem (18) *has at least one weak solution provided the Landesman–Lazer type condition*

$$\int_0^\pi g^{+\infty}(x) v_m^+(x)\, dx - \int_0^\pi g_{-\infty}(x) v_m^-(x)\, dx < \int_0^\pi f(x) v_m(x)\, dx <$$

$$< \int_0^\pi g_{-\infty}(x) v_m^+(x)\, dx - \int_0^\pi g^{+\infty}(x) v_m^-(x)\, dx$$

holds.

Proof. The proof is based on the Leray-Schauder degree theory (see [4]).

3.2. The Navier boundary value problem

Let us consider the boundary value problem

$$\begin{cases} u^{\mathrm{IV}}(x) - \lambda_m u(x) + g(x, u(x)) = f(x), & x \in [0, \pi], \\ u(0) = u''(0) = u(\pi) = u''(\pi) = 0, \end{cases} \tag{23}$$

where $g : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is the *Carathéodory function* satisfying the assumptions from Section 3.1, the right hand side $f \in \mathrm{L}^1(0, \pi)$, $\lambda_m = m^4$ for $m \in \mathbb{N}$ is the eigenvalue of the boundary value problem (11).

Let us denote $\mathrm{H} = \{u \in \mathrm{W}^{2,2}(0, \pi);\ u(0) = u(\pi) = 0\} = \mathrm{W}^{2,2}(0, \pi) \cap \mathrm{W}_0^{1,2}(0, \pi)$ the space with the inner product and the norm

$$(u, v) = \int_0^\pi [u''(x) v''(x) + u(x) v(x)]\, dx, \quad \text{and} \quad \|u\| = \sqrt{(u, u)}, \quad \text{respectively.}$$

We say that u is the *weak solution* of the boundary value problem (23), if $u \in \mathrm{H}$ and the integral identity

$$\int_0^\pi u''(x) v''(x)\, dx - \lambda_m \int_0^\pi u(x) v(x)\, dx + \int_0^\pi g(x, u(x)) v(x)\, dx = \int_0^\pi f(x) v(x)\, dx$$

holds for all $v \in \mathrm{H}$.

Theorem 3 (Sublinear growth). *Let us suppose that the Carathéodory function* $g = g(x, s)$ *satisfies* (19) – (22). *Then the boundary value problem* (23) *has at least one weak solution provided the Landesman–Lazer type condition*

$$\int_0^\pi g^{+\infty}(x) (\sin mx)^+\, dx - \int_0^\pi g_{-\infty}(x) (\sin mx)^-\, dx < \int_0^\pi f(x) \sin mx\, dx <$$

$$< \int_0^\pi g_{-\infty}(x)(\sin mx)^+ \, dx - \int_0^\pi g^{+\infty}(x)(\sin mx)^- \, dx$$

holds.

Proof. The proof is analogous to the proof of Theorem 2 (see [4]).

3.3. The periodic boundary value problem

In this section, we will consider the periodic boundary value problem

$$\begin{cases} u^{\mathrm{IV}}(x) - \lambda_m u(x) + g(x, u(x)) = f(x), \quad x \in [0, 2\pi], \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi), \end{cases} \tag{24}$$

where $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ is the *Carathéodory function*, the right hand side $f \in \mathrm{L}^1(0, 2\pi)$, $\lambda_m = m^4$ for $m \in \mathbb{N}$ is the eigenvalue of the boundary value problem (2). Moreover, let us suppose that the function $g = g(x, s)$ satisfies all assumptions for the function g in the Section 3.1 with $[0, \pi]$ replaced by $[0, 2\pi]$. In particular, this means that the growth condition (19) and the conditions (20), (21) hold with the replacement of the interval $[0, \pi]$ by $[0, 2\pi]$.

Let us denote $\mathrm{H} = \{u \in \mathrm{W}^{2,2}(0, 2\pi);\ u(0) = u(2\pi),\ u'(0) = u'(2\pi)\}$ the space with the inner product and the norm

$$(u, v) = \int_0^{2\pi} [u''(x)v''(x) + u(x)v(x)] \, dx \quad \text{and} \quad \|u\| = \sqrt{(u, u)}, \quad \text{respectively.}$$

We say that u is the *weak solution* of the boundary value problem (24), if $u \in \mathrm{H}$ and the integral identity

$$\int_0^{2\pi} u''(x)v''(x) \, dx - \lambda_m \int_0^{2\pi} u(x)v(x) \, dx + \int_0^{2\pi} g(x, u(x))v(x) \, dx = \int_0^{2\pi} f(x)v(x) \, dx$$

holds for all $v \in \mathrm{H}$.

Theorem 4 (Sublinear growth). *Let us suppose that the function $g = g(x, s)$ satisfies all assumptions stated above and, moreover, the growth condition (22) holds uniformly for a. a. $x \in [0, 2\pi]$. Then the boundary value problem (24) has at least one weak solution provided the Landesman–Lazer type condition*

$$\int_{v>0} g^{+\infty}(x)v(x) \, dx + \int_{v<0} g_{-\infty}(x)v(x) \, dx < \int_0^{2\pi} f(x)v(x) \, dx$$

holds for all $v \in \mathrm{Span}\{\cos mx, \sin mx\} \setminus \{0\}$.

Proof. The proof is analogous to the proof of Theorem 2 (see [4]).

3.4. The reverse growth of the nonlinearity

Let us suppose that in the case of the Dirichlet boundary value problem (18) the function $g = g(x, s)$ satisfies

$$g(x,s) \le k(x) \quad \text{for all } s \le K \text{ and for a. a. } x \in [0,\pi], \tag{25}$$

$$g(x,s) \ge l(x) \quad \text{for all } s \ge L \text{ and for a. a. } x \in [0,\pi], \tag{26}$$

instead of the conditions (20) and (21). The meaning of k, K, l and L is the same as in the Section 3.1. Note that the hypotheses (25), (26) are in a certain sense *dual* to the assumptions (20), (21). In this case we can formulate the *dual version* of Theorem 2.

Theorem 5 (Sublinear growth). *Let us suppose that the function $g = g(x, s)$ satisfies (19), (22) and the conditions (25), (26). Then the boundary value problem (18) has at least one weak solution provided the Landesman–Lazer type condition*

$$\int_0^\pi g^{-\infty}(x) v_m^+(x)\,dx - \int_0^\pi g_{+\infty}(x) v_m^-(x)\,dx < \int_0^\pi f(x) v_m(x)\,dx <$$

$$< \int_0^\pi g_{+\infty}(x) v_m^+(x)\,dx - \int_0^\pi g^{-\infty}(x) v_m^-(x)\,dx$$

holds, where

$$g^{-\infty}(x) = \limsup_{s\to-\infty} g(x,s), \qquad g_{+\infty}(x) = \liminf_{s\to+\infty} g(x,s).$$

Proof. The proof follows the lines of that of Theorem 2 (see [4]).

The main difference between Theorem 2 and its *dual version* Theorem 5 is in different form of the Landesman-Lazer type condition. For the *dual* formulations in the cases of our two remaining boundary value problems see thesis [4].

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SUFFICIENT CONDITIONS FOR NONOSCILLATION OF NEUTRAL EQUATIONS

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Abstract. Sufficient conditions are given under which the first order neutral differential equation with constant coefficients has a nonoscillatory solution.

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1. Introduction

Consider the neutral differential equation

$$\frac{d}{dt}[x(t) + px(t-\tau)] + qx(t-\sigma) = 0, \quad t \geq t_0, \tag{1}$$

where

(i) p, q, τ, σ are positive real numbers.

Note that a nontrivial solution of an equation we call oscillatory if it has arbitrarily large zeros, and call it nonoscillatory otherwise, and next we shall say that an equation is oscillatory provided all its (nontrivial) solutions are oscillatory, and call it nonoscillatory otherwise.

A basic result on the oscillation of equation (1) says that every solution of equation (1) is oscillatory if and only if its characteristic equation

$$\lambda + p\lambda e^{-\lambda\tau} + qe^{-\lambda\sigma} = 0 \tag{2}$$

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has no real roots. Such result we can find in the book [1] and for more general equations in the book [2] and in the paper [3]. But to determine if equation (2) has a real root is quite a problem itself. Therefore an effort of many authors is to derive other conditions for oscillation and nonoscillation of considered equation which can be easily applied than previous one. In a literature we can find several sufficient conditions for every solution of equation (1) to be oscillatory (see e.g. [1] and [4]) but less conditions for the existence of nonoscillatory solution of (1).

The aim of this contribution is to present new well-applicable conditions for the existence of nonoscillatory solution of (1). The method is based on a transformation of the equation (1) by a transformation of the independent variable.

The straight consideration about the existence of a real root of characteristic equation (2) enables us to obtain the following result.

Theorem 1. *Assume the condition* (*i*) *holds true and* $\tau \geq \sigma$. *Then equation* (1) *has nonoscillatory solution* $x(t) = e^{\lambda t}$, $\lambda \in (-\frac{q}{p}, 0)$.

Proof. According to assumptions it is clear that if the equation (2) has a real root so it must be negative. Thus we define

$$F(\lambda) = \lambda + p\lambda e^{-\lambda\tau} + qe^{-\lambda\sigma} \quad \text{for} \quad \lambda \leq 0$$

and put $F(\lambda) = H_1(\lambda) + H_2(\lambda)$, where $H_1(\lambda) = \lambda + p\lambda e^{-\lambda\tau}$, $H_2(\lambda) = qe^{-\lambda\sigma}$. Then we have

$$\lim_{\lambda\to 0^-} H_1(\lambda) = 0, \quad \lim_{\lambda\to -\infty} H_1(\lambda) = -\infty, \quad H_1'(\lambda) = 1 + pe^{-\lambda\tau}(1 - \lambda\tau) > 0$$
$$\text{and} \quad \lim_{\lambda\to 0^-} H_2(\lambda) = q, \quad \lim_{\lambda\to -\infty} H_2(\lambda) = \infty, \quad H_2'(\lambda) = -q\sigma e^{-\lambda\sigma} < 0$$

from which we see that for $\tau \geq \sigma$ we have $F(-\frac{q}{p}) = -\frac{q}{p} + q(e^{\frac{q}{p}\sigma} - e^{\frac{q}{p}\tau}) < 0$. Since $F(0) = q > 0$ so we know that the equation (2) has the root $\lambda \in (-\frac{q}{p}, 0)$, the function $x(t) = e^{\lambda t}$ is the solution of (1) and the proof is complete.

Another way how to gain sufficient conditions for the existence of nonoscillatory solution of equation (1) we present in the following sections.

2. Preliminaries

Consider the equation (1) but instead of condition (i) we suppose that

(ii) p, q, τ, σ are real numbers different from zero.

We transform the equation (1) by the transformation of the independent variable. We put $s = at$, $y(s) = x(\frac{1}{a}s)$ where $a > 0$. Then the equation (1) acquires the form

$$\frac{d}{ds}[y(s) + py(s - a\tau)] + \frac{1}{a}qy(s - a\sigma) = 0, \quad s \geq s_0, \tag{3}$$

where $s_0 = at_0$.

It is clear the following holds true.

Note 1. A function $x(t)$ is a solution of the equation (1) for $t \geq t_0$ if and only if the function $y(s) = x(\frac{1}{a}s)$ is a solution of the equation (3) for $s \geq s_0$ and thus the equation (1) is oscillatory if and only if equation (3) is oscillatory.

Since equation (3) is of the same form as equation (1) is so it is oscillatory if and only if its characteristic equation

$$a\eta + pa\eta e^{-a\eta\tau} + qe^{-a\eta\sigma} = 0 \tag{4}$$

has no real roots and we can decide about solutions of (1) by the roots of the equation (4).

Now we analyse this position.

(a) First of all we see that the number $\lambda = 0$ is not the root of equation (2).

(b) Suppose that equation (2) has a positive root λ. Then we can take $a = \lambda$ and equation (4) will be of the form $\lambda\eta + p\lambda\eta e^{-\lambda\eta\tau} + qe^{-\lambda\eta\sigma} = 0$ and we see that $\eta = 1$ is the root of this equation. It means that equation

$$\frac{d}{ds}[y(s) + py(s - \lambda\tau)] + \frac{q}{\lambda}y(s - \lambda\sigma) = 0, \quad s \geq s_0,$$

has nonoscillatory solution $y(s) = e^s$.

(c) Now suppose that equation (2) has a negative root λ. So if we take $a = -\lambda$, equation (4) will be of the form $-\lambda\eta - p\lambda\eta e^{\lambda\eta\tau} + qe^{\lambda\eta\sigma} = 0$ and we see that $\eta = -1$ is the root of this equation. It means that equation

$$\frac{d}{ds}[y(s) + py(s + \lambda\tau)] - \frac{q}{\lambda}y(s + \lambda\sigma) = 0, \quad s \geq s_0,$$

has nonoscillatory solution $y(s) = e^{-s}$.

We conclude this consideration in the following note.

Note 2. To every equation of the form (1), the characteristic equation of which has a positive (negative) root, we can coordinate an equation of the same form with the characteristic root 1 (-1). On the other hand, if we take an equation of the form (1) with the solution $y(s) = e^s$ (similarly with the solution $y(s) = e^{-s}$) and we choose some positive number λ (a negative number λ) so we can write the equation of the same form with the solution $x(t) = e^{\lambda t}$ ($x(t) = e^{\lambda t}$).

3. Conditions for nonoscillatory solutions

Theorem 2. *Assume that* $p \neq 0,\ q > 0,\ \tau > 0,\ \sigma > 0$.

(I) Let there exist numbers $q_1 > 0,\ \tau_1 > 0,\ \sigma_1 > 0$ *such that the conditions*

$$1 + pe^{-\tau_1} + q_1 e^{-\sigma_1} = 0 \quad \textit{and} \quad \frac{\tau_1}{\tau} = \frac{\sigma_1}{\sigma} = \frac{q}{q_1} = \frac{1}{a} \tag{5}$$

are satisfied. Then equation (1) *has nonoscillatory solution* $x(t) = e^{\frac{1}{a}t}$.

(II) Let there exist numbers $q_2 > 0,\ \tau_2 > 0,\ \sigma_2 > 0$ *such that the conditions*

$$-1 - pe^{\tau_2} + q_2 e^{\sigma_2} = 0 \quad \textit{and} \quad \frac{\tau_2}{\tau} = \frac{\sigma_2}{\sigma} = \frac{q}{q_2} = \frac{1}{a}, \tag{6}$$

are satisfied. Then equation (1) *has nonoscillatory solution* $x(t) = e^{-\frac{1}{a}t}$.

Proof. Consider the equation

$$\frac{d}{dz}[u(z) + p_1 u(z - \tau_1)] + q_1 u(z - \sigma_1) = 0, \quad z \geq z_0, \tag{7}$$

with $p_1 \neq 0, q_1 > 0, \tau_1 > 0, \sigma_1 > 0$, which has the solution $u(z) = e^z$, i.e. such that its characteristic equation $\mu + p_1\mu e^{-\mu\tau_1} + q_1 e^{-\mu\sigma_1} = 0$ has the root $\mu = 1$, i.e. such that $1 + p_1 e^{-\tau_1} + q_1 e^{-\sigma_1} = 0$. The equation (7) we can transform to the equation (1) by a suitable $a > 0$. In other words, there exists a number $a > 0$ such that the transformation of (7) by $t = az$, $x(t) = u(\frac{1}{a}t)$ gives the equation (1) in the formal form

$$\frac{d}{dt}[x(t) + p_1 x(t - a\tau_1)] + \frac{1}{a} q_1 x(t - a\sigma_1) = 0.$$

So we have $p = p_1$, and next

$$q = \frac{1}{a} q_1, \quad \tau = a\tau_1, \quad \sigma = a\sigma_1. \tag{8}$$

The conditions (8) we can write in the form

$$\frac{\tau_1}{\tau} = \frac{\sigma_1}{\sigma} = \frac{q}{q_1} = \frac{1}{a}.$$

The straight computation shows that the number $\frac{1}{a}$ is the root of the equation (2).

The similar arguments hold true if we take the equation

$$\frac{d}{dz}[u(z) + p_2 u(z - \tau_2)] + q_2 u(z - \sigma_2) = 0, \quad z \geq z_0, \tag{9}$$

where $p_2 \neq 0, q_2 > 0, \tau_2 > 0, \sigma_2 > 0$ with the solution $u(z) = e^{-z}$. The theorem is proved.

Now using Theorem 2 we study the problem of the existence of nonoscillatory solutions of the equation (1) under the condition (i).

The assumption (i) ensures that the equation (2) has not nonnegative root i.e. the equation (1) has not the solution of the form $x(t) = e^{\lambda t},\ \lambda \geq 0$ and thus there do not exist positive numbers q_1, τ_1, σ_1 satisfying the first condition from (5). Therefore we devote our attention to the case (II) of Theorem 2.

Let the numbers $q_2 > 0,\ \tau_2 > 0,\ \sigma_2 > 0$ be such that the first condition from (6) is satisfied (note that such numbers always exist) and for some $\sigma_2 > 0$ we choose $q_2 > 0$ and $\tau_2 > 0$ such that

$$q_2 = \frac{q\sigma}{\sigma_2} \quad \text{and} \quad \tau_2 = \frac{\tau\sigma_2}{\sigma}. \tag{10}$$

Then the numbers $q_2,\ \tau_2,\ \sigma_2$ satisfy the second condition from (6) and the problem of the existence of trinity of numbers for which the first condition from (6) is satisfied is reduced to the problem of the existence of one such number.

Now we define the function $G(\sigma_2) = \frac{1}{\sigma_2} e^{\sigma_2}, \quad \sigma_2 > 0$. Then

$$G'(\sigma_2) = \frac{1}{\sigma_2^2} e^{\sigma_2}(\sigma_2 - 1), \quad G''(\sigma_2) = \frac{1}{\sigma_2^3} e^{\sigma_2}((\sigma_2 - 1)^2 + 1),$$

from which we see that for every $\sigma_2 > 0$ we have $G(\sigma_2) \geq e$.

Now suppose that $q\sigma > \frac{1}{e}$. Then for every $\sigma_2 > 0$ we have

$$\frac{1}{q\sigma} < e \leq \frac{1}{\sigma_2} e^{\sigma_2}.$$

Therefore, according to (10) we have $-1 + q_2 e^{\sigma_2} > 0$ and the first condition from (6) will be satisfied if and only if $\frac{\tau}{\sigma}\sigma_2 = \ln \frac{q_2 e^{\sigma_2} - 1}{p}$ or

$$\frac{\tau}{\sigma}\sigma_2 + \ln p = \ln\left(\frac{q\sigma}{\sigma_2} e^{\sigma_2} - 1\right) \tag{11}$$

for some $\sigma_2 > 0$.

The existence of a positive root of the equation (11) we investigate now by the auxiliary function

$$F(\sigma_2) = \frac{\ln(q\sigma \frac{1}{\sigma_2} e^{\sigma_2} - 1)}{\frac{\tau}{\sigma}\sigma_2 + \ln p},$$

defined

- for $\sigma_2 \in (0, \infty)$ if $p \geq 1$
- for $\sigma_2 \in ((0, -\frac{\sigma}{\tau}\ln p) \cup (-\frac{\sigma}{\tau}\ln p, \infty))$ if $0 < p < 1$.

Then for $p > 0$ we have $\lim_{\sigma_2 \to \infty} F(\sigma_2) = \frac{\sigma}{\tau}$, and

$$\lim_{\sigma_2 \to 0^+} F(\sigma_2) = \begin{cases} \infty & \text{if} \quad p \geq 1 \\ -\infty & \text{if} \quad 0 < p < 1. \end{cases}$$

In the case $0 < p < 1$ we compute one-side limits of the function F at the point $-\frac{\sigma}{\tau}\ln p$ and we obtain

$$\lim_{\sigma_2 \to -\frac{\sigma}{\tau}\ln p^-} F(\sigma_2) = \begin{cases} -\infty & \text{if} \quad q\tau + 2p^{\frac{\sigma}{\tau}}\ln p > 0 \\ \infty & \text{if} \quad q\tau + 2p^{\frac{\sigma}{\tau}}\ln p < 0 \\ c \in \mathbb{R} & \text{if} \quad q\tau + 2p^{\frac{\sigma}{\tau}}\ln p = 0 \end{cases}$$

and

$$\lim_{\sigma_2 \to -\frac{\sigma}{\tau}\ln p^+} F(\sigma_2) = \begin{cases} \infty & \text{if} \quad q\tau + 2p^{\frac{\sigma}{\tau}}\ln p > 0 \\ -\infty & \text{if} \quad q\tau + 2p^{\frac{\sigma}{\tau}}\ln p < 0 \\ c \in \mathbb{R} & \text{if} \quad q\tau + 2p^{\frac{\sigma}{\tau}}\ln p = 0. \end{cases}$$

This investigation and the continuity of F enables us to formulate the following results.

Theorem 3. *Let the condition (i) hold true and let*

$$0 < p < 1, \quad q\sigma > \frac{1}{e}, \quad q\tau + 2p^{\frac{\sigma}{\tau}} \ln p < 0.$$

Then there exists $\sigma_2 \in (0, -\frac{\sigma}{\tau} \ln p)$ *such that* (11) *holds true, i.e. the equation* (9) *has the solution* $x(t) = e^{-t}$ *and the equation* (1) *has the nonoscillatory solution* $x(t) = e^{-\frac{\sigma_2}{\sigma} t}$.

Theorem 4. *Let the condition (i) hold true and let*

$$0 < p < 1, \quad q\sigma > \frac{1}{e}, \quad q\tau + 2p^{\frac{\sigma}{\tau}} \ln p < 0 \quad \textit{and} \quad \frac{\sigma}{\tau} > 1.$$

Then there exists $\sigma_2 \in (-\frac{\sigma}{\tau} \ln p, \infty)$ *such that* (11) *holds true i.e. the equation* (9) *has the solution* $x(t) = e^{-t}$ *and the equation* (1) *has the nonoscillatory solution* $x(t) = e^{-\frac{\sigma_2}{\sigma} t}$.

Theorem 5. *Let the condition (i) hold true and let*

$$0 < p < 1, \quad q\sigma > \frac{1}{e}, \quad q\tau + 2p^{\frac{\sigma}{\tau}} \ln p > 0 \quad \textit{and} \quad \frac{\sigma}{\tau} < 1.$$

Then there exists $\sigma_2 \in (-\frac{\sigma}{\tau} \ln p, \infty)$ *such that* (11) *holds true i.e. the equation* (9) *has the solution* $x(t) = e^{-t}$ *and the equation* (1) *has nonoscillatory solution* $x(t) = e^{-\frac{\sigma_2}{\sigma} t}$.

Remark 1. One can see that the above presented method can be used in many other cases not only in the case when the condition (i) is satisfied.

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ON SOME SPECIFIC NON-LINEAR ORDINARY DIFFERENCE EQUATIONS

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Abstract. It is proved that some specific non-linear ordinary difference equations, which appear in various applications, have a unique solution in the Banach space l_1. Moreover a bound of the solutions and a region of attraction of their equilibrium points are found. The obtained results improve some previous known results.

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Keywords. Non-linear difference equations, bounded solutions, asymptotic stability

1. Introduction

In this paper, we study the homogeneous, non-linear difference equation:

$$f(n+2) = \lambda f(n+1) + pf(n)e^{-\sigma f(n)}, n = 1, 2, ... \tag{1.1}$$

where $0 < \lambda < 1$, $\sigma > 0$, $0 < p < (1-\lambda)e^{\frac{2-\lambda}{1-\lambda}}$, $p \neq 1-\lambda$ and the non-homogeneous, non-linear difference equations:

$$\begin{aligned} f(n+1) = &-\frac{b_1(n+1)}{\alpha_1(n+1)} + \frac{h_1(n+1)}{\alpha_1(n+1)} f(n+2)f(n+1)f(n) + \\ &+\frac{d_1(n+1)}{\alpha_1(n+1)} f(n+2)f(n), n = 1, 2, ... \end{aligned} \tag{1.2}$$

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$$f(n+2) = \frac{\alpha_2(n+1)}{h_2(n+1)} + \frac{b_2(n+1)}{h_2(n+1)}[f(n+1)]^2 - \\ - \frac{1}{h_2(n+1)} f(n+2)[f(n)]^2, n = 1, 2, ... \tag{1.3}$$

$$f(n+1) = h_3(n) + [f(n)]^2, n = 0, 1, ... \tag{1.4}$$

$$f(n+1) = h_4(n) + \mu f(n) \left[1 - \frac{1}{K} f(n)\right], n = 1, 2, ... \tag{1.5}$$

where $\mu \in \mathbb{R} \setminus \{1\}$, $K > 0$ and $\alpha_1(n+1)$, $b_1(n+1)$, $h_1(n+1)$, $d_1(n+1)$, $\alpha_2(n+1)$, $b_2(n+1)$, $h_2(n+1)$, $h_3(n)$ and $h_4(n)$ are suitably defined complex sequences.

Our aim is to prove that the equations (1.1)-(1.5) have a unique solution in the Banach space:

$$l_1 = \{f(n) : \mathbb{N} \to \mathbb{C} / \|f(n)\|_{l_1} = \sum_{n=1}^{\infty} |f(n)| < +\infty\}, \tag{1.6}$$

For the motivation of seeking solutions of non-linear difference equations in l_1 see [1, pp. 84-112], [6]. Also it is known, see [11] and the references therein, that, under various conditions, a positive generated, ordered Banach space is order-isomorphic to l_1. Finally, we would like to point out that, the real space $l_1|_{\mathbb{R}}$, i.e.

$$l_1|_{\mathbb{R}} = \{f(n) : \mathbb{N} \to \mathbb{R} \ / \sum_{n=1}^{\infty} |f(n)| < +\infty\}, \tag{1.7}$$

is suitable for problems of population dynamics, since the condition:

$$\sum_{n=1}^{\infty} |f(n)| < +\infty,$$

represents the realistic fact that the population $f(n)$ is finite in every time instant n.

The method we use is a functional analytic method developed by E. K. Ifantis in [6] and used recently by P. D. Siafarikas and the author in [9], [10] for more general forms of non-linear difference equations. Using this method, equations (1.1)-(1.5) are reduced equivalenlty to operator equations on an abstract Banach space H_1. For our approach we also need the following result of Earle and Hamilton [2]:

If $f : X \to X$ is holomorphic, i.e. its Fréchet derivative exists, and $f(X)$ lies strictly inside X, then f has a unique fixed point in X, where X is a bounded, connected and open subset of a Banach space E.

By saying that a subset X' of X lies strictly inside X we mean that there exists an $\epsilon_1 > 0$ such that $\|x' - y\| > \epsilon_1$ for all $x' \in X'$ and $y \in E - X$.

All our results except those concerning equation (1.5) for $|\mu| > 1$, follow from a general theorem (Theorem 2.1), which was proved in [10] and which we state for the sake of completeness in Section 2.

2. Preliminaries

In the following, H is used to denote an abstract separable Hilbert space with the orthonormal basis $e_n, n = 1, 2, 3, \ldots.$ We use the symbols $(\cdot,\cdot)$ and $\|\cdot\|$ to denote scalar product and norm in H respectively. By H_1 we mean the Banach space consisting of those elements f in H which satisfy the condition $\sum_{n=1}^{\infty}|(f,e_n)| < +\infty$. The norm in H_1 is denoted by $\|f\|_1 = \sum_{n=1}^{\infty}|(f,e_n)|$. By $f(n)$ we mean an element of the Banach space l_1 and by $f = \sum_{n=1}^{\infty} f(n)e_n$ we mean that element in H_1 generated by $f(n) \in l_1$. Finaly by V we mean the shift operator on H

$$V : Ve_n = e_{n+1}, n = 1, 2, \ldots$$

and by V^* its adjoint

$$V^* : V^*e_n = e_{n-1}, n = 2, 3, \ldots, V^*e_1 = 0.$$

It can easily be proved that the function

$$\phi : H_1 \to l_1$$

which is defined as follows:

$$\phi(f) = (f, e_n) = f(n)$$

is an isomorphism from H_1 onto l_1. We call f the abstract form of $f(n)$.

In general, if G is a mapping in l_1 and N is a mapping in H_1, we call $N(f)$ the abstract form of $G(f(n))$ if

$$G(f(n) = (N(f), e_n).$$

It follows easily that V^*f is the abstract form of $f(n+1)$.

We state now the basic theorem that we use.

Theorem 2.1. [10] *Consider the $m-th$ order non-homogeneous, nonlinear difference equation:*

$$\begin{aligned}
f(n+m)+&\sum_{p=1}^{m}(\alpha_p+\beta_p(n))f(n+m-p) = g(n)+\sum_{s=2}^{\infty}c_s(n)[f(n+q)]^s + \\
&+\sum_{i=1}^{N}\sum_{k=1}^{\infty}d_{ik}(n)[f(n+q_{i1})f(n+q_{i2})]^k + \\
&+\sum_{t=1}^{\Lambda}\sum_{k=1}^{\infty}b_{tk}(n)[f(n+q_{t3})f(n+q_{t4})f(n+q_{t5})]^k + \\
&+\sum_{j=1}^{M}\sum_{k=1}^{\infty}l_{jk}(n)[A_jf(n+q_{j6})+B_jf(n+q_{j7})]^k f(n+q_{j8})
\end{aligned} \tag{2.1}$$

where m, N, M, Λ *positive integers,* q, q_{i1}, q_{i2}, $i = 1, ..., N$, q_{t3}, t_{t4}, q_{t5}, $t = 1, ..., \Lambda$, q_{j6}, q_{j7}, q_{j8}, $j = 1, ..., M$ *non-negative integers and* α_p, $p = 1, ..., m$ *in general complex numbers. Assume that* $\lim_{n\to\infty} \beta_p(n) = 0$, $\forall p = 1, ..., m$, *the complex sequences* $c_s(n)$, $d_{ik}(n)$, $b_{tk}(n)$, *and* $l_{jk}(n)$, $s = 2, 3, ...$, $i = 1, ..., N$, $t = 1, ..., \Lambda$, $j = 1, ..., M$, $k = 1, 2, 3, ...$ *satisfy the conditions*

$$\sup_n |c_s(n)| \leq \gamma_s, \quad \sup_n |d_{ik}(n)| \leq \delta_{ik}, \quad \sup_n |b_{tk}(n)| \leq \beta_{tk}, \quad \sup_n |l_{jk}(n)| \leq \lambda_{jk}$$

and the functions

$$G_0(w) = \sum_{s=2}^{\infty} \gamma_s w^s, \quad G_i(w) = \sum_{k=1}^{\infty} \delta_{ik} w^{2k},$$
$$T_t(w) = \sum_{k=1}^{\infty} \beta_{tk} w^{3k}, \quad F_j(w) = \sum_{k=1}^{\infty} \lambda_{jk} (|A_j| + |B_j|)^k w^{k+1}$$

are entire functions, or they have a sufficiently large radius of convergence. Assume also that the roots of the algebraic equation

$$r^m + \alpha_1 r^{m-1} + ... + \alpha_m = 0$$

satisfy the conditions $|r_p| < 1$, $p = 1, 2, ..., m$. *Then there exist positive numbers* R_0 *and* P_0 *such that for*

$$\begin{aligned} |u| + \|g(n)\|_{l_1} =& |u_1| + |\alpha_1 u_1 + u_2| + ... + \\ &+ |\alpha_{m-1} u_1 + \alpha_{m-2} u_2 + ... + u_m| + \|g(n)\|_{l_1} < P_0, \end{aligned} \tag{2.2}$$

where

$$f(p) = u_p, \quad p = 1, ..., m \tag{2.3}$$

the equation (2.1) *together with the initial conditions* (2.3) *has a unique solution* $f(n)$ *in* l_1. *Moreover*

$$\sum_{n=1}^{\infty} |f(n)| < R_0. \tag{2.4}$$

Remark 1. The numbers R_0 and P_0 predicted by the above theorem are precisely determined due to the constructive character of Theorem 2.1. In particular R_0 is the point at which the function

(2.5)
$$P_1(R) = L^{-1} R \left[1 - LR \left(M_0(R) + \sum_{i=1}^{N} M_i(R) + R \sum_{t=1}^{\Lambda} \Delta_t(R) + \sum_{j=1}^{M} Q_j(R) \right) \right],$$

attains a maximum and $P_0 = P_1(R_0)$. In (2.5)

$$M_0(R) = \sum_{s=2}^{\infty} \gamma_s R^{s-2}, M_i(R) = \sum_{k=1}^{\infty} \delta_{ik} R^{2k-2}, \tag{2.6}$$

$$\Delta_t(R) = \sum_{k=1}^{\infty} \beta_{tk} R^{3k-3}, Q_j(R) = \sum_{k=1}^{\infty} \lambda_{jk}(|A_j| + |B_j|)^k R^{k-1}, \tag{2.7}$$

$1 \leq i \leq N, 1 \leq t \leq \Lambda, 1 \leq j \leq M$ are positive, continuous and increasing functions of R in an open interval suitably defined and L is the norm or a bound of the norm of the operator Γ^{-1}, where

$$\Gamma = (I - r_1 V)(I - r_2 V)...(I - r_m V) + V^m \sum_{p=1}^{m} B_p V^{*m-p}.$$

Remark 2. From (2.4) it follows that:

$$|f(n)| < R_0.$$

3. Applications

1) Consider the difference equation:

$$f(n+2) = \lambda f(n+1) + pf(n)e^{-\sigma f(n)}, n = 1, 2, ... \tag{3.1}$$

where $0 < \lambda < 1$, $\sigma > 0$, $0 < p < (1-\lambda)e^{\frac{2-\lambda}{1-\lambda}}$, $p \neq 1 - \lambda$. Equation (3.1) is the discrete version of a population model described by a differential equation [7].

The equilibrium points of (3.1) are:

$$\varrho_1 = 0, \quad \varrho_2 = \frac{1}{\sigma} \ln \frac{p}{1-\lambda} > 0.$$

For the equilibrium point $\varrho_1 = 0$ equation (3.1) can also be written as follows:

$$f(n+2) - \lambda f(n+1) - pf(n) = \sum_{s=2}^{\infty} \frac{(-1)^{s-1} p \sigma^{s-1}}{(s-1)!} [f(n)]^s. \tag{3.2}$$

Equation (3.2) results from equation (2.1) for:

$$m = 2, \quad \alpha_1 = -\lambda, \quad \alpha_2 = -p, \quad \beta_1(n) \equiv \beta_2(n) \equiv 0, \quad g(n) \equiv 0,$$

$$d_{ik}(n) \equiv b_{tk}(n) \equiv l_{jk}(n) \equiv 0, \quad c_s(n) = \frac{(-1)^{s-1} p \sigma^{s-1}}{(s-1)!}, \quad q = 0.$$

In this case $\gamma_s = \dfrac{p\sigma^{s-1}}{(s-1)!}$ and $G_0(s) = \displaystyle\sum_{s=2}^{\infty} \frac{p\sigma^{s-1}}{(s-1)!} w^s$ is an entire function. Also the roots of the algebraic equation $r^2 - \lambda r - p = 0$ are

$$r_1 = \frac{\lambda + \sqrt{\lambda^2 + 4p}}{2} \in (0,1), \quad r_2 = \frac{\lambda - \sqrt{\lambda^2 + 4p}}{2} \in (-1, 0),$$

for $0 < p < 1 - \lambda$. Then

$$\Gamma = (I - r_1 V)(I - r_2 V), \quad L = \frac{1}{1 + p - \sqrt{\lambda^2 + 4p}},$$
$$P_1(R) = \frac{R}{L} - R^2 \sum_{s=2}^{\infty} \frac{p\sigma^{s-1}}{(s-1)!} R^{s-2}.$$

It follows easily from Theorem 2.1 that for

$$|f(1)| + |f(2) - \lambda f(1)| < P_1(R_0), \tag{3.3}$$

equation (3.2) has a unique solution in l_1, where R_0 is the point at which $P_1(R)$ attains a maximum. Thus $\lim_{n\to\infty} f(n) = 0$ and $\varrho_1 = 0$ is a locally asymptotically stable equilibrium point of (3.2) with region of attraction given by (3.3). Also

$$|f(n)| < R_0.$$

For the equilibrium point $\varrho_2 = \frac{1}{\sigma} \ln \frac{p}{1-\lambda}$ we set

$$f(n) = F(n) + \varrho_2$$

and (3.2) becomes:

$$\begin{aligned} F(n+2) - \lambda F(n+1) + p(\varrho_2\sigma - 1)e^{-\sigma\varrho_2} F(n) = \\ = \sum_{s=2}^{\infty} \frac{(-1)^{s-1} p e^{-\sigma\varrho_2} \sigma^{s-1}(s-\sigma)}{s!} [F(n)]^s. \end{aligned} \tag{3.4}$$

Equation (3.4) results from equation (2.1) for:

$$m = 2, \quad \alpha_1 = -\lambda, \quad \alpha_2 = p(\varrho_2\sigma - 1)e^{-\sigma\varrho_2}, \quad \beta_1(n) \equiv \beta_2(n) \equiv 0, \quad g(n) \equiv 0,$$
$$d_{ik}(n) \equiv b_{tk}(n) \equiv l_{jk}(n) \equiv 0, \quad c_s(n) = \frac{(-1)^{s-1} p e^{-\sigma\varrho_2} \sigma^{s-1}(s-\sigma)}{s!}, \quad q = 0.$$

In this case

$$\gamma_s = \frac{p e^{-\sigma\varrho_2} \sigma^{s-1} |s-\sigma|}{s!} = \frac{(1-\lambda)\sigma^{s-1}|s-\sigma|}{s!}$$

and $G_0(s) = \sum_{s=2}^{\infty} \frac{(1-\lambda)\sigma^{s-1}|s-\sigma|}{s!} w^s$ is an entire function. Also the roots of the algebraic equation

$$r^2 - \lambda r + p(\varrho_2\sigma - 1)e^{-\sigma\varrho_2} = 0 \Leftrightarrow r^2 - \lambda r + (1-\lambda)(\ln \frac{p}{1-\lambda} - 1) = 0$$

are

i)

$$r_1 = \frac{\lambda + \sqrt{\lambda^2 + 4(1-\lambda)(1 - \ln\frac{p}{1-\lambda})}}{2} \in (0,1),$$
$$r_2 = \frac{\lambda - \sqrt{\lambda^2 + 4(1-\lambda)(1 - \ln\frac{p}{1-\lambda})}}{2} \in (-1,0)$$

for $1-\lambda < p < e(1-\lambda)$,
ii)

$$r_1 = \frac{\lambda + \sqrt{\lambda^2 + 4(1-\lambda)(1 - \ln\frac{p}{1-\lambda})}}{2} \in (0,1),$$
$$r_2 = \frac{\lambda - \sqrt{\lambda^2 + 4(1-\lambda)(1 - \ln\frac{p}{1-\lambda})}}{2} \in (0,1)$$

for $e(1-\lambda) \le p < (1-\lambda)e^{1+\frac{\lambda^2}{4(1-\lambda)}}$,
iii) $r_1 = r_2 = \dfrac{\lambda}{2} \in (0,1)$ for $p = (1-\lambda)e^{1+\frac{\lambda^2}{4(1-\lambda)}}$ and
iv)

$$r_{1,2} = \frac{\lambda \pm i\sqrt{-\lambda^2 - 4(1-\lambda)(1 - \ln\frac{p}{1-\lambda})}}{2} \text{ and}$$
$$|r_{1,2}| = \sqrt{(1-\lambda)(\ln\frac{p}{1-\lambda} - 1)} < 1$$

for $(1-\lambda)e^{1+\frac{\lambda^2}{4(1-\lambda)}} < p < (1-\lambda)e^{\frac{2-\lambda}{1-\lambda}}$. Then

$$\varGamma = (I - r_1 V)(I - r_2 V),$$

and the corresponding bounds of $\varGamma^{-1}$ are
i) $L = \dfrac{1}{(1-\lambda)(1-\ln\frac{p}{1-\lambda})}$, ii) $L = \dfrac{1}{(1-\lambda)(\ln\frac{p}{1-\lambda} - 1)}$,
iii) $L = \dfrac{4}{(2-\lambda)^2}$, iv) $L = \dfrac{1}{\left(1 - \sqrt{(1-\lambda)(\ln\frac{p}{1-\lambda} - 1}\right)^2}$, respectively.

Thus

$$P_1(R) = \frac{R}{L} - R^2 \sum_{s=2}^{\infty} \frac{p e^{-\sigma\varrho_2}\sigma^{s-1}|s-\sigma|}{s!} R^{s-2}.$$

It follows easily from Theorem 2.1 that for

$$|F(1)| + |F(2) - \lambda F(1)| < P_1(R_0), \tag{3.5}$$

equation (3.4) has a unique solution in l_1, where R_0 is the point at which $P_1(R)$ attains a maximum. Thus $\lim\limits_{n\to\infty} F(n) = 0$ and 0 is a locally asymptotically stable equilibrium point of (3.4) with region of attraction given by (3.5). Thus $\varrho_2 = \frac{1}{\sigma}\ln\frac{p}{1-\lambda}$ is a locally asymptotically stable equilibrium point of (3.1) with region of attraction given by:

$$|f(1) - \frac{1}{\sigma}\ln\frac{p}{1-\lambda}| + |f(2) - \lambda f(1) + \frac{\lambda-1}{\sigma}\ln\frac{p}{1-\lambda}| < P_1(R_0), \tag{3.6}$$

Also

$$|f(n)| \le |F(n)| + \varrho_2 \Leftrightarrow |f(n)| < R_0 + \frac{1}{\sigma}\ln\frac{p}{1-\lambda}$$

and equation (3.1) has a unique solution in $l_1 + \left\{\frac{1}{\sigma}\ln\frac{p}{1-\lambda}\right\}$.

Remark 3. Equation (3.1) is a particular case (for $\nu = 1$) of the equation:

$$f(n+\nu+1) = \lambda f(n+\nu) + pf(n)e^{-\sigma f(n)}, \tag{3.7}$$

which was studied, among other things, in [7]. It was shown there that any solution of (3.7) converges to its positive equilibrium point ϱ_2 as $n \to \infty$ if $p \in (1-\lambda, (1-\lambda)e]$. Notice that this is a subset of $(1-\lambda, (1-\lambda)e^{\frac{2-\lambda}{1-\lambda}}]$.

Remark 4. Relations (3.3) and (3.5) describe the region of attraction for the equilibrium points ϱ_1 and ϱ_2 respectively. Note that these inequalities do not give explicitly the regions of attraction, because we do not know the point R_0, but we can achieve that by truncating the power series, of which $P_1(R)$ is consisted.

Remark 5. If the initial conditions $f(1)$, $f(2)$ are positive numbers then every real solution of (3.1) is positive.

2) Consider the difference equation:

$$\begin{aligned} f(n+1) = &-\frac{b_1(n+1)}{\alpha_1(n+1)} + \frac{h_1(n+1)}{\alpha_1(n+1)}f(n+2)f(n+1)f(n) + \\ &+\frac{d_1(n+1)}{\alpha_1(n+1)}f(n+2)f(n), n = 1, 2, ... \end{aligned} \tag{3.8}$$

where $\frac{b_1(n+1)}{\alpha_1(n+1)} \in l_1$, $\sup_n|\frac{h_1(n+1)}{\alpha_1(n+1)}| \le \beta$ and $\sup_n|\frac{d_1(n+1)}{\alpha_1(n+1)}| \le \delta$.

Equation (3.2) appears often in various applications. In this case $\Delta_1(R) = \beta$, $M_1(R) = \delta$ are entire functions and $\Gamma = I$, $L = 1$. Thus

$$P_1(R) = R - \delta R^2 - \beta R^3.$$

It follows easily that $R_0 = \frac{\sqrt{\delta^2+3\beta}-\delta}{2}$ and $P_0 = \frac{(2\delta^2+6\beta)(\sqrt{\delta^2+3\beta}-\delta)}{27\beta^2} - \frac{\delta}{9\beta}$. By applying Theorem 2.1 to equation (3.8) we find that for

$$|f(1)| + \|\frac{b_1(n+1)}{\alpha_1(n+1)}\|_{l_1} < \frac{(2\delta^2+6\beta)(\sqrt{\delta^2+3\beta}-\delta)}{27\beta^2} - \frac{\delta}{9\beta},$$

equation (3.8) has a unique bounded solution in l_1 with bound:

$$|f(n)| < \frac{\sqrt{\delta^2 + 3\beta} - \delta}{2}.$$

In the special case where $d_1(n+1) \equiv 1$ and $h_1(n+1) \equiv 0$, equation (3.8) becomes:

$$f(n+1) = -\frac{b_1(n+1)}{\alpha_1(n+1)} + \frac{1}{\alpha_1(n+1)} f(n+2)f(n), \tag{3.9}$$

which is the well-known non-autonomous Lyness equation. As before, we find that $\Gamma = I$, $L = 1$ and $P_1(R) = R - \delta R^2$. Thus $R_0 = \dfrac{1}{2\delta}$ and $P_0 = \dfrac{1}{4\delta}$. By applying Theorem 2.1 to equation (3.3) we find that for

$$|f(1)| + \|\frac{b_1(n+1)}{\alpha_1(n+1)}\|_{l_1} < \frac{1}{4\delta},$$

equation (3.9) has a unique bounded solution in l_1 with bound:

$$|f(n)| < R_0 = \frac{1}{2\delta}.$$

Remark 6. In the case when equation (3.8) has positive solutions and $\alpha_1(n+1)$, $b_1(n+1)$, $h_1(n+1)$, $d_1(n+1)$ are constants, equation (3.8) was studied in [4]. Invariants for equation (3.8) have been found in [3], in the case when $\alpha_1(n+1)$, $b_1(n+1)$, $h_1(n+1)$, $d_1(n+1)$, are periodic sequences of positive numbers and the initial conditions are positive numbers. The non-autonomous Lyness equation (3.9) was studied, among other things, in [5]. In particular it was shown there that under some different, than those we used, but more complicated conditions on the sequences $\alpha_1(n+1)$ and $b_1(n+1)$, every positive solution of (3.9) is bounded.

3) Consider the difference equation:

$$\begin{aligned} f(n+2) = &\frac{\alpha_2(n+1)}{h_2(n+1)} + \frac{b_2(n+1)}{h_2(n+1)}[f(n+1)]^2 - \\ &-\frac{1}{h_2(n+1)} f(n+2)[f(n)]^2, n = 1, 2, ... \end{aligned} \tag{3.10}$$

where $\dfrac{\alpha_2(n+1)}{h_2(n+1)} \in l_1$, $\sup\limits_n |\dfrac{b_2(n+1)}{h_2(n+1)}| \leq \gamma$ and $\sup\limits_n |\dfrac{1}{h_2(n+1)}| \leq \lambda$.

In this case $M_0(R) = \gamma$, $Q_1(R) = \lambda R$ are entire functions and $\Gamma = I^2 = I$, $L = 1$. Thus

$$P_1(R) = R - \gamma R^2 - \lambda R^3.$$

It follows easily that $R_0 = \dfrac{\sqrt{\gamma^2 + 3\lambda} - \gamma}{2}$ and $P_0 = \dfrac{(2\gamma^2 + 6\lambda)(\sqrt{\gamma^2 + 3\lambda} - \gamma)}{27\lambda^2} - \dfrac{\gamma}{9\lambda}$. By applying Theorem 2.1 to equation (3.10) we find that for

$$|f(1)| + |f(2)| + \|\frac{\alpha_2(n+1)}{h_2(n+1)}\|_{l_1} < \frac{(2\gamma^2 + 6\lambda)(\sqrt{\gamma^2 + 3\lambda} - \gamma)}{27\lambda^2} - \frac{\gamma}{9\lambda},$$

equation (3.10) has a unique bounded solution in l_1 with bound:

$$|f(n)| < \frac{\sqrt{\gamma^2 + 3\lambda} - \gamma}{2}.$$

Remark 7. Equation (3.10) has been studied in [8] for $\alpha_2(n+1)$, $b_2(n+1)$ and $h_2(n+1)$ constants.

4) Consider the difference equation:

$$f(n+1) = h_3(n) + [f(n)]^2, n = 1, 2, \dots \tag{3.11}$$

where $h_3(n) \in l_1$.

In this case $M_0(R) = 1$ is an entire function and $\Gamma = I$, $L = 1$. Thus

$$P_1(R) = R - R^2.$$

It follows easily that $R_0 = \dfrac{1}{2}$ and $P_0 = \dfrac{1}{4}$. By applying Theorem 2.1 to equation (3.11) we find that for

$$|f(1)| + \|h_3(n)\|_{l_1} < \frac{1}{4}, \tag{3.12}$$

equation (3.11) has a unique bounded solution in l_1 with bound:

$$|f(n)| < \frac{1}{2}.$$

Also notice that (3.11) can also be written as:

$$\frac{f(n+1)}{f(n)} = \frac{h_3(n)}{f(n)} + f(n).$$

Thus if $K = \lim\limits_{n\to\infty} \dfrac{h_3(n)}{f(n)}$ exists then $\lim\limits_{n\to\infty} \dfrac{f(n+1)}{f(n)} = K$ and the generating analytic function $f(z) = \sum\limits_{n=1}^{\infty} f(n) z^{n-1}$ converges absolutely for $|z| < \dfrac{1}{K}$.

Remark 8. In the case where $h_3(n) \equiv h \notin l_1$, equation (3.11) becomes the well-known equation from which the Mandlebrot and the Julia sets are deduced. More particularly, the set of all points h for which the solution $f(n)$ of (3.11) with $f(1) = 0$ stays bounded as $n \to \infty$ is called the Mandlebrot set (M) and for a given parameter $h =$ constant, the set of initial values f(0) for which $f(n)$ stays bounded is the so-called filled-in Julia set (J_c). (The Julia set proper consists of the boundary points of J_c.)

Thus for $f(1) = 0$ we obtain from (3.12):

$$\|h_3(n)\|_{l_1} < \frac{1}{4},$$

which can be considered as a generalized Mandelbrot set.

Also for $h_3(n)$ a given sequence of l_1, relation (3.12) can be considered as a generalized Julia set.

Notice that when $h_3(n) \equiv h =$ constant, our method can not be applied, because h does not belong in l_1.

5) Consider the difference equation:

$$f(n+1) = h_4(n) + \mu f(n)\left[1 - \frac{1}{K} f(n)\right], n = 1, 2, ... \tag{3.13}$$

where $\mu \in \mathbb{R} \setminus \{1\}$, $K > 0$ and $h_4(n) \in l_1$.

Equation (3.13) describes the development of a single species population $f(n)$, where μ is the parameter related to the growth or death rate, $K > 0$ is the carrying capacity and $h_4(n)$ represents the harvest/stock [12].

We shall distinguish the following two cases:

1) First case: $|\mu| < 1$.

Here $M_0(R) = \frac{|\mu|}{K}$ is an entire function and $\Gamma = I - \mu V$, $L = \frac{1}{1-|\mu|}$. Thus

$$P_1(R) = (1 - |\mu|)R - \frac{|\mu|}{K} R^2.$$

It follows easily that $R_0 = \dfrac{(1-|\mu|)K}{2|\mu|}$ and $P_0 = \dfrac{(1-|\mu|)^2 K}{4|\mu|}$. By applying Theorem 2.1 to equation (3.13) we find that for

$$|f(1)| + \|h_4(n)\|_{l_1} < \frac{(1-|\mu|)^2 K}{4|\mu|}, \quad |\mu| < 1$$

equation (3.13) has a unique bounded solution in l_1 with bound:

$$|f(n)| < \frac{(1-|\mu|)K}{2|\mu|}, \quad |\mu| < 1.$$

2) Second case: $|\mu| > 1$.

In this case, Theorem 2.1 can not be applied to equation (3.13) because the unique solution of the algebraic equation

$$r - \mu = 0$$

is $r = \mu$ and $|\mu| > 1$.

Notice that equation (3.13) can also be written as:

$$f(n) - \frac{1}{\mu} f(n+1) = -\frac{1}{\mu} h_4(n) + \frac{1}{K}[f(n)]^2, n = 1, 2, ... \tag{3.14}$$

According to the representation presented in Section 2, the abstract form of (3.14) in H_1 is:

$$\left(I - \frac{1}{\mu}V^*\right) f = N(f) - \frac{1}{\mu}h_4, \tag{3.15}$$

where h_4 is the abstract form of $h_4(n)$ and $N(f) = \frac{1}{K}(f, e_n)(f, e_n)e_n$, is a Fréchet differentiable operator defined on all H_1 with $\|N(f)\|_1 \leq \|f\|_1^2$ ([9] or [10]).

Since $|\mu| > 1$, the operator $\left(I - \frac{1}{\mu}V^*\right)^{-1}$ is uniquely determined on H_1 and bounded, with bound:

$$\| \left(I - \frac{1}{\mu}V^*\right) \|_1 < \frac{|\mu|}{|\mu| - 1}.$$

Thus (3.15) becomes

$$f = \left(I - \frac{1}{\mu}V^*\right)^{-1} \left[N(f) - \frac{1}{\mu}h_4\right]. \tag{3.16}$$

Following a technique similar to the one used in [6], [9], [10] we define the function:

$$\phi(f) = \left(I - \frac{1}{\mu}V^*\right)^{-1} \left[N(f) - \frac{1}{\mu}h_4\right].$$

Let $\|f\|_1 \leq R < \bar{R} < +\infty$, where $\bar{R}$ is as large as we want, but finite. Then from (3.16) we obtain:

$$\|\phi(f)\|_1 \leq \frac{|\mu|}{|\mu| - 1}\left[\frac{R^2}{K} + \frac{1}{|\mu|}\|h_4\|_1\right]. \tag{3.17}$$

Since $\bar{R}$ is sufficienlty large, there exists an $\bar{R}_1 \in [0, \bar{R}]$ such that

$$\frac{|\mu|}{|\mu| - 1}\frac{\bar{R}_1}{K} > 1.$$

Thus the value $\bar{R}_2 = \dfrac{(|\mu| - 1)K}{|\mu|}$ is a zero of the function

$$P(R) = 1 - \frac{|\mu|}{|\mu| - 1}\frac{\bar{R}_1}{K}.$$

So the continuous function

$$P_1(R) = \frac{|\mu| - 1}{|\mu|}RP(R)$$

satisfies $P_1(0) = P_1(\bar{R}_2) = 0$ and therefore attains a maximum at the point

$$R_0 = \frac{(|\mu| - 1)K}{2|\mu|} \in (0, \bar{R}_2).$$

Now for every $\epsilon > 0$, $R = R_0$ and

$$\|h_4\|_1 \leq \frac{(|\mu|-1)^2 K}{4|\mu|} - (|\mu|-1)\epsilon$$

we find from (3.17)

$$\|\phi(f)\|_1 \leq \frac{(|\mu|-1)K}{2|\mu|} - \epsilon = R_0 - \epsilon < R_0$$

for $\|f\|_1 < R_0$. This means that for

$$\|h_4\|_1 < \frac{(|\mu|-1)^2 K}{4|\mu|}$$

ϕ is a holomorphic map from $B\left(0, \frac{(|\mu|-1)K}{2|\mu|}\right)$ strictly inside $B\left(0, \frac{(|\mu|-1)K}{2|\mu|}\right)$. Thus applying the fixed point theorem of Earle and Hamilton [2] we find that equation $\phi(f) = f$ has a unique fixed point in H_1. This means equivalently that for

$$\|h_4(n)\|_{l_1} < \frac{(|\mu|-1)^2 K}{4|\mu|}, \quad |\mu| > 1$$

equation (3.14) has a unique bounded solution in l_1 with bound:

$$|f(n)| < \frac{(|\mu|-1)K}{2|\mu|}, \quad |\mu| > 1.$$

Remark 9. In [12] the real periodic solutions of (3.14) have been investigated for $\mu \in (1,2)$ and $h_4(n) : \mathbb{N} \to \mathbb{R}$ an ω periodic number sequence with $\omega \geq 1$ which satisfies the relation:

$$\|h_4\| < \frac{(|\mu|-1)^2 K}{4|\mu|}, \quad \mu \in (1,2)$$

where $\|h_4\| = \max_n |h_4(n)|$. Moreover it was found in [12] that the predicted periodic solution satisfies:

$$|f(n)| < \left(1 - \frac{1}{\mu}\right) K r_0, \quad r_0 \in (0, 1/2), \quad \mu \in (1,2).$$

Remark 10. Our results, concerning all five applications hold also, if we consider the Banach space $l_1|_{\mathbb{R}}$ instead of l_1.

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STABILITY ZONES FOR DISCRETE TIME HAMILTONIAN SYSTEMS

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Abstract. The discrete version of the Hamiltonian system

$$\dot{x} = \lambda J H(t) x$$

with $H(t) = H^*(t) = H(t+T)$ is considered. Following the line of M.G. Krein the stability zones with respect to the parameter λ are considered: the side zones have to be estimated from multiplier traffic rules while the central stability zone from the discrete version of the skew - periodic boundary value problem.

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1. Introduction and motivation

The object of this paper is the stability analysis of the discrete version of the linear periodic Hamiltonian system:

$$\dot{x} = \lambda J H(t) x \tag{1}$$

where $H(t) = H^*(t) = H(t+T), T > 0$; $H(t)$ has complex entries and is Hermitian. Also J is defined by

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \tag{2}$$

and λ is, generally speaking, a complex parameter. System (1) is a generalization encompassing a lot of by now classical systems that go back to Sturm, Liapunov and Žukovskii. M.G. Krein [1] give a strong generalization of many classical results and opening new fields of research issued from the interaction of several apparently independent mathematical domains. The long line of research opened by Krein is summarized in the monograph of Yakubovich and Staržinskii [2]. As pointed out by Krein and Yakubovich [3] various problems in contemporary engineering and physics (e.g. dynamic stability of structures, parametric resonance in high-capacity electrical transmission lines, motion of particles in accelerators) lead to the investigation of Hamiltonian systems with periodic coefficients.

Another field of origin for periodic Hamiltonian systems is calculus of variations and optimal control. Here a long list of papers may be mentioned but we mention here only the papers of Yakubovich [4] where linear periodic Hamiltonians are considered in the context of linear optimal feedback (minimizing a quadratic integral performance index) and quadratic Liapunov functions.

A crucial difference between these two directions of research exists. The first one, developed mainly by Krein is concerned with stable Hamiltonian systems whose multipliers are located on the unit circle. On the contrary linear quadratic control requires a dichotomic i.e. totally unstable Hamiltonian system whose multipliers *are not* on the unit circle. This last property is robust (i.e. it is preserved against structural perturbations) while the first one is not robust (generally speaking). The search for robustness of stable Hamiltonian systems led Krein to the introduction of strong stability, to the discovery of "traffic rules" on the unit circle for the multipliers, and to new results about the λ zones of stability. Since the central zone is estimated using the eigenvalues of a certain self adjoint boundary value problem, the research on stability met the old Sturm-Liouville framework which also generates problems for Hamiltonian systems. A good reference on these problems together with variational calculus and optimal control is the book of Kratz [5].

In the last few years a new field of research emerged - discrete time Hamiltonian systems. A basic reference is the book of Ahlbrandt and Peterson [6]. We shall mention here some papers [7], [8], [9], from the long list belonging to Bohner and Došlý. Their topics are oscillation, disconjugacy and transformation of Hamiltonian systems, both continuous and discrete time. The study of discrete-time Hamiltonian systems in connection with linear - quadratic optimal control may be found in the paper of Halanay and Ionescu [10]. Applications of dichotomic periodic linear Hamiltonian systems (i.e. totally unstable), both continuous and discrete-time to forced nonlinear oscillations are to be found in [11].

This paper is concerned with strong stability (in the sense of Krein) of discrete-time Hamiltonian systems. Such systems may arise from sampling (1). *Since stability is, generally speaking, not preserved by sampling* (not always) this problem is of interest. On the other hand, not all results of the continuous time fields may migrate, *mutatis-mutandis*, to the discrete-time field even in the conditions of the new emerging theory on time scales [12], [13],[14]; this will become clear throughout the paper. Let us consider system (1) with $H(t)$ as follows

$$H(t) = \begin{pmatrix} A(t) & B^*(t) \\ B(t) & D(t) \end{pmatrix} \tag{3}$$

with $A(.)$ and $B(.)$ Hermitian matrices. We perform the usual Euler discretization of the derivatives with the step $h = T/N$ but using forward difference in the first equation and the backward difference in the second one; it is necessary to observe this rule if we want to obtain a discrete-time Hamiltonian. We deduce

$$\begin{aligned} \tfrac{y((k+1)h)-y(kh)}{h} &= \lambda B(kh)y(kh) + \lambda D(kh)z(kh) \\ \tfrac{z(kh)-z((k-1)h)}{h} &= -\lambda A(kh)y(kh) - \lambda B^*(kh)z(kh) \end{aligned} \tag{4}$$

where y,z are the m-dimensional sub-vectors of the $2m$ vector x. Denoting $y(kh) = y_k$, $z(kh) = z_{k+1}$, $A(kh) = A_k$, $B(kh) = B_k$, $D(kh) = D_k$ and, with an abuse of notations, λh by λ we obtain the discrete-time linear periodic Hamiltonian system:

$$\begin{aligned} y_{k+1} - y_k &= \lambda B_k y_k + \lambda D_k z_{k+1} \\ z_{k+1} - z_k &= -\lambda A_k - \lambda B_k^* z_{k+1} \end{aligned} \tag{5}$$

with A_k, B_k, D_k being N-periodic. Remark that this system may be also written as:

$$\begin{pmatrix} y_{k+1} - y_k \\ z_{k+1} - z_k \end{pmatrix} = \lambda J H_k \begin{pmatrix} y_k \\ z_{k+1} \end{pmatrix} \tag{6}$$

with $H_k = \begin{pmatrix} A_k & B_k^* \\ B_k & D_k \end{pmatrix}$ and J as previously. Also system (5) may be given the Cauchy form

$$x_{k+1} = C_k(\lambda) x_k \tag{7}$$

with

$$C_k(\lambda) = \begin{pmatrix} I & -\lambda D_k \\ 0 & I + \lambda B_k^* \end{pmatrix}^{-1} \begin{pmatrix} I + \lambda B_k & 0 \\ -\lambda A_k & I \end{pmatrix} \tag{8}$$

and this is true for any $\lambda \in \mathbb{C}$ except a finite member of eigenvalues of B_k^*. If the eigenvalues of B_k are also excluded, then $C_k(\lambda)$ is invertible and the solution of (5) may be constructed for all integers $k \in \mathbb{Z}$ (i.e. forward and backward); only in this case stability and strong stability have sense.

Definition 1. A point λ_0 is called a λ-point of stability of system (5) if for $\lambda = \lambda_0$ all solutions of the system are bounded on $\mathbb{Z}$. If, moreover, for $\lambda = \lambda_0$, all solutions of any system of (6) type having H_k replaced by $\hat{H}_k$ (N-periodic and Hermitian) sufficiently close to H_k (in some well-defined sense) are also bounded on $\mathbb{Z}$, then we call $\lambda = \lambda_0$ a λ-point of strong stability of (6).

It will be shown in the paper that, as in the continuous time case [1] the set of λ-points of strong stability of (6) is an open set and thus if it is nonempty it decomposes into a finite or infinite system of disjoint intervals that are called *λ-zones of stability*.

In the following we shall deal with the theory of the λ-zones of stability for system (6) following the line of [1], relating the existence and estimation of these zones to the multiplier problem (as in the pioneering papers of Liapunov).

2. The monodromy matrix and the multipliers

We may compute $C_k(\lambda)$ from (8) and find that

$$C_k^*(\lambda)JC_k(\lambda) - J = (\overline{\lambda} - \lambda)Q_k(\lambda) \tag{9}$$

where $Q_k(\lambda)$ is Hermitian. We deduce that $C_k(\lambda)$ is J-unitary for real λ and Hermitian H_k and J - orthogonal (symplectic) if H_k is symmetric. We may also write

$$x_k(\lambda) = C_{k-1}(\lambda)...C_0(\lambda)x_0 = U_k(\lambda)x_0 \tag{10}$$

thus defining the transition matrix(fundamental matrix of solutions) which results J-unitary or symplectic accordingly. It follows that the monodromy matrix $U_N(\lambda)$ will be also J-unitary or symplectic. In the terminology of [2] systems with complex coefficients and J-unitary matrix $C_k(\lambda)$ are called *Hamiltonian* while systems with real coefficients and symplectic matrix $C_k(\lambda)$ are called canonical.

The eigenvalues of the monodromy matrix i.e. the roots $\rho_i(\lambda)$ of the characteristic equation

$$det(U_N(\lambda) - \rho I) = 0 \tag{11}$$

are called multipliers of (5) (or (6)). The following result of Poincaré-Liapunov type may be proved following, e.g.,[2].

Theorem 1. *a) If H_k is Hermitian the spectrum of $U_N(\lambda)$ is located symmetrically with respect to the unit circle i.e. the multipliers occur in pairs $(\rho, \overline{\rho}^{-1})$ including their multiplicities as roots of (11).*

b) If H_k is symmetric the spectrum of $U_N(\lambda)$ is located skew-symmetrically with respect to the unit circle, i.e., the multipliers occur in pairs (ρ, ρ^{-1}).

c) If H_k and λ are real and H_k is symmetric the multipliers occur in groups of four, being symmetric with respect to both unit circle and imaginary axis.

From here we may deduce:

Proposition 1. *All solutions of (5) are bounded on $\mathbb{Z}$ iff all multipliers of the system are of modulus one (located on the unit circle) and are of simple type (its root space coincides with its eigenspace) or, equivalently, have simple elementary divisors.*

Since we are concerned with robust(strong) stability, it is useful to analyze parameter dependence (on λ) of the multipliers. *Unlike in the continuous-time case* $U_N(\lambda)$ is not of entire but of meromorphic type being rational with respect to λ. For λ sufficiently close to the origin we may consider the McLaurin expansion of $C_k(\lambda)$

$$C_k(\lambda) = I_{2m} + \lambda J H_k + o(\lambda)$$

and of $U_N(\lambda)$

$$U_N(\lambda) = I_{2m} + \lambda J \sum_0^{N-1} H_k + o(\lambda)$$

It follows that in a sufficiently small neighborhood of $\lambda = 0$ the holomorphic matrix-valued logarithm is well defined

$$\varGamma(\lambda) = ln\, U_N(\lambda) = \varGamma_0 + \varGamma_1 \lambda + o(\lambda)$$

such that $U_N(\lambda) = e^{\varGamma(\lambda)}$. We deduce that $\varGamma_0 = 0, \varGamma_1 = J \sum_0^{N-1} H_k$. With an appropriate indexing we shall have $\rho_j(\lambda) = exp(\gamma_j(\lambda)), j = \overline{1,n}$, with $\rho_j(\lambda)$ being system's multipliers and $\gamma_j(\lambda)$ the eigenvalues of $\varGamma(\lambda)$. Following the line of [1] and [15] we may prove.

Theorem 2. *Assume that $\sum_0^{N-1} H_k > 0$ and has distinct eigenvalues. Then there exists an interval $(-l, l)$ such that for $\lambda \in (-l, l)$ all solutions of* (6) *are bounded on Z.*

Remark that this is the first result asserting existence of a central λ-zone of stability for (6). In the following we shall extend the result to the case of non-distinct eigenvalues and obtain estimates for l.

3. Self-adjoint boundary value problems for the canonical system

In this section we shall consider the boundary value problem for (6) defined by the boundary condition

$$x_N - G x_0 = 0 \tag{12}$$

with G some J-unitary matrix ($G^* J G = J$). Following [1] and [15] it can be proved.

Theorem 3. *Let $H_k \geq 0, k = \overline{0, N-1}, \sum_0^{N-1} H_k > 0$. Then the eigenvalues (characteristic numbers) of the boundary value problem defined by* (6) *and* (12) *are real.*

We point out also the following facts

1. Any root of the equation

$$det(U_N(\lambda) - G) = 0 \tag{13}$$

   is a characteristic number of the boundary value problem and is real. Therefore all roots of (13), if any, are real.
2. The number $\lambda = 0$ is a characteristic number iff $det(I - G) = 0$ (iff G has 1 as eigenvalue).

We may also prove

Theorem 4. *The multiplicity k_j of any characteristic number of (6), (12) coincides with the number of linearly independent associated solutions of the problem.*

The proof of this theorem follows the line of Theorem 3.4 in [1] and Theorem 3.3 in [15] but in this case $U_N(\lambda) - G$ is, generally speaking, rational and we need the Smith-McMillan form of a rational matrix in order to obtain the result.

In order to obtain strong (i.e. robust) stability using the properties of the boundary value problem we shall need a result concerning the dependence of the characteristic numbers λ_j on the matrix H_k, dependence that is symbolized by $\lambda_j(H)$.

Theorem 5. *Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the positive characteristic numbers of the boundary value problem and let $0 > \lambda_{-1} > \lambda_{-2} > \ldots$ be the negative ones (it is assumed that each characteristic number occurs in the corresponding sequence the number of times equal to its multiplicity as a root of (13)). Let H_k^1, H_k^2 be such that $H_k^i \geq 0, \sum_0^{N-1} H_k^i > 0, i = 1, 2$ and assume that $H_k^1 \leq H_k^2, k = \overline{0, N-1}$. Then $\lambda_j(H^1) \geq \lambda_j(H^2), \lambda_{-j}(H^1) \leq \lambda_{-j}(H^2)$.*

The proof follows the line of [1] and [15].

4. Multipliers of 1st and 2nd kind: analytic properties and the strong (robust) stability

We shall return to Proposition 1 which states that (6) is stable provided all its multipliers are located on the unit circle and are of simple type. Generally speaking, a *J-unitary matrix with the eigenvalues on the unit circle and of simple type is called of stable type*. The matrices of stable type have an interesting property: *all J-unitary matrices that belong to a δ-neighborhood of a matrix of stable type are also of stable type* ([1],Theorem 1.2). This property suggests the approach to be token in the analysis of strong stability for Hamiltonian systems.

Definition 2. A Hamiltonian system is said to be strongly stable if it is stable and all Hamiltonian systems belonging to a neighborhood of it are also stable.

In fact we may follow [1] and [15] and use some arguments of [16] to show that if the Hamiltonian system

$$x_{k+1} - x_k = JH_k \begin{pmatrix} y_k \\ z_{k+1} \end{pmatrix}$$

is stable (of stable type) then there exists some $\delta > 0$ such that all Hamiltonian systems with H_k replaced by $\tilde{H}_k$ with $\sum_0^{N-1} |H_k - \tilde{H}_k| < \delta$ are also of stable type.

This robustness result has the following consequences:

A. If we consider (6) we may obtain neighboring Hamiltonian systems by modifying the parameter λ; but λ has to take real values in order that monodromy matrices be J-unitary.
B. Since stability is expressed through the properties of the multipliers and strong stability means preservation of this property with respect to parameter λ variations (among other perturbations that preserve the Hamiltonian character of the system) it would be of interest to discuss multiplier properties with respect to λ.

The first remark hints to the λ-zones of stability for real λ. The other one indicates that multiplier dependence on λ may help in strong stability studies even for complex λ. Indeed, for complex λ we may state and prove

Theorem 6. *Consider (6) with complex λ i.e. with $Im\ \lambda \neq 0$. Then half of system's multipliers have moduli less than 1 and the other half have them larger than 1 provided $H_k \geq 0, \sum_0^{N-1} H_k > 0$.*

The proof relies on the fact that $U_N(\lambda)$ is nonsingular and also either J-increasing (for $Im\ \lambda > 0$) or J-decreasing (for $Im\ \lambda < 0$); then Theorem 1.1 of [1] is used.

Definition 3. a) Let ρ_0 with $|\rho_0| = 1$ be a simple eigenvalue of a J-unitary matrix and e_0 the associated eigenvector. If e_0 is a plus-vector (with $i(Je_0, e_0) > 0$) the eigenvalue is called of 1st kind and if e_0 is a minus-vector (with $i(Je_0, e_0) < 0$) the eigenvalue is called of 2nd kind.

b) Let ρ_0 with $|\rho_0| = 1$ be a non-simple eigenvalue of a J-unitary matrix and let $\mathcal{L}_{\rho_0}$ be the corresponding proper subspace. If $\mathcal{L}_{\rho_0}$ contains plus-vectors only, then ρ_0 is of 1st kind and if $\mathcal{L}_{\rho_0}$ contains minus-vectors only, then ρ_0 is of 2nd kind. If $\mathcal{L}_{\rho_0}$ contains at least a null-vector (with $i(Je_0, e_0) = 0$) then ρ_0 is of mixed (indefinite type).

c) Let ρ_0 with $|\rho_0| \neq 1$ be a non simple eigenvalue of a J-unitary matrix: if $|\rho_0| > 1$ it is called of 1st kind and if $|\rho_0| < 1$ it is called of 2nd kind.

The main feature of this classification is the fact that it relies on the sign of the associated eigenvectors. This allows the extension of the notions to matrices that *are not* J-unitary. Indeed we already known [1], [15] that $U_N(\lambda)$ - the monodromy matrix - whose eigenvalues, the multipliers, are of interest - is J-increasing for

$Im\ \lambda > 0$ and J-decreasing for $Im\ \lambda < 0$. It is also known [1], that for J-increasing matrices an eigenvalue with modulus larger than 1 has its eigenvectors plus vectors thus being of 1st kind; accordingly, the eigenvalues with modulus lower than 1 (located inside the unit disk) are of 2nd kind. For J-decreasing matrices, the eigenvalues inside the unit disk are of 1st kind etc.

The dependence of multipliers' properties on λ may be followed using arguments from analytic function theory as in [1] and especially in [2]. The multipliers equation:

$$\Delta(\rho;\lambda) \equiv det(U_N(\lambda) - \rho I) = 0$$

takes the form

$$\rho^{2m} + A_{2m-1}(\lambda)\rho^{2m-1} + \cdots + A_1(\lambda)\rho + A_0(\lambda) = 0$$

where $A_k(\lambda)$ are rational functions and $A_0(\lambda) = det\ U_N(\lambda)$. From a basic representation lemma of Weierstrass it follows that in a neighborhood of $\lambda_0 \in R$ the multipliers $\rho_j(\lambda)$ that coincide for $\lambda \to \lambda_0$ with a multiplier ρ_0 of definite kind (1st or 2nd but not mixed) are analytic functions of λ i.e. the expansion of $\rho_j(\lambda)$ contains only integer powers of $(\lambda - \lambda_0)$. Further, we may follow [2] and obtain more specific information on the expansions of $\rho_j(\lambda)$, $\rho_j(\lambda)$ being considered branches of some analytic function coinciding in ρ_0 for $\lambda \to \lambda_0$.

From this information on expansion's coefficients we may deduce the so-called *Krein traffic rules* for the multipliers on the unit circle. We shall give below an account on these traffic rules that remain unchanged in the discrete-time case.

1. Let $\lambda_0 \in R$ and ρ_0 be a multiplier i.e an eigenvalue of $U_N(\lambda_0)$ with $|\rho_0| = 1$ and of multiplicity r. Consider a sufficiently small disk $\gamma : \{\rho : |\rho - \rho_0| < \epsilon\}$ such that there are no other eigenvalues of $U_N(\lambda_0)$ inside it. There will then exist some $\delta(\epsilon) > 0$ such that for all λ satisfying $|\lambda - \lambda_0| < \delta$ there will exist exactly r multipliers (eigenvalues of $U_N(\lambda)$ with their multiplicities) which are located inside the disk γ considered above. If $\lambda = \lambda_0 + ih, 0 < h < \delta$, $U_N(\lambda)$ is J-increasing and, therefore, the multipliers which are in γ and inside the unit disk are of 2nd kind while those which are in γ and outside the unit disk are of 1st kind. It was shown [1], [2] that this distribution of multipliers does not change as long as λ does not cross the real axis of the (λ) plane.
   Consequently we may say that in ρ_0 coincide for $\lambda = \lambda_0$ e.g. r_1 of 1st kind and $r - r_1$ of 2nd kind. The multiplier ρ_0 is thus of mixed type.
2. Consider a multiplier of definite type on the unit circle e.g. a multiplier of 1st kind (with its eigenvectors - plus-vectors) with multiplicity r, corresponding to λ_0. In its neighborhood one may find only multipliers of 1st kind. Let us assume that λ takes real values on the interval $(\lambda_0-\delta, \lambda_0+\delta)$. For $\lambda \neq \lambda_0$ the multipliers that coincided in ρ_0 *split off* in r multipliers describing r branches of the corresponding analytic function. Nevertheless the resulting multipliers remain on the unit circle and *move clockwise for increasing* λ. Were this not true, if a multiplier of 1st kind occurs (outside the unit disk) it will be accompanied by the occurrence of a multiplier of 2nd kind due to multipliers' symmetry; but in this case ρ_0 would not be of definite kind.

Obviously *for multipliers of 2nd kind the motion for increasing* λ *is counter-clockwise* when the multiplier splits off.

3. The multipliers of mixed type from the unit circle split off in multipliers of different kinds and they may, for some real λ to leave the circle in a symmetrical way: one outside and one inside.

We may now represent the multiplier traffic on the unit circle. The multipliers of definite kinds split and move clockwise and counter-clockwise, they met and separate, but do not leave the circle as $\lambda \in R$ increases or decreases. When two multipliers of different kind met they generate a multiplier of mixed kind which will split in multipliers of different kind again leaving the circle symmetrically (an equal number entering the unit disk and leaving it) thus generating instability.

5. The stability zones of the Hamiltonian system with parameter

In this section we shall consider that the neighboring Hamiltonians of the strong stability problem are generated by modifying the parameter λ.

Theorem 7. *The strong stability points of* (6) *form an open set which is not empty when* (6) *is of positive type, i.e., when* $H_k \geq 0, \sum_0^{N-1} H_k > 0$.

The proof goes as in [1] and [15] with $\lambda_0 H_k$ and λH_k as $\tilde{H}_k$: if $|\lambda - \lambda_0| < \delta$ then we are in the basic case of neighboring Hamiltonians.

If $\lambda_0 \in R$ is a point of strong stability, the set of strong stability points is open: we start with the interval $(\lambda_0 - \delta, \lambda_0 + \delta)$ and afterwards we consider neighborhoods of the points of this interval ("continuations"). The open intervals thus obtained are the λ-stability zones.

Non emptiness is connected with the central stability zone (around $\lambda = 0$) which is nonempty at least in the case of Theorem 2. The central stability zone will be again considered in the next section. Now we shall focus on side zones in the positive type case, when $H_k \geq 0, \sum_0^{N-1} H_k > 0$.

The main tool of the analysis is an inequality that follows from the analytic properties of the multipliers:

$$-\frac{d}{d\lambda} arg\ \rho_j(\lambda)|_{\lambda=\lambda_0} \geq \sum_{k=0}^{N-1} \sigma_k^{min}(\lambda_0) \tag{14}$$

where $\rho_j(\lambda)$ is any branch of the analytic functions defined by multiplier dependence on λ [1] and σ_k^{min} is the lowest eigenvalue of a nonnegative matrix. It has been shown by a simple example that, *unlike in the continuous-time case*, a strictly positive lower bound that is independent of λ_0 does not exist. Therefore it is not possible to obtain, even in the simplest case, an estimate of the width of any side zone that is independent of its position with respect to the central zone [15].

We may however choose some interval $(-\Lambda_0, \Lambda_0)$ and compute a lower bound for the smallest eigenvalue that is independent of λ_0 but depends on the chosen interval i.e. on Λ_0. Let $\chi_k(\Lambda_0)$ be this lower bound. Since the system is of positive type, $\chi_k(\Lambda_0) \geq 0$ but $\sum_0^{N-1} \chi_k(\Lambda_0) > 0$ and (14) becomes

$$-\frac{d}{d\lambda} arg\ \rho_j(\lambda)|_{\lambda=\lambda_0} \geq \sum_0^{N-1} \chi_k(\Lambda_0) \tag{15}$$

This inequality is similar to (5.12) of [1]; the dependence on some interval width Λ_0 that may include the central zone and, possibly, some side zones, is not very restrictive: any numerical results are obtained for finite intervals, finite sums etc.

Theorem 8. *If $H_k \geq 0, \sum_0^{N-1} H_k > 0$ then the width of any λ-zone of stability included in some interval $(-\Lambda_0, \Lambda_0)$ does not exceed $\pi(\sum_0^{N-1} \chi_k(\Lambda_0))^{-1}$ where $\chi_k(\Lambda_0) = inf_{|\lambda| \leq \Lambda_0} \sigma_k^{min}(\lambda)$.*

The proof follows at once by applying the "traffic rules" [1],[15]. Note that the width of any of two parts of the central zone also does not exceed the above estimate.

6. The central zone of stability for a Hamiltonian system of positive type

We shall consider here the boundary value problem for (6) defined by (12) with $G = -I$. Its characteristic numbers are real: their existence follows from the fundamental theorem of Algebra provided $det(U_N(\lambda) + I) \neq$ const. and their number is finite. Let Λ_+ be the smallest (first) positive characteristic number and Λ_- the largest (first) negative one. We shall have

Theorem 9. *Assume that $H_k \geq 0, \sum_0^{N-1} H_k > 0$. The open interval (Λ_-, Λ_+) belongs to the central zone of stability of (6); moreover, if H_k are real, this interval and the central zone of stability coincide.*

The proof of this result goes as in [1], [15] and relies on Theorem 2.3; the restriction on distinct eigenvalues is removed by a perturbation argument.

The only remaining point of the entire construction is existence of the characteristic numbers of opposite sign for the skew-symmetric (with $G = -I$) boundary value problem. The complex function argument of [1] was valid in the case of [15] but it can not be used in general since $U_N(\lambda)$ is not, generally speaking, of entire type and the contradiction obtained in [1] which proved existence of characteristic numbers of opposite signs fails. Krein himself was aware of the fact that complex function arguments were perhaps too strong [1] and suggested to apply the theory of weighted integral equations [17]; later this theory was incorporated in the theory of Volterra operators on Hilbert spaces [18]. In the discrete-time case this may reduce to some (possibly less) known results on determinants. Application of the theory on time scales [12], [13], [14] may be of great interest.

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A NOTE ON THE PERIODICITY IN DIFFERENCE EQUATIONS

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Abstract. Sufficient conditions are obtained for the existence of a unique periodic solution of a linear first order difference equation in a Banach space.

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Let $\langle X, \|\cdot\|\rangle$ be a Banach space, $\ell^\infty(X)$ be the Banach space of bounded sequences $x = (x_k)_{k=0}^{+\infty} \subset X$ with the norm $\|x\|_\infty := \sup_{k\geq 0}\|x_k\|$, and $\ell^1(X)$ be the Banach space of summable sequences $x = (x_k)_{k=0}^{+\infty} \subset X$ with the norm $\|x\|_1 := \sum_{k\geq 0}\|x_k\|$.

Consider the linear difference equation

$$x(n+1) - x(n) = (Lx)(n) + f(n), \qquad n = 0, 1, 2, \dots \tag{1}$$

together with the N-periodic ($N \geq 1$) condition

$$x(n+N) = x(n), \qquad n = 0, 1, 2, \dots. \tag{2}$$

In (1), $f \in \ell^1(X)$, and $L : \ell^\infty(X) \to \ell^1(X)$ is a linear continuous operator. Here and below, L is assumed to leave invariant the subspace of sequences having property (2), and f is supposed to satisfy (2).

Remark 1. The use of special sequence spaces when posing problem (1), (2), in fact, can be avoided by restricting the consideration to problem (3), (4) or equation (6); see below. We have began with such a problem setting in order to note at this point that results similar to those to follow can be obtained for problems other than the periodic one.

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The invariance condition above implies that there is a one-to-one correspondence between solutions of (1), (2) and those of the problem

$$x(n+1)-x(n)=\sum_{\nu=0}^{N-1} L_{n,\nu}x(\nu)+f(n), \qquad 0\le n\le N-1, \tag{3}$$

$$x(N)=x(0), \tag{4}$$

where $(L_{n,\nu})_{n,\nu=0}^{N} \subset \mathcal{B}(X)$ are certain linear operators such that

$$L_{N,\nu}=L_{0,\nu} \text{ for all } \nu\in\{0,1,\dots,N-1\}. \tag{5}$$

Here and below, the symbol $\mathcal{B}(X)$ stands for the algebra of all bounded linear operators in X.

Due to property (5), knowing solutions of problem (3), (4), one can reconstruct those of (1), (2) by extending them periodically to all the non-negative integers. Furthermore, the periodic nature of problem (3), (4) allows one to consider it as a single linear equation with operator "matrices" acting in the space X^N of "vectors" $(x(0),x(1),\dots,x(N-1))$:

$$(\varDelta x)(n)=\sum_{\nu=0}^{N-1} L_{n,\nu}x(\nu)+f(n), \qquad 0\le n\le N-1, \tag{6}$$

where

$$(\varDelta x)\,(n):=\begin{cases} x(n+1)-x(n) & \text{for } 0\le n<N-1,\\ x(0)-x(n) & \text{for } n=N-1.\end{cases} \tag{7}$$

The latter circumstance will be essentially used below; we shall even identify L with the appropriate mapping $X^N\to X^N$:

$$(Lx)(n)=\sum_{\nu=0}^{N-1} L_{n,\nu}x(\nu), \qquad 0\le n<N.$$

Lemma 1. *Assume that the operator $\varLambda_{L,N}: X\to X$ defined with the formula*

$$\varLambda_{L,N}:=\sum_{n=0}^{N-1}\sum_{\nu=0}^{N-1} L_{n,\nu} \tag{8}$$

is invertible. Then $x=(x(0),x(1),\dots,x(N-1))$ is a solution of equation (6) if, and only if there exists some $a\in X$ such that the equalities

$$x(n)=\left(H_{L,N,l}Lx\right)(n)+f_{L,N,l}(n)+a, \quad 0\le n\le N-1, \tag{9}$$

$$\sum_{n=0}^{N-1}\left[\sum_{\nu=0}^{N-1} L_{n,\nu}x(\nu)+f(n)\right]=0 \tag{10}$$

hold with some $l \in \{0, 1, \ldots, N-1\}$*, where the linear mapping* $H_{L,N,l} : X^N \to X^N$ *is defined with the formula*

$$(H_{L,N,l}x)(n) := \begin{cases} \displaystyle\sum_{k=l}^{N-1} \Big[x(k) - \sum_{\nu=0}^{N-1} L_{k,\nu} \Lambda_{L,N}^{-1} \sum_{j=0}^{N-1} x(j) \Big] & \text{for } n = 0, \\ \displaystyle\sum_{k=l}^{n-1} \Big[x(k) - \sum_{\nu=0}^{N-1} L_{k,\nu} \Lambda_{L,N}^{-1} \sum_{j=0}^{N-1} x(j) \Big] & \text{for } 0 < n < N, \end{cases} \tag{11}$$

and

$$f_{L,N,l} := H_{L,N,l} f. \tag{12}$$

Proof. Assume that $x = (x(0), x(1), \ldots, x(N-1))$ satisfies (9) and (10). Then, for $1 \leq n < N-1$, we have

$$\begin{aligned} x(n) &= a + f_{L,N,l}(n) + \sum_{k=l}^{n-1} \sum_{\nu=0}^{N-1} L_{k,\nu} x(\nu) \\ &\quad - \sum_{k=l}^{n-1} \sum_{\nu=0}^{N-1} L_{k,\nu} \Lambda_{L,N}^{-1} \sum_{k=0}^{N-1} \sum_{\nu=0}^{N-1} L_{k,\nu} x(\nu) \qquad (13) \\ &= a + f_{L,N,l}(n) + \sum_{k=l}^{n-1} \sum_{\nu=0}^{N-1} L_{k,\nu} x(\nu) + \sum_{k=l}^{n-1} \sum_{\nu=0}^{N-1} L_{k,\nu} \Lambda_{L,N}^{-1} \sum_{k=0}^{N-1} f(k), \qquad (14) \end{aligned}$$

whence

$$\begin{aligned} x(n+1) - x(n) = \sum_{\nu=0}^{N-1} L_{n,\nu} x(\nu) + \sum_{\nu=0}^{N-1} L_{n,\nu} \Lambda_{L,N}^{-1} \sum_{k=0}^{N-1} f(k) \\ + f_{L,N,l}(n+1) - f_{L,N,l}(n). \end{aligned} \tag{15}$$

It is easy to see from definition (11) that, when $1 \leq n \leq N-1$, (12) is equivalent to the relation

$$f_{L,N,l}(n) = \sum_{k=l}^{n-1} \Big[f(k) - \sum_{\nu=0}^{N-1} L_{k,\nu} \Lambda_{L,N}^{-1} \sum_{j=0}^{N-1} f(j) \Big], \tag{16}$$

whence

$$f_{L,N,l}(n+1) - f_{L,N,l}(n) = f(n) - \sum_{\nu=0}^{N-1} L_{n,\nu} \Lambda_{L,N}^{-1} \sum_{k=0}^{N-1} f(k) \tag{17}$$

for $0 < n < N$. Combining (15) and (17), we show that (6) holds for $1 \leq n < N-1$. The case $n = 0$ is considered analogously.

Let us now suppose that equality (6) holds. Then, we have

$$\sum_{n=0}^{N-2}\left(\sum_{\nu=0}^{N-1} L_{n,\nu}x(\nu)+f(n)\right)=\sum_{n=0}^{N-2}[x(n+1)-x(n)]=x(N-1)-x(0). \tag{18}$$

According to definition (7), equation (6) for $n=N-1$ means that

$$\sum_{\nu=0}^{N-1} L_{N-1,\nu}x(\nu)+f(N-1)=x(0)-x(N-1),$$

which, combined with (18), implies (10).

Furthermore, in view of (11) and (10), for $n\in\{1,2,\dots,N-1\}$, we have

$$\begin{aligned}(H_{L,N,l}Lx)(n)&=\sum_{k=l}^{n-1}\left[\sum_{\nu=0}^{N-1} L_{k,\nu}x(\nu)-\sum_{\mu=0}^{N-1} L_{k,\mu}\varLambda_{L,N}^{-1}\sum_{j=0}^{N-1}\sum_{\nu=0}^{N-1} L_{j,\nu}x(\nu)\right]\\ &=\sum_{k=l}^{n-1}\left[\sum_{\nu=0}^{N-1} L_{k,\nu}x(\nu)+\sum_{\mu=0}^{N-1} L_{k,\mu}\varLambda_{L,N}^{-1}\sum_{j=0}^{N-1} f(j)\right].\end{aligned} \tag{19}$$

Carrying out the manipulations marked as (13), (14), and (15) in the reverse order and taking into account (19), we find that equality (9) holds for $0<n\le N-1$. When $n=0$, in view of (11), identity (19) is replaced by the relation

$$(H_{L,N,l}Lx)(0)=\sum_{k=l}^{N-1}\left[\sum_{\nu=0}^{N-1} L_{k,\nu}x(\nu)+\sum_{\mu=0}^{N-1} L_{k,\mu}\varLambda_{L,N}^{-1}\sum_{j=0}^{N-1} f(j)\right],$$

and a similar argument leads one to (9) in this case as well. □

Remark 2. Lemma 1 is similar to some statements from [3], [4], and [5].

Lemma 2. *The identity*

$$(H_{L,N,l}Lx)(n)=\varOmega_{L,N,l}\begin{pmatrix}x(0)\\ x(1)\\ \vdots\\ x(N-1)\end{pmatrix} \tag{20}$$

holds for $0 \le n < N$*, where* $\Omega_{L,N,l} : X^N \to X^N$ *is given by the matrix*

$$\Omega_{L,N,l} = \begin{bmatrix} \sum\limits_{k=l}^{N-1}\Big(L_{k,0} - L_k^{\#}\sum\limits_{j=0}^{N-1} L_{j,0}\Big) \dots \sum\limits_{k=l}^{N-1}\Big(L_{k,N-1} - L_k^{\#}\sum\limits_{j=0}^{N-1} L_{j,N-1}\Big) \\ \sum\limits_{k=l}^{0}\Big(L_{k,0} - L_k^{\#}\sum\limits_{j=0}^{N-1} L_{j,0}\Big) \dots \sum\limits_{k=l}^{0}\Big(L_{k,N-1} - L_k^{\#}\sum\limits_{j=0}^{N-1} L_{j,N-1}\Big) \\ \sum\limits_{k=l}^{1}\Big(L_{k,0} - L_k^{\#}\sum\limits_{j=0}^{N-1} L_{j,0}\Big) \dots \sum\limits_{k=l}^{1}\Big(L_{k,N-1} - L_k^{\#}\sum\limits_{j=0}^{N-1} L_{j,N-1}\Big) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum\limits_{k=l}^{N-2}\Big(L_{k,0} - L_k^{\#}\sum\limits_{j=0}^{N-1} L_{j,0}\Big) \dots \sum\limits_{k=l}^{N-2}\Big(L_{k,N-1} - L_k^{\#}\sum\limits_{j=0}^{N-1} L_{j,N-1}\Big) \end{bmatrix} \tag{21}$$

and

$$L_k^{\#} := \sum_{\nu=0}^{N-1} L_{k,\nu}\Lambda_{L,N}^{-1}, \quad 0 \le k \le N-1. \tag{22}$$

Proof. Considering (11), it is not difficult to verify by computation that, for $1 \le n \le N-1$,

$$\left(H_{L,N,l}Lx\right)(n) = \sum_{\nu=0}^{N-1}\left[\sum_{k=l}^{n-1} L_{k,\nu} - \sum_{j=0}^{N-1}\sum_{k=l}^{n-1} L_k^{\#}L_{j,\nu}\right]x(\nu), \tag{23}$$

where $L_k^{\#}$ $(0 \le k \le N-1)$ are the linear operators given by (22) and (8). This, together with a similar observation for $n = 0$, leads one to formula (21) for the operator "matrix" $\Omega_{L,N,l}$ in equality (20). □

Introduce the notation

$$\operatorname{diag} X^N := \Big\{\big(\underbrace{a, a, \dots, a}_{N}\big) \;:\; a \in X\Big\}. \tag{24}$$

Lemma 3. $\operatorname{diag} X^N \subset \ker H_{L,N,l}L$.

Proof. According to equality (23) established in the proof of Lemma 2, we have

$$\begin{aligned} \left(H_{L,N,l}La\right)(n) &= \sum_{\nu=0}^{N-1}\left[\sum_{k=l}^{n-1} L_{k,\nu} - \sum_{j=0}^{N-1}\sum_{k=l}^{n-1} L_k^{\#}L_{j,\nu}\right]a \\ &= \sum_{k=l}^{n-1}\left[\sum_{\nu=0}^{N-1} L_{k,\nu} - L_k^{\#}\sum_{j=0}^{N-1}\sum_{\nu=0}^{N-1} L_{j,\nu}\right]a, \end{aligned}$$

whence, by definitions (8) and (22),

$$(H_{L,N,l}La)(n) = \sum_{k=l}^{n-1}\left[\sum_{\nu=0}^{N-1} L_{k,\nu} - L_k^{\#}\Lambda_{L,N}\right]a = 0$$

for all $a \in X$ and $n \in \{1, 2, \dots, N-1\}$.

The remaining case when $n = 0$ is considered in a similar way. □

Let us now put $\rho_L(N) := r(\Omega_{L,N,l})$, the spectral radius of the linear operator $\Omega_{L,N,l} : X^N \to X^N$ defined with formula (21). The notation is justified by the following

Lemma 4. *$\rho_L(N)$ is independent of l.*

Proof. Let us first prove the following CLAIM: *If $A : X^N \to X^N$ and $B : X^N \to X^N$ are bounded linear mappings such that $\sigma(B) \subset \sigma(A)$ and $\operatorname{im} B \subset \ker A$, then $\sigma(A + B) = \sigma(A)$.*

Indeed, let $\lambda \notin \sigma(A)$ be a regular point for A. Then the equation

$$Ax - \lambda x = y - \phi$$

has the unique solution $x(y - \phi, \lambda) := -\lambda^{-1}[y - \phi + \lambda^{-1}A(y - \phi) + \dots]$ for all y and ϕ. Consider the equation

$$\phi = Bx(y - \phi, \lambda), \tag{25}$$

or, which is the same,

$$\phi = \lambda^{-1}B\sum_{\nu=0}^{+\infty}\lambda^{-\nu}A^{\nu}(\phi - y).$$

Since, obviously, we are seeking for a ϕ in $\operatorname{im} B$, the assumption that $\operatorname{im} B \subset \ker A$ yields $\sum_{\nu=0}^{+\infty}\lambda^{-\nu}A^{\nu}\phi = \phi$ and, therefore, equation (25) rewrites as

$$B\phi - \lambda\phi = B\sum_{\nu=0}^{+\infty}\lambda^{-\nu}A^{\nu}y. \tag{26}$$

Since $\lambda \not\ni \sigma(A) \supset \sigma(B)$, we see that (26), and hence (25), has a unique solution, say $\phi(y, \lambda)$. Thus, for every y, the equation

$$Ax - \lambda x = y - \phi(y, \lambda) \tag{27}$$

has a unique solution and, moreover, by virtue of the form of equation (25), the solution $\Xi(y, \lambda) := x(y - \phi(y, \lambda), \lambda)$ of (27) also satisfies the equation

$$Ax - \lambda x = y - Bx. \tag{28}$$

Let us prove that (28) cannot have any other solutions. Indeed, in the contrary case, when (28) has another solution, say z, the difference $\delta := \Xi(y,\lambda) - z$ satisfies the equality

$$A\delta - \lambda\delta = -B\delta. \tag{29}$$

Since, by assumption, $\operatorname{im} B$ is contained in $\ker A$, relation (29) implies that $A^2\delta = \lambda A\delta$. Therefore, $A\delta = 0$, because otherwise $A\delta$ would be an eigen-vector of A with the eigen-value λ, which has been assumed to be regular for A. The same equality (29) then yields $B\delta = \lambda\delta$, which can be the case only when $\delta = 0$, because $\lambda \notin \sigma(B)$. Hence, z and $\Xi(y,\lambda)$ coincide.

The argument above shows that, for $\lambda \notin \sigma(A)$ and arbitrary y, equation (28) has a unique solution, whose continuous dependence upon y is obvious. Therefore, $\sigma(A) \supset \sigma(A+B)$.

Conversely, if $\lambda \notin \sigma(A+B)$, then there exists a bounded inverse operator $(A+B-\lambda I)^{-1}$, where I stands for the unity in $\mathcal{B}(X)$. Since, by assumption, $AB = 0$, we have

$$(A-\lambda I)(B-\lambda I) = -\lambda[A+B-\lambda I], \tag{30}$$

an invertible operator. Assume that $B - \lambda I$ is non-invertible. Then, according to a well-known criterion (see, e. g., Theorem 2 in [1, p. 209]), there is some sequence $(u_k)_{k=1}^{+\infty}$ such that $\|u_k\| = 1$ and $\|Bu_k - \lambda u_k\| \le \frac{1}{k}$ for all $k \ge 1$. On the other hand, since operator (30) is invertible, the same reasoning shows the existence of a constant $c \in (0,+\infty)$ such that $\|(A-\lambda I)(B-\lambda I)x\| \ge c\|x\|$ for all x. Combining these two statements, we obtain that, for all $k \ge 1$,

$$c \le \|(A-\lambda I)(B-\lambda I)u_k\| \le \|A-\lambda I\| \cdot \|Bu_k - \lambda u_k\| \le \frac{\|A-\lambda I\|}{k},$$

which is impossible. Therefore, $B - \lambda I$ is invertible and, by (30), so does $A - \lambda I$, i. e., $\lambda \notin \sigma(A)$. Hence, $\sigma(A+B) \supset \sigma(A)$, and the proof of the CLAIM is complete.

Returning to our lemma, one can readily check that matrix (21) corresponding to operator (11) has the property

$$[\Omega_{L,N,l_1}x - \Omega_{L,N,l_2}x](n) = \sum_{k=l_1}^{l_2}\sum_{\nu=0}^{N-1}\left[L_{k,\nu} - L_k^{\#}\sum_{j=0}^{N-1}L_{j,\nu}\right]x(\nu)$$

for all $n \in \{0,1,\dots,N-1\}$. It is then easy to verify that $\sigma\left(\Omega_{L,N,l_1} - \Omega_{L,N,l_2}\right) = \sigma(\beta)$, where $\beta := \sum_{k=l_1}^{l_2}\sum_{\nu=0}^{N-1}[L_{k,\nu} - L_k^{\#}\sum_{j=0}^{N-1}L_{j,\nu}]$. Recalling notations (8) and (22), we see that, in fact, $\beta = 0$.

Finally, putting $A := \Omega_{L,N,l_1}$ and $B := \Omega_{L,N,l_2} - \Omega_{L,N,l_1}$ in the CLAIM above, we obtain that $\sigma(\Omega_{L,N,l_1}) = \sigma(\Omega_{L,N,l_2})$ for all l_1 and l_2 in $\{0,1,\dots,N-1\}$. □

Lemma 5. *$\rho_L(N) = r(Q_{L,N})$, where $Q_{L,N} : X^{N-1} \to X^{N-1}$ is given by*

$$(Q_{L,N}x)(n) := \sum_{k=0}^{n-1}\sum_{\nu=0}^{N-1}\left(L_{k,\nu} - L_k^{\#}\sum_{j=0}^{N-1}L_{j,\nu}\right)x(\nu), \qquad 1 \le n \le N-1. \tag{31}$$

Proof. By virtue of Lemma 4, we can put $l = 0$ in (11), in which case, as is easy to see, the first row of matrix (21) is filled with zeroes. Thus, $\Omega_{L,N,0} = \left[\begin{smallmatrix} 0 & 0 \\ M & Q_{L,N} \end{smallmatrix}\right]$ with a certain M and, obviously, $r(\Omega_{L,N,0}) = r(Q_{L,N})$. □

Now we can apply the above lemmata to obtain the following theorem.

Theorem 1. *Assume that operator (8) is invertible and, moreover, $\rho_L(N) < 1$. Then equation (6) has a unique solution for every $f : \{0, 1, \ldots, N-1\} \to X$.*

Proof. By Lemma 1, every solution of (6), if there are any, satisfies relations (9) and (10) for some $a \in X$ and, conversely, a solution of (9) is also that of (6) whenever a is such that (10) holds. Let us fix some $a \in X$ and consider the corresponding equation (9).

Introduce the sequence

$$y_{m+1}(n) = a + f_{L,N,l}(n) + (H_{L,N,l} L y_m)(n), \quad 0 \le n < N, \ m \ge 0,$$

where $f_{L,N,l} : \{0, 1, \ldots, N-1\} \to X$ is defined by (12) and the starting member is arbitrary. We have:

$$\begin{aligned} y_{m+1} &= a + f_{L,N,l} + H_{L,N,l} L y_m \\ &= a + f_{L,N,l} + H_{L,N,l} L \left[a + f_{L,N,l} + H_{L,N,l} L y_{m-1}\right], \end{aligned}$$

which, by Lemma 3, yields

$$y_{m+1} = a + f_{L,N,l} + H_{L,N,l} L f_{L,N,l} + (H_{L,N,l} L)^2 y_{m-1}.$$

Proceeding similarly, we arrive at the equality

$$y_{m+1} = a + \sum_{\nu=0}^{m} (H_{L,N,l} L)^{\nu} f_{L,N,l} + (H_{L,N,l} L)^{m+1} y_0.$$

It follows immediately from Lemma 2 that $r(H_{L,N,l} L) = \rho_L(N)$ and, therefore, our assumption implies the convergence of the series $\sum_{\nu=0}^{+\infty} (H_{L,N,l} L)^{\nu} f_{L,N,l}$, which means that equation (9) has a unique solution for every $a \in X$.

Furthermore, according to Lemma 1, a certain $x : \{0, 1, \ldots, N-1\} \to X$ is a solution of equation (6) if, and only if

$$x = a + \sum_{\nu=0}^{+\infty} (H_{L,N,l} L)^{\nu} f_{L,N,l} \tag{32}$$

with some $a \in X$ such that (10) holds. However, it is easy to see that, for x given by (32), relation (10) is equivalent to the equality

$$a = -\Lambda_{L,N}^{-1} \sum_{n=0}^{N-1} \left(f(n) + \left[L \sum_{\nu=0}^{+\infty} (H_{L,N,l} L)^{\nu} f_{L,N,l} \right](n) \right). \tag{33}$$

Inserting (33) into (32) and expanding notation (12), we obtain the unique solution of equation (6) in the form of the series

$$x = \sum_{\nu=0}^{+\infty}\Big[(H_{L,N,l}L)^{\nu} H_{L,N,l}f - \Lambda_{L,N}^{-1}\sum_{k=0}^{N-1}\left[L\,(H_{L,N,l}L)^{\nu} H_{L,N,l}f\right](k)\Big] - \Lambda_{L,N}^{-1}\sum_{k=0}^{N-1} f(k), \tag{34}$$

and the proof of the theorem is thus complete. □

Remark 3. Theorem 1 is in the spirit of Corollary 5.2 from [2] and Corollary 4.2.1 from [6] established for linear systems of ordinary differential equations.

Let us say that some problem *does not possess uniqueness property* if it either has no solutions or has more than one solution.

Corollary 1. *Assume that $\{L_{k,\nu}\}_{k,\nu=0}^{N-1} \subset \mathcal{B}(X)$ are some linear operators such that the corresponding mapping (8) is invertible. Then, for the boundary value problem*

$$x(n+1) - x(n) = \lambda \sum_{\nu=0}^{N-1} L_{n,\nu}x(\nu) + f(n), \qquad 0 \le n \le N-1, \tag{35}$$

$$x(N) = x(0) \tag{36}$$

not to possess the uniqueness property for some $f : \{0,1,2,\dots,N-1\} \to X$, it is necessary that the parameter $\lambda \in (-\infty,+\infty)$ satisfy the inequality

$$|\lambda| \ge 1/\rho_L(N).$$

Proof. It suffices to replace system (35), (36) by an equation of type (6) and apply Theorem 1. □

Corollary 2. *Assume that the operators $\{L_{k,\nu}\}_{k,\nu=0}^{N-1} \subset \mathcal{B}(X)$ satisfy the condition*

$$\sum_{\nu=0}^{N-1} L_{n,\nu} = A \quad \textit{for all } n \in \{0,1,\dots,N-1\} \tag{37}$$

with some invertible $A \in \mathcal{B}(X)$ and, moreover, the spectral radius of the operator

$$\begin{bmatrix} L_{1,1} - \dfrac{1}{N}\displaystyle\sum_{j=0}^{N-1} L_{j,1} & \dots & L_{1,N-1} - \dfrac{1}{N}\displaystyle\sum_{j=0}^{N-1} L_{j,N-1} \\ \displaystyle\sum_{k=0}^{1} L_{k,1} - \dfrac{2}{N}\sum_{j=0}^{N-1} L_{j,1} & \dots & \displaystyle\sum_{k=0}^{1} L_{k,N-1} - \dfrac{2}{N}\sum_{j=0}^{N-1} L_{j,N-1} \\ \dots & \dots & \dots \\ \displaystyle\sum_{k=0}^{N-2} L_{k,1} - \dfrac{N-1}{N}\sum_{j=0}^{N-1} L_{j,1} & \dots & \displaystyle\sum_{k=0}^{N-2} L_{k,N-1} - \dfrac{N-1}{N}\sum_{j=0}^{N-1} L_{j,N-1} \end{bmatrix} \tag{38}$$

is less than one. Then, for every $f \in \operatorname{diag} X^{N+1}$*, problem* (3), (4) *has a unique solution, and this solution belongs to* $\operatorname{diag} X^{N+1}$*:*

$$x(n) = -A^{-1} f \qquad \text{for all } n \in \{0, 1, 2, \ldots, N\}.$$

Proof. As before, instead of (3), (4), we consider equation (6).

Taking into account notations (22) and (8), it is not difficult to verify that, under assumption (37), $\Lambda_{L,N} = N \cdot A$ and $L_k^{\#} = \frac{1}{N} I$ $(0 \leq k \leq N-1)$, whence we see that the operator defined by matrix (38) is nothing but $Q_{L,N}$ given by (31). Theorem 1, together with Lemma 5, then guarantees the unique solvability of equation (6), whose solution can be represented as series (34).

By Lemma 3, the relation $f \in \operatorname{diag} X^N$ yields $H_{L,N,l} f = 0$, whence, considering (34), we conclude that the solution of (6) is equal identically to $-\Lambda_{L,N}^{-1} \sum_{k=0}^{N-1} f(k)$. Returning to problem (3), (4), we obtain the conclusion desired. □

Remark 4. The condition imposed on $\rho_L(N)$ in Theorem 1, generally speaking, cannot be weakened. Indeed, consider the simplest scalar difference equation

$$x(n+1) = -x(n) \quad (n \geq 0). \tag{39}$$

The 2-periodic boundary value problem for equation (39) can be interpreted as (6) with $N = 2$, $f(0) = f(1) = 0$, $L_{0,1} = L_{1,0} = 0$, and $L_{0,0} = L_{1,1} = -2$. It is obvious that, in this case, $\Omega_{L,N,0} = \left[\begin{smallmatrix} 0 & 0 \\ -1 & 1 \end{smallmatrix}\right]$ and, thus, $\rho_L(2) = 1$. On the ther hand, every non-trivial solution of (39) is periodic with period 2. Hence, the corresponding inhomogeneous problem does not have uniqueness property and, therefore, the inequality $\rho_L(2) < 1$ in Theorem 1 [resp., $|\lambda| \geq \rho_L(2)$ in Corollary 1] cannot be replaced by $\rho_L(2) \leq 1$ [resp., $|\lambda| > \rho_L(2)$].

One can also construct similar examples for an arbitrary period $N \geq 2$ (this is not done here).

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ON NON-LINEAR BOUNDARY VALUE PROBLEMS CONTAINING PARAMETERS

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Abstract. We consider a boundary value problem containing two parameters both in the non-linear ordinary differential equation and in the non-linear boundary conditions. By using a suitable change of variables, we bring the given problem to a family of those with linear boundary conditions (plus some non-linear determining equations), and apply an iterative method to approximately find its solution.

Keywords. Parametrised boundary value problems, non-linear boundary conditions, numerical-analytic methods, successive approximations, determining equations.

1. Introduction

An analysis of the publications concerning the iterative methods in the theory of boundary value problems shows that various numerical-analytic methods, in particular, those based upon successive approximations, are now widely used and developed (see, e. g., [5] for a review).

According to the basic idea of the latter group of methods, the given boundary value problem is replaced by a problem for a "perturbed" differential equation containing some artificially introduced parameter, whose value should be determined later. The solution of the "perturbed" problem is sought for in the analytic form by iteration with all the iterations depending upon the parameter mentioned.

As to the way how the auxiliary problem is constructed, it is essential that the form of the "perturbation term" yields a certain system of (algebraic or transcendental) "determining equations," which give the numerical values of the parameter corresponding to the solutions sought-for. By studying these determining equations, it is possible to establish existence results for the original problem.

It is worth mentioning that, earlier, the parametrised boundary value problems were studied mostly in the case of the linear boundary conditions [4], or even in the case when the parameters are contained only in the differential equation [1,2].

It has been an open problem to find out how one can construct a numerical-analytic scheme suitable for problems with parameters both in the equation and in non-linear boundary conditions. Here, we give a possible approach to this question following the method from [3].

2. Problem setting

We consider the non-linear two-point parameterized boundary value problem

$$y'(t) = f(t, y(t), \lambda_1, \lambda_2), \qquad t \in [0, T], \tag{1}$$

$$g(y(0), y(T), \lambda_1, \lambda_2) = 0, \tag{2}$$

$$y_1(0) = y_{10}, \quad y_2(0) = y_{20}, \tag{3}$$

containing the parameters λ_1 and λ_2 both in Eq. (1) and in condition (2).

Here, we suppose that the functions $f : [0, T] \times G \times [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}^n$ $(n \geq 3)$ and $g : G \times G \times I_1 \times I_2 \to \mathbb{R}^n$ are continuous, $G \subset \mathbb{R}^n$ is a closed, connected, and bounded domain, and $\lambda_k \in I_k := [a_k, b_k]$ $(k = 1, 2)$ are unknown scalar parameters.

Assume that, for $t \in [0, T]$, $\lambda_1 \in I_1$, and $\lambda_2 \in I_2$ fixed, the function f satisfies the Lipschitz condition

$$|f(t, u, \lambda_1, \lambda_2) - f(t, v, \lambda_1, \lambda_2)| \leq K|u - v| \tag{4}$$

for all $\{u, v\} \subset G$ and some non-negative matrix $K = (K_{kl})_{k,l=1}^n$. In (4), as well as in similar relations below, the signs $|\cdot|$ and $\leq$ are understood component-wise.

The problem is to find the values of the parameters λ_1 and λ_2 such that problem (1), (2) has a classical solution satisfying the additional conditions (3). Thus, a solution is the triple $\{y, \lambda_1, \lambda_2\}$ and, therefore, (1)–(3) is similar, in a sense, to an eigen-value problem.

3. A reduction to the parametrised boundary value problem with linear conditions

Let us introduce the substitution

$$y(t) = x(t) + w, \tag{5}$$

where $w = \mathrm{col}(w_1, w_2, \ldots, w_n) \in \Omega \subset \mathbb{R}^n$ is an unknown parameter. The domain Ω is chosen so that

$$D + \Omega \subset G,$$

whereas the new variable, x, is supposed to have range in D, the closure of a bounded subdomain of G.

Substitution (5) allows one to rewrite problem (1)–(3) as

$$x'(t) = f(t, x(t) + w, \lambda_1, \lambda_2), \qquad t \in [0, T], \tag{6}$$

$$g(x(0) + w, x(T) + w, \lambda_1, \lambda_2) = 0, \tag{7}$$

$$x_1(0) = y_{10} - w_1, \quad x_2(0) = x_{20} - w_2. \tag{8}$$

Let us bring the boundary condition (7) to the form

$$Ax(0) + Bx(T) = \Phi(x(0) + w, x(T) + w, \lambda_1, \lambda_2) = [A + B]w,$$

where $\Phi(u, v, \lambda_1, \lambda_2) := Au + Bv + g(u, v, \lambda_1, \lambda_2)$ and A, B are fixed square n-dimensional matrices such that $\det B \neq 0$.

The parameter w is natural to be determined from the determining equation

$$\Phi(x(0) + w, x(T) + w, \lambda_1, \lambda_2) = [A + B]w$$

or, equivalently,

$$Ax(0) + Bx(T) + g(x(0) + w, x(T) + w, \lambda_1, \lambda_2) = 0.$$

Thus, the essentially non-linear problem (1)–(3) turns out to be equivalent to

$$x'(t) = f(t, x(t) + w, \lambda_1, \lambda_2), \qquad t \in [0, T], \tag{9}$$

$$Ax(0) + Bx(T) + g(x(0) + w, x(T) + w, \lambda_1, \lambda_2) = 0, \tag{10}$$

$$x_1(0) = y_{10} - w_1, \quad x_2(0) = x_{20} - w_2. \tag{11}$$

On the other hand, system (9), (10), (11) can be regarded as a collection of problems

$$x'(t) = f(t, x(t) + w, \lambda_1, \lambda_2), \qquad t \in [0, T], \tag{12}$$

$$Ax(0) + Bx(T) = 0, \tag{13}$$

$$x_1(0) = y_{10} - w_1, \quad x_2(0) = x_{20} - w_2. \tag{14}$$

parametrised by the unknown vector w and considered together with the determining equation (10).

The essential advantage obtained thereby is that the boundary condition (13) is linear.

It follows from the consideration above that family (12)–(14) can be studied by using the numerical-analytic method developed in [5].

Assume that

$$D_\beta := \{x \in \mathbb{R}^n : B(x, \beta(x)) \subset D\} \neq \emptyset, \tag{15}$$

where

$$\beta(x) := \frac{T}{2}\delta_G(f) + |(B^{-1}A + E_n)x|$$

and

$$\delta_G(f) := \frac{1}{2}\Big[\max_{(t,x,\lambda_1,\lambda_2)\in[0,T]\times\Omega\times I_1\times I_2} f(t,x,\lambda_1,\lambda_2) - \min_{(t,x,\lambda_1,\lambda_2)\in[0,T]\times\Omega\times I_1\times I_2} f(t,x,\lambda_1,\lambda_2)\Big]. \tag{16}$$

Moreover, we suppose that K in (4) satisfies

$$r(K) < \frac{10}{3T}. \tag{17}$$

Set

$$D_1 := \big\{u \in \mathbb{R}^{n-2} : z \equiv \operatorname{col}(y_{10} - w_1, y_{20} - w_2, u_1, u_2, \ldots, u_{n-2}) \in D_\beta\big\}$$

and introduce the sequence of functions

$$\begin{aligned} x_{m+1}(t,w,u,\lambda_1,\lambda_2) := z &+ \int_0^t f\left(s, x_m(s,w,u,\lambda_1,\lambda_2) + w, \lambda_1,\lambda_2\right) ds \\ &- \frac{t}{T}\int_0^T f\left(s, x_m(s,w,u,\lambda_1,\lambda_2) + w, \lambda_1,\lambda_2\right) ds \\ &- \frac{t}{T}[B^{-1}A + E_n]z, \end{aligned} \tag{18}$$

where $m \geq 0$ and $x_0(t,w,u,\lambda_1,\lambda_2) \equiv z$.

Note that $x_m(0,w,u,\lambda_1,\lambda_2) = z$ for all m.

It can be verified that all the members of sequence (18) satisfy conditions (13) and (14) for arbitrary $u \in D_1$, $w \in \Omega$, and $\lambda_k \in I_k$ $(k = 1, 2)$.

By virtue of (13), every solution, x, of (12)–(14) satisfies

$$x(T) = -B^{-1}Ax(0).$$

Therefore, Eq. (10) can be rewritten as

$$g(x(0) + w, -B^{-1}Ax(0) + w, \lambda_1, \lambda_2) = 0. \tag{19}$$

So, we conclude that problem (9)–(14) is equivalent to the following family of boundary value problems with linear conditions:

$$x'(t) = f(t, x(t) + w, \lambda_1, \lambda_2), \qquad t \in [0,T], \tag{20}$$

$$Ax(0) + Bx(T) = 0, \tag{21}$$

$$x_1(0) = y_{10} - w_1, \quad x_2(0) = x_{20} - w_2 \tag{22}$$

considered together with the determining equation (19).

We suggest to solve the latter system sequentially: first solve (20)–(22), and then try to find out whether (19) can simultaneously be fulfilled.

Theorem 1. *Assume conditions (4), (15), and (17). Then:*

1. *Sequence (18) converges to the function $x^* = x^*(\cdot, w, u, \lambda_1, \lambda_2)$ as $m \to +\infty$ uniformly in $(w, u, \lambda_1, \lambda_2) \in \Omega \times D_1 \times I_1 \times I_2$.*
2. *The limit function $x^*(\cdot, w, u, \lambda_1, \lambda_2)$ is the unique solution of the "perturbed" parametrised boundary value problem*

$$\begin{gathered} x'(t) = f(t, x(t) + w, \lambda_1, \lambda_2) + \Delta(w, u, \lambda_1, \lambda_2), \qquad t \in [0, T], \\ Ax(0) + Bx(T) = 0, \\ x_1(0) = y_{10} - w_1, \quad x_2(0) = x_{20} - w_2 \end{gathered} \tag{23}$$

having the initial value $x^*(0, w, u, \lambda_1, \lambda_2) = z$, where

$$\begin{aligned} \Delta(w, u, \lambda_1, \lambda_2) := &-\frac{1}{T}[B^{-1}A + E_n]z \\ &- \frac{1}{T}\int_0^T f\left(s, x^*(s, w, u, \lambda_1, \lambda_2) + w, \lambda_1, \lambda_2\right) ds. \end{aligned}$$

3. *The following error estimate holds:*

$$|x_m(t, w, u, \lambda_1, \lambda_2) - x^*(t, w, u, \lambda_1, \lambda_2)| \le h(t, w, u, \lambda_1, \lambda_2), \tag{24}$$

where

$$\begin{aligned} h(t, w, u, \lambda_1, \lambda_2) := \frac{20t}{9}\left(1 - \frac{t}{T}\right) Q^{m-1}\left(E_n - Q\right)^{-1} \Big[&Q\delta_G(f) \\ &+ K\left|\left(B^{-1}A + E_n\right)z\right|\Big], \end{aligned}$$

the vector $\delta_G(f)$ is given by (16), and $Q := \frac{3T}{10}K$.

Proof. It can be carried out similarly to that of Theorem 2.1 from [5, p. 34].

The following statement shows the relation of the function $x^*(\cdot, w, u, \lambda_1, \lambda_2)$ to the solution of problem (20)–(22).

Theorem 2. *Under the assumptions of Theorem 1, the function*

$$x^*(\cdot, w^*, u^*, \lambda_1^*, \lambda_2^*)$$

is a solution of the parametrised boundary value problem (20)–(22) if, and only if the triplet $\{u^*, \lambda_1^*, \lambda_2^*\}$ satisfies the system of determining equations

$$[B^{-1}A + E_n]z + \int_0^T f\left(s, x^*(s, w, u, \lambda_1, \lambda_2) + w, \lambda_1, \lambda_2\right) ds = 0,$$

where w is considered as a parameter.

Proof. Analogous to that of Theorem 2.3 from [5, p. 40].

Theorem 3. *Assume conditions* (4), (15), *and* (17). *Then, for the function*

$$y^* := x^*(\cdot, w^*, u^*, \lambda_1^*, \lambda_2^*) + w^* \tag{25}$$

to be a solution of the given parametrised problem (1)–(3), *it is necessary and sufficient that* $\{w^*, u^*, \lambda_1^*, \lambda_2^*\}$ *satisfy the system of determining equations*

$$\begin{gathered} z + \int_0^T f\left(s, x^*(s, w, u, \lambda_1, \lambda_2) + w, \lambda_1, \lambda_2\right) ds = 0, \\ g\left(z + w, -B^{-1}Az + w, \lambda_1, \lambda_2\right) = 0. \end{gathered} \tag{26}$$

Proof. It is easily seen from the form substitution (5) that Eqns. (26) hold whenever the transformed boundary value problem (23) is equivalent to the original problem (1)–(3).

Remark 1. Considering function (25), one can set

$$y_m := x_m(\cdot, w_m, u_m, \lambda_{1,m}, \lambda_{2,m}) + w_m \tag{27}$$

and regard (27) as the mth approximation to function (25), which solves the boundary value problem (1)–(3).

In Eq. (27), x_m is given by (18), whereas w_m, u_m, $\lambda_{1,m}$, and $\lambda_{2,m}$ are solutions of

$$\begin{gathered} z + \int_0^T f\left(s, x_m(s, w, u, \lambda_1, \lambda_2) + w, \lambda_1, \lambda_2\right) ds = 0, \\ g\left(z + w, -B^{-1}Az + w, \lambda_1, \lambda_2\right) = 0. \end{gathered} \tag{28}$$

We do not consider the strict substantiation of the above idea, referring to [5] where similar techniques are described.

Example 1. Let us consider the third order parametrised differential equation

$$y'''(t) + \frac{1}{2}(y''(t))^2 + \lambda_1 y(t) = \left(\lambda_2 + \frac{3}{4}\right)\frac{t^2}{16}, \qquad t \in [0,1] \tag{29}$$

with the following non-linear boundary conditions containing parameters:

$$\begin{gathered} y'(1)y'(0) + \lambda_1 y(1) = \frac{1}{32}, \\ y(1)y'(0) + \lambda_2 y'(0) + \lambda_2 y''(1) = \frac{1}{16}, \\ \frac{1}{2}y'(0) + \left(\frac{1}{2} - \lambda_1\right) y'(1) = 0, \\ y(0) = -\frac{1}{16}, \qquad y'(0) = 0. \end{gathered} \tag{30}$$

Equivalently, equation (29) can be rewritten as

$$
\begin{aligned}
y_1'(t) &= y_2(t),\\
y_2'(t) &= y_3(t),\\
y_3'(t) &= \frac{t^2}{16} - \frac{1}{2}y_3^2(t) - \lambda_1 y_1(t)
\end{aligned}
\tag{31}
$$

together with the boundary conditions

$$
\begin{gathered}
y_2(1)y_2(0) + \lambda_1 y_1(1) = \frac{1}{32},\\
y_1(1)y_2(0) + \lambda_2 y_2(0) + \lambda_2 y_3(1) = \frac{1}{16},\\
\frac{1}{2}y_2(0) + \left(\frac{1}{2} - \lambda_1\right) y_2(1) = 0,\\
y_1(0) = -\frac{1}{16}, \qquad y_2(0) = 0.
\end{gathered}
\tag{32}
$$

One can verify that, for problem (31), (32), conditions (4), (15), and (17) are fulfilled with $(t, y_2, y_2, \lambda_1, \lambda_2) \in [0,1]\times G\times I_1\times I_2$, $\lambda_1 \in I_1 := [0,1]$, $\lambda_2 \in I_2 := [0,1]$, $A := B := E_3 := \operatorname{diag}(1,1,1)$, $K := \left[\begin{smallmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & \frac{1}{3}\end{smallmatrix}\right]$, and

$$
G := \left\{ (y_1, y_2, y_3) \quad : \quad |y_1| \le \frac{1}{2},\ |y_2| \le \frac{1}{2},\ |y_3| \le \frac{1}{3}, \right\},
$$

because, in this case, $r(K) = 0.9$,

$$
\delta_G(f) \le \begin{pmatrix} \frac{1}{2}\\ \frac{1}{3}\\ \frac{53}{144}\end{pmatrix},
$$

and

$$
\beta(x) = \frac{T}{2}\delta_G(f) + \left|(B^{-1}A + E_3)x\right| \le \begin{pmatrix} \frac{1}{4}\\ \frac{1}{6}\\ \frac{53}{288}\end{pmatrix} + 2|x|.
$$

Substitution (5) brings (31) to the form

$$
\begin{aligned}
x_1'(t) &= x_2(t) + w_2,\\
x_2'(t) &= x_3(t) + w_3,\\
x_3'(t) &= \frac{t^2}{16} - \frac{1}{2}(x_3 + w_3)^2(t) - \lambda_1\left(x_1(t) + w_1\right),\\
x_1(0) &= -\frac{1}{16} - w_1, \qquad x_2(0) = -w_2.
\end{aligned}
$$

The computation performed according to (18) shows that the components of the first iteration have the form

$$x_{1,1}(t,\, w_1,\, w_2,\, w_3,\, u,\, \lambda_1,\, \lambda_2) = -\frac{1}{16} - w_1 + \frac{1}{8}\,t + 2\,t\,w_1,$$
$$x_{1,2}(t,\, w_1,\, w_2,\, w_3,\, u,\, \lambda_1,\, \lambda_2) = -w_2 + 2\,t\,w_2,$$

and

$$x_{1,3}(t,\, w_1,\, w_2,\, w_3,\, u,\, \lambda_1,\, \lambda_2) = u + \frac{1}{48}t^3\lambda_2 - \frac{1}{48}t\lambda_2 + \frac{1}{64}t^3 - \frac{1}{64}t - 2\,u\,t,$$

where $x_m = \operatorname{col}(x_{m,1}, x_{m,2}, x_{m,3})$.

Similarly, for the second iteration, we have the first

$$x_{2,1}(t,\, w_1,\, w_2,\, w_3,\, u,\, \lambda_1,\, \lambda_2) = -\frac{1}{16} - w_1 + w_2\,t^2 - t\,w_2 + \frac{1}{8}\,t + 2\,t\,w_1,$$

the second

$$\begin{aligned} x_{2,2}(t,\, w_1,\, w_2,\, w_3,\, u,\, \lambda_1,\, \lambda_2) = -w_2 + \frac{1}{192}\,t^4\,\lambda_2 + \frac{1}{256}\,t^4 - t^2\,u - \frac{1}{96}\,t^2\,\lambda_2 \\ - \frac{1}{128}\,t^2 + u\,t + 2\,t\,w_2, \end{aligned}$$

and the third

$$\begin{aligned} x_{2,3}(t,\, w_1,\, w_2,\, w_3,\, u,\, \lambda_1,\, \lambda_2) &:= -\frac{1}{256}\,w_3\,t - \frac{1679}{107520}\,t + u + t\,\lambda_1\,w_1 \\ &- \frac{1}{192}\,t\,w_3\,\lambda_2 + \frac{1}{2880}\,t\,u\,\lambda_2 - t\,w_3\,u + \frac{1}{96}\,t^2\,w_3\,\lambda_2 + t^2\,w_3\,u \\ &+ \frac{1}{96}\,t^2\,u\,\lambda_2 - t^2\,\lambda_1\,w_1 - \frac{1}{72}\,t^3\,u\,\lambda_2 - \frac{1}{192}\,t^4\,w_3\,\lambda_2 \\ &- \frac{1}{192}\,t^4\,u\,\lambda_2 + \frac{1}{60480}\,t\,{\lambda_2}^2 + \frac{1}{120}\,t^5\,u\,\lambda_2 - \frac{1}{32256}\,t^7\,{\lambda_2}^2 \\ &- \frac{1}{21504}\,t^7\,\lambda_2 - \frac{1}{256}\,t^4\,u - \frac{1}{256}\,t^4\,w_3 + t^2\,u^2 + \frac{1}{128}\,t^2\,w_3 \\ &- \frac{1}{3}\,t\,u^2 + \frac{1}{16}\,\lambda_1\,t - \frac{7679}{3840}\,u\,t - \frac{839}{40320}\,t\,\lambda_2 + \frac{191}{9216}\,t^3\,\lambda_2 \\ &+ \frac{383}{24576}\,t^3 + \frac{1}{128}\,t^2\,u + \frac{1}{20480}\,t^5 - \frac{1}{57344}\,t^7 + \frac{1}{160}\,t^5\,u \\ &+ \frac{1}{11520}\,t^5\,{\lambda_2}^2 + \frac{1}{7680}\,t^5\,\lambda_2 - \frac{1}{13824}\,t^3\,{\lambda_2}^2 - \frac{2}{3}\,t^3\,u^2 - \frac{1}{96}\,t^3\,u \\ &- \frac{1}{16}\,t^2\,\lambda_1 \end{aligned}$$

components of the function x_2.

Solving the approximate determining equations (28) gives us the approximate values of the unknown parameters. More precisely, we have

$$w_1 = 0,\ w_2 \approx .1250000000,\ w_3 \approx .2552083572$$
$$\lambda_1 = \frac{1}{2},\ \lambda_2 \approx .2500045836,\ u = \frac{-1 + 16\,w_3\,\lambda_2}{16\lambda_2} \approx .005212940674$$

for $m = 1$ and

$$w_1 = 0,\ w_2 \approx .127331555,\ w_3 \approx .2547074002$$
$$\lambda_1 = \frac{1}{2},\ \lambda_2 \approx .2458952578,\ u = \frac{-1 + 16\, w_3\, \lambda_2}{16\lambda_2} \approx .2458952578$$

for $m = 2$.

Therefore, in the first approximation, the solution of parametrised problem (29), (30) is

$$\begin{gathered} y_{1,1}(t) = -\frac{1}{16} + \frac{1}{8}\,t, \quad t \in [0,1], \\ \lambda_1 = \frac{1}{2},\ \lambda_2 \approx .2500045836 \end{gathered} \tag{33}$$

and, in the second approximation,

$$\begin{gathered} y_{2,1}(t) \approx -\frac{1}{16} + .1273315558\, t^2 - .0023315558\, t, \quad t \in [0,1], \\ \lambda_1 = \frac{1}{2},\ \lambda_2 \approx .2458952578. \end{gathered} \tag{34}$$

Note that

$$\begin{gathered} y(t) = \frac{t^2}{8} - \frac{1}{16}, \quad t \in [0,1], \\ \lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{4} \end{gathered} \tag{35}$$

is an exact solution of problem (29), (30). Computation by using `Maple` shows that (33) and (34) provide good enough approximations to (35).

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METHOD OF LOWER AND UPPER SOLUTIONS FOR A GENERALIZED BOUNDARY VALUE PROBLEM

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Abstract. A method of lower and upper solutions is used to prove the existence of a solution of a boundary value problem with generalized boundary conditions given by continuous linear functionals. The cases of Dirichlet, Neumann, multipoint and integral conditions are covered.

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The method of lower and upper solutions is, in connection with the topological degree theory, widely used to prove the existence or multiplicity results for various types of boundary value problems. See [1] – [8].

The aim of this paper is to extend the method of lower and upper solutions to the case of boundary conditions given by the continuous linear functionals. Such conditions are given by Riemann-Stjeltjes integrals.

We consider the second order differential equation

$$x'' = f(t, x, x') \tag{1}$$

with the generalized boundary conditions

$$\begin{aligned} x(a) &= \int_a^b x(t)\, dg_1(t) + k_1 x'(a) \\ x(b) &= \int_a^b x(t)\, dg_2(t) - k_2 x'(b), \end{aligned} \tag{2}$$

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where $f : I = [a,b] \times R^2 \to R$ is a continuous function, $g_i(t)$ are nondecreasing functions with bounded variation, $1 \geq g_i(b) - g_i(a)$ and $k_i \geq 0$.

We assume that g_i, k_i, are such that the boundary conditions are linearly independent. Our purpose is to extend some existence results of [6] to the case of the problem (1), (2).

Definition 1. The function $\alpha(t)$ is called a lower solution for the problem (1), (2) if

$$\begin{aligned} \alpha''(t) &\geq f(t, \alpha(t), \alpha'(t)), \\ \alpha(a) &\leq \int_a^b \alpha(t)\, dg_1(t) + k_1 \alpha'(a) \\ \alpha(b) &\leq \int_a^b \alpha(t)\, dg_2(t) - k_2 \alpha'(b), \end{aligned} \tag{3}$$

Similarly the function $\beta(t)$ is called an upper solution for the problem (1), (2) if

$$\begin{aligned} \beta''(t) &\leq f(t, \beta(t), \beta'(t)), \\ \beta(a) &\geq \int_a^b \beta(t)\, dg_1(t) + k_1 \beta'(a) \\ \beta(b) &\geq \int_a^b \beta(t)\, dg_2(t) - k_2 \beta'(b), \end{aligned} \tag{4}$$

If the strict inequalities for α'', β'' hold α, β are called strict lower and upper solutions.

Remark 1. In the case of Dirichlet conditions $x(a) = x(b) = 0$, continuity of the function f implies that for $\varepsilon > 0$ sufficiently small $\alpha(t) - \varepsilon$, $\beta(t) + \varepsilon$ are strict lower and upper solutions satisfying the strict inequalities (3), (4).

Therefore below we assume that in the case of Dirichlet conditions the strict lower and upper solutions satisfy also the strict inequalities (3), (4).

Lemma 1. *[8, p. 214] Let $h(s)$ be a positive continuous function such that*

$$\int^{\infty} \frac{s}{h(s)} ds = \infty, \tag{5}$$

f be a continuous function satisfying

$$|f(t,x,y)| \leq h(|y|) \quad \text{for each } |x| \leq r,\ t \in I,$$

and let $x(t)$ be a solution of the problem (1), (2) such that $\|x\| \leq r$. Then there is a constant $\rho_0 > 0$ such that $\|x'\| < \rho_0$.

Lemma 2. *Let α, β be a strict lower and upper solutions and $u(t)$ be a solution of the problem (1), (2).*

Then $\alpha(t) \leq u(t)$ implies $\alpha(t) < u(t)$ and $\beta(t) \geq u(t)$ implies $\beta(t) > u(t)$.

Proof. Let $0 = u(t_0) - \beta(t_0)$ at $t_0 \in (a,b)$. Then

$$0 \geq u(t_0)'' - \beta(t_0)'' = f(t_0, u(t_0), u'(t_0)) - \beta(t_0)'' \geq f(t_0, \beta(t_0), \beta'(t_0)) - \beta(t_0)'' > 0,$$

a contradiction.

Let $0 = u(a) - \beta(a)$, $u(t) < \beta(t)$ for $t \in (a,b)$. If $u'(a) = \beta'(a)$ we obtain the same contradiction as above. Suppose $u'(a) < \beta'(a)$.

We consider several cases.

Let $k_1 > 0$. Then

$$u(a) - \beta(a) < \int_a^b u(t) - \beta(t)\, dg_1(t) \leq (g_1(b) - g_1(a)) \max_{t \in I} (u(t) - \beta(t)) \leq 0,$$

a contradiction.

Let $k_1 = 0$. If g_1 is nonconstant on a subinterval $[c,d] \subset (a,b)$ then

$$u(a) - \beta(a) \leq \int_a^b u(t) - \beta(t)\, dg_1(t) < (g_1(b) - g_1(a)) \max_{t \in I} (u(t) - \beta(t)) \leq 0,$$

a contradiction.

If g_1 is constant on $(a,b]$ then the first condition of (2) is reduced to Dirichlet condition. With respect to Remark 1 we assume $\beta(a) > 0$. Then $u(a) - \beta(a) < 0$, a contradiction.

If g_1 is constant on $[a,b)$ then $u(a) - \beta(a) \leq c(u(b) - \beta(b))$, $c \leq 1$. That means $u(a) = \beta(a)$ implies $u(b) = \beta(b)$. Using the boundary condition at point b and considering the same cases as above we obtain a contradiction with the equality $u(b) - \beta(b) = 0$. The last case g_2 is constant on $(a,b]$ leads either to the Dirichlet conditions case, or to the linear dependance of boundary conditions.

Let $X = C^1(I)$, $\operatorname{dom} L = \{x(t) \in C^2(I),\ x \text{ satisfies } (2)\}$, $Z = C(I)$. We denote

$$L : \operatorname{dom} L \subset X \to Z, \qquad Lx = x'',$$
$$N : X \to Z, \qquad Nx(t) = f(t, x(t), x'(t)).$$

The problem (1), (2) is equivalent to the operator equation

$$Lx = Nx,$$

where the operator N is L-compact [2].

We denote

$$\Omega_{r,\rho} = \{x(t) \in C^1(I), \quad \|x\| < r, \quad \|x'\| < \rho\}.$$

Lemma 3. *Let*

(i) there is a constant $r > 0$ such that $f(t, r, 0) > 0$ and $f(t, -r, 0) < 0$,

(ii) $|f(t,x,y)| \leq h(|y|)$, $h \geq \varepsilon > 0$ satisfies (5), for each $t \in I$, $|x| < r$.

Then there is $\rho_0 > 0$ such that the topological degree

$$D(L, N, \Omega_{r,\rho}) = 1 \qquad (\operatorname{mod} 2)$$

for each $\rho > \rho_0$ i.e. there is a solution $x(t)$ of (1), (2) such that $|x(t)| < r$, $|x'(t)| < \rho$.

Proof. We consider the homotopy

$$Lx = \tilde{N}(x, \lambda)$$

defined by the parametric system of equations

$$x'' = \lambda f(t, x, y) + (1 - \lambda)x, \qquad (2). \tag{6}$$

Now $-r$, r are a strict lower and upper solutions of the problem (6).

As $|\lambda f(t, x, y) + (1-\lambda)x| \leq h(|y|) + r$, the assumptions of Lemma 1 are satisfied for the function $\lambda f(t, x, y) + (1 - \lambda)x$. Then the a priori bound of derivative and Lemma 2 imply that no solution of (6) lies on the boundary of $\partial \Omega_{r,\rho}$, $\rho \geq \rho_0$.

By the generalized Borsuk theorem [3]

$$D(L, \tilde{N}(., 1), \Omega_{r,\rho}) = D(L, \tilde{N}(., 0), \Omega_{r,\rho}) = 1 \qquad (\mathrm{mod}\ 2)$$

and Lemma 3 is proved.

Theorem 1. *Let*

(i) $\alpha(t) < \beta(t)$ be a lower and upper solutions of the problem (1), (2).

(ii) $|f(t, x, y)| \leq h(|y|)$, for each (t, x, y), $t \in I$, $\alpha(t) \leq x \leq \beta(t)$, $y \in R$, where $h \geq \varepsilon > 0$ satisfies (5),

Then there is a constant ρ_0 such that for each $\Omega = \{x(t) \in C^1(I), \quad \alpha(t) < x(t) < \beta(t), \quad \|x'\| < \rho\}$, $\rho > \rho_0$ there is a solution $x \in \bar{\Omega}$ of (1), (2).

Moreover if $\alpha(t)$, $\beta(t)$ are strict lower and upper solutions then

$$D(L, N, \Omega) = 1 \qquad (\mathrm{mod}\ 2).$$

Proof. Let $r = \max\{\|\alpha\|, \|\beta\|\}$, $M > \max|f(t, x, 0)|$ for $t \in I$, $|x| \leq r$.

We define a perturbation

$$f^*(t, x, y) = \begin{cases} f(t, \beta(t), y) + M(r - \beta(t)) + M & x > r + 1, \\ f(t, \beta(t), y) + M(x - \beta(t)) & \beta(t) < x \leq r + 1, \\ f(t, x, y) & \alpha(t) \leq x \leq \beta(t), \\ f(t, \alpha(t), y) - M(\alpha(t) - x) & -r - 1 \leq x < \alpha(t), \\ f(t, \alpha(t), y) - M - M(\alpha(t) + r) & x < -r - 1. \end{cases}$$

The function f^* satisfies the Nagumo condition as well as the assumptions of Lemma 3 for $\Omega_{r+1,\rho}$, $\rho > \rho_0$ where ρ_0 is a constant from Lemma 1 for the function f^*.

Suppose $u(t) \in \Omega_{r+1,\rho}$ is a solution of the problem

$$x'' = f^*(t, x, x'), \quad (2). \tag{7}$$

We show that $\alpha \leq u \leq \beta$.

Let $v(t) = u(t) - \beta(t)$ attains its maximum $v_{max} > 0$. Then $\beta(t) + v_{max}$ is a strict upper solution of (7). Lemma 2 implies $u(t) < \beta(t) + v_{max}$ a contradiction.

That means $u(t)$ is a solution of (1), (2).

If $\alpha(t)$, $\beta(t)$ are a strict lower and upper solutions then moreover

$$D(L, N^*, \Omega_{r,\rho}) = D(L, N^*, \Omega) = D(L, N, \Omega) = 1 \qquad (\text{mod } 2).$$

Theorem 2. *Let*

(i) $|f(t, x, y)| < M$,

(ii) α, β, $\beta(t) < \alpha(t)$, *be a strict lower and upper solutions for the problem* (1), (2).

Then there are constants $r, \rho > 0$ *such that*

$$D(L, N, \Omega) = 1 \qquad (\text{mod } 2)$$

where $\Omega = \{x(t) \in C^1(I),\ \exists t_x \in I,\ \beta(t_x) < x(t_x) < \alpha(t_x),\ \|x\| < r,\ \|x'\| < \rho\}$.

Proof. Let $\rho = (b - a)2M$ and $r = \max(\|\alpha\|, \|\beta\|) + (b - a)\rho$.

We define a perturbation f^* by

$$f^*(t, x, y) = \begin{cases} f(t, x, y) + M & x > r + 1, \\ f(t, x, y) + M(x - r) & r < x \leq r + 1, \\ f(t, x, y) & -r \leq x \leq r, \\ f(t, x, y) + M(x + r) & -r - 1 \leq x < -r, \\ f(t, x, y) - M & x < -r - 1. \end{cases}$$

Clearly $r + 1$, $-r - 1$ are a strict lower and upper solutions of the problem

$$x'' = f^*(t, x, x'), \quad (2). \tag{8}$$

As $|f^*| < 2M$ then for each solution of (8) the boundary conditions (2) imply that there is a constant ρ such that $|x'(t)| < \rho$.

Therefore

$$D(L, N^*, \Omega_{r+1,\rho}) = 1 \qquad (\text{mod } 2)$$

Let now

$$\begin{aligned} \Omega_l &= \{x(t) \in \Omega_{r+1,\rho}, \quad -r - 1 < x < \beta\}, \\ \Omega_u &= \{x(t) \in \Omega_{r+1,\rho}, \quad \alpha < x < r + 1\}. \end{aligned}$$

Then

$$D(L, N^*, \Omega_l) = D(L, N^*, \Omega_u) = 1 \qquad (\text{mod } 2)$$

Set $\Omega_m = \Omega_{r+1,\rho} \setminus \left(\overline{\Omega_l \cup \Omega_u}\right)$.

As $-r - 1$, α, $r + 1$, β are strict lower and upper solutions, Lemma 2 implies there is no solution $u \in \partial\Omega_m$.

The addition property of the degree means

$$D(L, N^*, \Omega_m) = 1 \qquad (\text{mod } 2)$$

on the set $\Omega_m = \Omega_{r+1} \setminus \left(\overline{\Omega_l \cup \Omega_u}\right)$, and finally the excision property implies

$$D(L, N^*, \Omega_m) = D(L, N^*, \Omega) = D(L, N, \Omega) = 1 \qquad (\text{mod } 2).$$

The Nagumo condition in Theorem 1 and the a priori bound of f in Theorem 2 are in the following theorems replaced by the one sided growth condition.

Theorem 3. *Let*

(i) $k_1 > 0, k_2 > 0$,

(ii) there is $M > 0$ *such that* $f(t, x, y) \leq M$ *for each* $t \in I$, *and each* $x, y \in R$.

(iii) α, β, $\alpha(t) < \beta(t)$, *be a strict lower and upper solutions of the problem* (1), (2).

Then there is $\rho_0 > 0$ *such that for each* $\rho > \rho_0$ *and* $\Omega = \{x(t) \in C^1(I),\ \alpha(t) < x(t) < \beta(t),\ \|x'\| < \rho\}$ *there is*

$$D(L, N, \Omega) = 1 \qquad (\mathrm{mod}\ 2).$$

Proof. Let $r = \max\{\|\alpha\|, \|\beta\|\}$.

Let $x(t)$ be a solution of (1), (2) such that $\|x\| < r$. Then the boundary conditions (2) imply $x'(a) \leq \dfrac{2r}{k_1}$ and $x'(b) \geq -\dfrac{2r}{k_2}$. Therefore $\|x'\| \leq \dfrac{2r}{k} + (b-a)M$, where $k = \min\{k_1, k_2\}$.

Let $\rho_1 = \dfrac{2r}{k} + (b-a)M + \max\{\|\alpha'\|, \|\beta'\|\}$.

We define

$$\chi(s, t) = \begin{cases} 1 & s \leq t \\ \frac{2t-s}{t} & t < s \leq 2t \\ 0 & s > 2t \end{cases}$$

and

$$f^* = \chi(\|x\|, r)\chi(\|y\|, \rho_1) f(t, x, y). \tag{9}$$

Now f^* is a bounded function and α, β, are strict lower and upper solutions of the problem

$$x'' = f^*(t, x, x'), \quad (2). \tag{10}$$

Theorem 1 implies that there is ρ_2 such that for each $\rho > \rho_2$

$$D(L, N^*, \Omega) = 1 \qquad (\mathrm{mod}\ 2).$$

We choose $\rho > \max\{\rho_1, \rho_2\} = \rho_0$. For each solution x of (10) such that $\|x\| < r$ there is $\|x'\| < \rho_1$. Then $f(t, x(t), x'(t)) = f^*(t, x(t), x'(t))$ and

$$D(L, N, \Omega) = D(L, N^*, \Omega) = 1 \qquad (\mathrm{mod}\ 2).$$

Theorem 4. *Let*

(i) $k_1, k_2 > 0$,

(ii) there is $M > 0$ *such that* $f(t, x, y) \leq M$ *for each* $t \in I$, *and each* $x, y \in R$.

(iii) α, β, $\beta(t) < \alpha(t)$, *be a strict lower and upper solutions of the problem* (1), (2).

Then there is $r, \rho > 0$ *such that*

$$D(L, N, \Omega) = 1 \qquad \pmod 2$$

where

$$\Omega = \{x(t) \in C^1(I),\ \exists t_x \in I\ \beta(t_x) < x(t_x) < \alpha(t_x),\ \|x\| < r,\ \|x'\| < \rho\}.$$

Proof. Let $m = \max\{\|\alpha\|, \|\beta\|\}$, $x(t)$ be a solution and let $\exists t_x \in I\ \beta(t_x) < x(t_x) < \alpha(t_x)$. Then $|x(t_x)| \leq m$.

Let t_1 be such that $\min x(t) = x(t_1)$, and suppose that $x(t_1) < -m$.

Let $t_1 < t_x$. Then either $x'(t_1) = 0$ or $t_1 = a$.

In the case $x'(t_1) = 0$ there is $x'(t) = \int_{t_1}^{t} x''(s)\, ds \leq (b - t_1)M$, for $t \geq t_1$. Then

$$x(t_1) = x(t_x) - \int_{t_1}^{t_x} x'(s)\, ds \geq -m - (t_x - t_1)(b - t_1)M.$$

If $t_1 = a$ then the boundary condition implies $x(a) > x(a)\,(g_1(b) - g_1(a)) + k_1 x'(a)$. Hence $k_1 x'(a) < (1 - (g_1(b) - g_1(a)))\, x(a)$ which implies $x'(a) < 0$, a contradiction.

Let $t_1 > t_x$. Then either $x'(t_1) = 0$ or $t_1 = b$.

Again $x'(t_1) = 0$ implies that $x'(t) = -\int_{t}^{t_1} x''(s)\, ds \geq -(t_1 - a)M$, for $t \leq t_1$. Then

$$x(t_1) = x(t_x) + \int_{t_x}^{t_1} x'(s)\, ds \geq -m - (t_1 - a)(b - t_x)M.$$

If $t_1 = b$ then $x(b) > x(b)\,(g_2(b) - g_2(a)) - k_2 x'(b)$ i.e. $k_2 x'(b) > -(1 - (g_2(b) - g_2(a)))x(b)$ which implies $x'(b) > 0$, a contradiction.

That means $x(t) > -m - (b - a)^2 M$.

Suppose that there is t_2 such that $\max x(t) = x(t_2) > m$.

Case $t_2 > t_1$.

There is $x'(t) = x'(t_1) + \int_{t_1}^{t} x''(s)\, ds \leq (t_2 - t_1)M$, for $t \in [t_1, t_2]$, and

$$x(t_2) = x(t_1) + \int_{t_1}^{t_2} x'(s)\, ds \leq m + (t_2 - t_1)^2 M.$$

Case $t_2 < t_1$.

There is $x'(t) = x'(t_1) - \int_{t}^{t_1} x''(s)\, ds \geq -(t_1 - t_2)M$, for $t \in [t_2, t_1]$, and

$$x(t_2) = x(t_1) - \int_{t_2}^{t_1} x'(s)\, ds \leq m + (t_2 - t_1)^2 M.$$

The above estimations give a priori bound of a solution

$$|x(t)| < r = m + (b - a)^2 M.$$

Arguing as in the proof of the preceeding theorem we obtain that

$$|x'(t)| \le \frac{2r}{k} + (b-a)M,$$

where again $k = \min\{k_1, k_2\}$ and we put $\rho_1 = \dfrac{2r}{k} + (b-a)M + \max\{\|\alpha'\|, \|\beta'\|\}$.

Using again the perturbation (9) and Theorem 2 we obtain that there is ρ_2 such that for each $\rho > \rho_2$

$$D(L, N^*, \Omega) = 1 \qquad (\mathrm{mod}\ 2),$$

where

$$\Omega = \{x(t) \in C^1(I),\ \exists t_x \in I\ \beta(t_x) < x(t_x) < \alpha(t_x),\ \|x\| < r,\ \|x'\| < \rho\}.$$

We choose $\rho > \max(\rho_1, \rho_2) = \rho_0$. A priori bounds of solutions imply

$$D(L, N, \Omega) = D(L, N^*, \Omega) = 1 \qquad (\mathrm{mod}\ 2).$$

Remark 2. It is possible to replace the inequality in the condition (ii) of Theorem 3 and 4 by $f(t, x, y) \ge -M$ for each $t \in I$, $x, y \in R$.

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ON PARABOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS IN UNBOUNDED DOMAINS

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Abstract. We shall consider weak solutions of initial–boundary value problems for nonlinear parabolic functional differential equations containing discontinuous terms in the unknown function. There will be proved the existence of solutions and formulated some properties of the solutions.

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1. Introduction

We shall consider initial-boundary value problems for the equation

$$\begin{aligned} &D_t u(t,x) - \sum_{j=1}^{n} D_j[f_j(t,x,u(t,x),\nabla u(t,x))] + f_0(t,x,u(t,x),\nabla u(t,x)) + \\ &\qquad + g(t,x,u(t,x)) + h(t,x,[H(u)](t,x)) = F(t,x), \\ &\qquad\qquad (t,x) \in Q_T = (0,T) \times \Omega \end{aligned} \tag{1}$$

where $\Omega \subset R^n$ is a (possibly unbounded) domain with sufficiently smooth boundary, H is a linear continuous operator in $L^p(Q_T)$, the functions f_j are measurable in (t,x), continuous with respect to $u(t,x)$, $\nabla u(t,x)$ but the functions g, h are assumed to be only measurable in all variables. Further, f_j, g, h have certain polynomial growth in $u(t,x)$, $\nabla u(t,x)$. The case when Ω is bounded, was considered, e.g., in [11] where certain terms were rapidly increasing in $u(t,x)$. In [13] there were considered equations of more general form where all the terms were continuous in $u(t,x)$ and $\nabla u(t,x)$.

The problem was motivated by the climate model considered by J.I. Díaz and G. Hetzer in [8] where a particular case of the equation (1) (also with discontinuous terms in u) was investigated on the unit sphere in R^3 (instead of Ω). Some qualitative properties of the solutions of the climate model (without delay terms) were proved in [1] and [7]. Functional partial differential equations arise also in population dynamics, plasticity, hysteresis (see, e.g., [2], [4], [10], [15]).

The aim of this work is to formulate and prove new results in the case of unbounded Ω. We shall formulate conditions which imply the existence of weak solutions of initial-boundary value problems for (1) and to show that in the case of unbounded Ω, the limit of solutions of problems in large bounded domains is a solution of the problem in Ω. There will also be proved the boundedness of the solutions under some conditions and a theorem on the stabilization of the solutions as $t \to \infty$. Our results can be easily extended to equations, containing higher order derivatives with respect to x.

2. Existence theorems

Let $\Omega \subset R^n$ be a (possibly unbounded) domain with sufficiently smooth boundary, $p \geq 2$. Denote by $W^{1,p}(\Omega)$ the usual Sobolev space with the norm

$$\| u \| = \left[\int_\Omega (\sum_{j=1}^n |D_j u|^p + |u|^p) \right]^{1/p}.$$

Let V be a closed linear subspace of $W^{1,p}(\Omega)$ and denote by $X_T = L^p(0,T;V)$ the Banach space of the set of measurable functions $u : (0,T) \to V$ such that $\| u \|^p$ is integrable. The dual space of $L^p(0,T;V)$ is $X_T^\star = L^q(0,T;V^\star)$ where $1/p + 1/q = 1$ and $V^\star$ is the dual space of V (see, e.g., [14]).

On functions f_j we assume that

A (i) $f_j : Q_T \times R^{n+1} \to R$ are measurable in $(t,x) \in Q_T$ and continuous in $\eta \in R, \zeta \in R^n$;

(ii) $|f_j(t,x,\eta,\zeta)| \leq c_1(|\eta|^{p-1} + |\zeta|^{p-1}) + k_1(x)$ with some constant c_1 and a function $k_1 \in L^q(\Omega)$ $(j = 0,1,...,n)$;

(iii) $\sum_{j=1}^n [f_j(t,x,\eta,\zeta) - f_j(t,x,\eta,\tilde{\zeta})](\zeta_j - \tilde{\zeta}_j) > 0$ if $\zeta \neq \tilde{\zeta}$;

(iv) $\sum_{j=1}^n f_j(t,x,\eta,\zeta)\zeta_j + f_0(t,x,\eta,\zeta)\eta \geq c_2[|\zeta|^p + |\eta|^p] - k_2(x)$ with some constant $c_2 > 0$ and $k_2 \in L^1(\Omega)$.

Remark 1. A simple example for f_j, satisfying A (i) - (iv) is

$$f_j(t,x,\eta,\zeta) = a_j(t,x)\zeta_j|\zeta_j|^{p-2}, \quad j = 1,...,n,$$

$$f_0(t,x,\eta,\zeta) = a_0(t,x)\eta|\eta|^{p-2},$$

where a_j are measurable functions, satisfying $0 < c_0 \le a_j(t,x) \le c_0'$ with some constants c_0, c_0'.

On functions g, h we assume that

B (i) $g = g_1 + g_2$, $g_j : Q_T \times R \to R$ and $h : Q_T \times R \to R$ are measurable functions;

(ii) $|g_1(t,x,\eta)| \le k_3(x)|\eta|^{p-1}$ and $g_1(t,x,\eta)\eta \ge 0$ with some function $k_3 \in L^1(\Omega) \cup |L^\infty(\Omega)$;

(iii)

$$|g_2(t,x,\eta)| \le k_3(x)k_4(|\eta|)|\eta|^{p-1} + k_5(x), \quad |h(t,x,\theta)| \le k_3(x)k_4(|\theta|)|\theta|^{p-1} + k_5(x)$$

where $k_5 \in L^q(\Omega)$ and k_4 is a continuous function, satisfying $\lim_\infty k_4 = 0$.

Further,

C $H : L^p(Q_T) \to L^p(Q_T)$ is a linear and continuous operator such that for any compact $K \subset \Omega$ there is a compact $\tilde{K} \subset \Omega$ with the following property: the restriction of $H(u)$ to $(0,t) \times K$ depends only on the restriction of u to $(0,t) \times \tilde{K}$ for all $t \in (0,T]$.

Remark 2. The operator H may have e.g. one of the following forms:

$$[H(u)](t,x) = \int_0^t \beta_0(s,t,x)u(s,x)ds \text{ or } [H(u)](t,x) = u(\tau(t),x)$$

with some $\beta_0 \in L^\infty((0,T) \times Q_T)$ and a continuously differentiable function τ satisfying $\tau' > 0, 0 < \tau(t) \le t$.

Since g_1 is locally bounded, for any $\epsilon > 0$ we may define (with fixed $(t,x) \in Q_T$)

$$\bar{g}_1^\varepsilon(t,x,\eta) = \text{ess sup}_{|\eta-\tilde{\eta}|<\varepsilon} g_1(t,x,\tilde{\eta}),$$
$$\underline{g}_1^\varepsilon(t,x,\eta) = \text{ess inf}_{|\eta-\tilde{\eta}|<\varepsilon} g_1(t,x,\tilde{\eta})$$

For fixed t,x,η $\bar{g}_1^\varepsilon(t,x,\eta)$ is nonincreasing and $\underline{g}_1^\varepsilon(t,x,\eta)$ is nondecreasing as ε is decreasing thus

$$\bar{g}_1(t,x,\eta) = \lim_{\varepsilon\to 0} \bar{g}_1^\varepsilon(t,x,\eta), \quad \underline{g}_1(t,x,\eta) = \lim_{\varepsilon\to 0} \underline{g}_1^\varepsilon(t,x,\eta)$$

exist. Similarly may be defined $\bar{g}_2, \underline{g}_2, \bar{h}, \underline{h}$ (by functions g_2, h, respectively).

Theorem 1. *Assume* **A** *(i) - (iv) and* **B** *(i) - (iii) and* **C**. *Then for each $F \in X_T^\star, u_0 \in V$ there exists $u \in X_T$ with $D_t u \in X_T^\star$ and $\varphi_1, \varphi_2, \psi \in L^q(Q_T)$ such that*

(2) $$u(0,\cdot) = u_0,$$

for arbitrary $v \in V$ we have

(3)

$$\langle D_t u(t,\cdot), v\rangle + \sum_{j=1}^n \int_\Omega f_j(t,x,u(t,x),\nabla u(t,x))D_j v(x)dx +$$
$$\int_\Omega f_0(t,x,u(t,x),\nabla u(t,x))v(x)dx + \int_\Omega [\varphi_1(t,x) + \varphi_2(t,x) + \psi(t,x)]v(x)dx =$$
$$\langle F(t,\cdot), v\rangle \textit{ for a.e. } t \in [0,T]$$

and for a.e. $(t,x) \in Q_T$

$$\underline{g}_l(t,x,u(t,x)) \le \varphi_l(t,x) \le \bar{g}_l(t,x,u(t,x)), \quad l=1,2 \tag{4}$$

$$\underline{h}(t,x,[H(u)](t,x)) \le \psi(t,x) \le \bar{h}(t,x,[H(u)](t,x)).$$

Proof. Consider the function $j \in C_0^\infty(R)$ supported by $[-1,1]$ with the properties $j \ge 0$, $\int_R j = 1$ and for any positive integer k define the functions j_k by $j_k(\eta) = kj(k\eta)$. Then the convolutions (with fixed t,x) $g_{l,k} = g_l \star j_k \quad (l=1,2), \quad h_k = h \star j_k$ are smooth functions (of η, θ, respectively). Further, define functions

$$\tilde{g}_{l,k}(t,x,\eta) = g_{l,k}(t,x,\eta) \text{ if } |x| \le k, \quad \tilde{g}_{l,k}(t,x,\eta) = 0 \text{ if } |x| > k$$

$$\tilde{h}_k(t,x,\theta) = h_k(t,x,\theta) \text{ if } |x| \le k, \quad \tilde{h}_k(t,x,\theta) = 0 \text{ if } |x| > k.$$

Then we may define operators $A, B_k, C_k : X_T \to X_T^\star$ by

$$[A(u),v] = \int_0^T \langle A(u)(t), v(t)\rangle dt,$$

$$\langle A(u)(t), v(t)\rangle = \sum_{j=1}^n \int_\Omega f_j(t,x,u,\nabla u) D_j v dx + \int_\Omega f_0(t,x,u,\nabla u) v dx,$$

$$[B_k^l(u),v] = \int_0^T \langle B_k^l(u)(t), v(t)\rangle dt = \int_{Q_T} \tilde{g}_{l,k}(t,x,u) v dt dx, \quad l=1,2,$$

$$[B_k(u),v] = [B_k^1(u),v] + [B_k^2(u),v],$$

$$[C_k(u),v] = \int_0^T \langle C_k(u)(t), v(t)\rangle dt = \int_{Q_T} \tilde{h}_k(t,x,H(u)) v dt dx, \quad u,v \in X_T.$$

By using the assumptions of our theorem, Hölder's inequality and Vitali's theorem it is not difficult to show that the operator $A+B_k+C_k : X_T \to X_T^\star$ is bounded (i.e. it maps bounded sets of X_T into bounded sets of $X_T^\star$) and demicontinuous, i.e.

$$(u_l) \to u \text{ in } X_T \text{ implies } (A+B_k+C_k)(u_l) \to (A+B_k+C_k)(u) \text{ weakly in } X_T^\star.$$

Further, by using compact imbedding theorems we obtain (as in [12]) that $A+B_k+C_k$ is pseudomonotone with respect to

$$D(L) = \{u \in X_T : D_t u \in X_T^\star, u(0) = 0\},$$

i.e. if $u_l, u \in D(L)$,

$$(u_l) \to u \text{ weakly in } X_T, \quad (D_t u_l) \to D_t u \text{ weakly in } X_T^\star \text{ and}$$

$$\limsup_{l\to\infty} [(A+B_k+C_k)(u_l), u_l - u] \le 0$$

then

$$(A + B_k + C_k)(u_l) \to (A + B_k + C_k)(u) \text{ weakly in } X_T^\star \text{ and}$$
$$\lim_{l\to\infty} [(A + B_k + C_k)(u_l), u_l - u] = 0.$$

Finally, we show that $A + B_k + C_k$ is coercive, i.e.

$$\lim_{\|u\|\to\infty} \frac{[(A + B_k + C_k)(u), u]}{\| u \|_{X_T}} = +\infty. \tag{5}$$

Assumption **A** (iv) implies

$$\int_0^t \langle A(u)(\tau), u(\tau)\rangle d\tau \geq c_2 \| u \|_{X_t}^p - t \int_\Omega k_2. \tag{6}$$

By **B** (ii)

$$\tilde{g}_{1,k}(t, x, \eta)\eta \geq 0 \text{ if } |\eta| > 1, \quad \tilde{g}_{1,k}(t, x, \eta)\eta \geq -k_3(x) \text{ if } |\eta| \leq 1$$

thus

$$\int_0^t \langle B_k^1(u)(\tau), u(\tau)\rangle d\tau \geq -t \int_\Omega k_3. \tag{7}$$

Let $a > 0$ be an arbitrary number. Since $\lim_\infty k_4 = 0$, there exists $b > 0$ such that $|\eta| \geq b$ implies $k_4(|\eta|) \leq a$. Hence, by using the notation $Q_t^b = \{(\tau, x) \in Q_t : |u(\tau, x)| \leq b\}$ we obtain from **B** (iii)

$$| \int_0^t \langle B_k^2(u)(\tau), u(\tau)\rangle d\tau| \leq \tag{8}$$

$$| \int_{Q_t^b} \tilde{g}_{2,k}(\tau, x, u)u d\tau dx| + | \int_{Q_t \setminus Q_t^b} \tilde{g}_{2,k}(\tau, x, u)u d\tau dx| \leq$$

$$C(a) + a \| k_3 \|_{L^\infty(\Omega)} \| u \|_{X_t}^p + \left[t \int_\Omega |k_5|^q \right]^{1/q} \| u \|_{X_t}$$

with a constant $C(a)$ (not depending on u).

One gets similarly

$$| \int_0^t \langle C_k(u)(\tau), u(\tau)\rangle d\tau| \leq \tag{9}$$

$$C(a) + a \| k_3 \|_{L^\infty(\Omega)} \| u \|_{X_t}^p + \left[t \int_\Omega |k_5|^q \right]^{1/q} \| u \|_{X_t} .$$

Choosing sufficiently small $a > 0$, from (6) - (9) we obtain for all $t \in [0, T]$

$$\int_0^t \langle (A + B_k + C_k)(u)(\tau), u(\tau)\rangle d\tau \geq c_2/2 \| u \|_{X_t}^p - c_2' \| u \|_{X_t} - c_3' \tag{10}$$

(with some constants c_2', c_3', not depending on u) which implies (5) since $p \geq 2$.

Thus, by Theorem 4 of [3], for any $F \in X_T^\star, u_0 \in V$ there exists $u_k \in X_T$ such that $D_t u_k \in X_T^\star$ and

$$D_t u_k + (A + B_k + C_k)(u_k) = F, \tag{11}$$

$$u_k(0) = u_0. \tag{12}$$

Since

$$\langle D_t u_k(t), u_k(t)\rangle = \frac{1}{2}\frac{d}{dt}\langle u_k(t), u_k(t)\rangle = \frac{1}{2}\frac{d}{dt}(u_k(t), u_k(t))_{L^2(\Omega)}$$

(see, e.g., [14]), applying both sides of (11) to u_k , we find by (10), (12)

$$\begin{aligned} 1/2 \parallel u_k(t) \parallel_{L^2(\Omega)}^2 -1/2 \parallel u_0 \parallel_{L^2(\Omega)}^2 +c_2/2 \parallel u_k \parallel_{X_t}^p \leq \\ [\parallel F \parallel_{X_T^\star} + c_2'] \parallel u_k \parallel_{X_t} + c_3', \quad t \in [0, T]. \end{aligned} \tag{13}$$

This inequality implies that

$$\parallel u_k \parallel_{X_T}, \parallel u_k \parallel_{L^\infty(0,T;L^2(\Omega))} \text{ are bounded.} \tag{14}$$

Hence the sequence $(A+B_k+C_k)(u_k)$ is bounded in $X_T^\star$ and so $(D_t u_k)$ is bounded in $X_T^\star$, too.

Consequently, there exist $u \in X_T$, $w \in X_T^\star$, $\varphi_l, \psi \in L^q(Q_T)$ and a subsequence of (u_k), again denoted by (u_k) such that

$$(u_k) \to u \text{ weakly in } X_T, \tag{15}$$

$$(u_k) \to u \text{ in } L^p((0,T) \times \Omega_0) \text{ for each fixed bounded } \Omega_0 \subset \Omega \text{ and a.e. in } Q_T; \tag{16}$$

thus by **C**

$$(H(u_k)) \to H(u) \text{ a.e. in } Q_T; \tag{17}$$

$$(A + B_k + C_k)(u_k) \to w \text{ weakly in } X_T^\star, \tag{18}$$

$$\tilde{g}_{l,k}(t, x, u_k) \to \varphi_l \text{ and } \tilde{h}_k(t, x, u_k) \to \psi \text{ weakly in } L^q(Q_T). \tag{19}$$

From (11), (12), (14), (15), (18), (19) it follows (see, e.g., [14]) $u(0) = u_0$,

$$D_t u + w + \varphi_1 + \varphi_2 + \psi = F. \tag{20}$$

Now we prove $w = A(u)$. Apply (11) to $(u_k - u)\zeta$ with arbitrary fixed $\zeta \in C_0^\infty(\Omega)$ having the properties : $\zeta \geq 0$, $\zeta(x) = 1$ in a compact subset K of Ω. So we obtain

$$\begin{aligned} [D_t u_k - D_t u, (u_k - u)\zeta] + [D_t u, (u_k - u)\zeta] + [A(u_k), (u_k - u)\zeta] + \\ [(B_k + C_k)(u_k), (u_k - u)\zeta] = [F, (u_k - u)\zeta]. \end{aligned} \tag{21}$$

For the first term we have

$$[D_t u_k - D_t u, (u_k - u)\zeta] = 1/2 \int_0^T \left[\frac{d}{dt} \int_\Omega (u_k(t) - u(t))^2 \zeta dx \right] dt = 1/2 \int_\Omega (u_k(T) - u(T))^2 \zeta dx \geq 0, \tag{22}$$

further, by (15), (16), (19)

$$\lim_{k\to\infty} [D_t u, (u_k - u)\zeta] = 0, \quad \lim_{k\to\infty} [(B_k + C_k)(u_k), (u_k - u)\zeta] = 0, \quad \lim_{k\to\infty} [F, (u_k - u)\zeta] = 0. \tag{23}$$

Thus (21) - (23) imply

$$\limsup_{k\to\infty} [A_k(u_k), (u_k - u)\zeta] \leq 0. \tag{24}$$

Since by **A** (ii) and (16)

$$\lim_{k\to\infty} \int_{Q_T} f_0(t, x, u_k, \nabla u_k)(u_k - u)\zeta dt dx = 0,$$

from (24) we obtain

$$\limsup_{k\to\infty} \sum_{j=1}^n \int_{Q_T} f_j(t, x, u_k, \nabla u_k)(u_k - u)\zeta dt dx \leq 0. \tag{25}$$

By using arguments of [5], we obtain from (25)

$$\nabla u_k \to \nabla u \text{ a.e. in } (0, T) \times K$$

(see [13]). Since K can be chosen as any compact subset of Ω, we find

$$\nabla u_k \to \nabla u \text{ a.e. in } Q_T. \tag{26}$$

Thus Vitali's theorem and Hölder's inequality imply

$$A(u_k) \to A(u) \text{ weakly in } X_T^\star$$

(see, e.g., [5]), i.e. $w = A(u)$.

In order to show the inequalities (4), one applies arguments of [9], by using (16), (17). (16) implies that for each positive a there exists a subset $\omega \subset Q_T$ with Lebesgue measure $\lambda(\omega) < a$ such that

$$(u_k) \to u \text{ uniformly on } Q_T \setminus \omega \text{ and } u \in L^\infty(Q_T \setminus \omega).$$

Thus for any $\varepsilon > 0$ there is k_0 such that $k_0 > 2/\varepsilon$ and $k > k_0$ implies

$$|u_k(t, x) - u(t, x)| < \varepsilon/2 \text{ if } (t, x) \in Q_T \setminus \omega. \tag{27}$$

Let $k > k_0$, $(t,x) \in Q_T \setminus \omega$. From $1/k < \varepsilon/2$, (27) and the definition of $g_{1,k}$, $\underline{g}_1^\varepsilon$, $\bar{g}_1^\varepsilon$ it easily follows

$$\underline{g}_1^\varepsilon(t,x,u(t,x)) \le g_{1,k}(t,x,u_k(t,x)) \le \bar{g}_1^\varepsilon(t,x,u(t,x)),$$

hence for sufficiently large k

$$\underline{g}_1^\varepsilon(t,x,u(t,x)) \le \tilde{g}_{1,k}(t,x,u_k(t,x)) \le \bar{g}_1^\varepsilon(t,x,u(t,x)).$$

Consequently, for any $\varphi \in C_0^\infty(Q_T)$ with $\varphi \ge 0$ we have

$$\int_{Q_T\setminus\omega} \underline{g}_1^\varepsilon(t,x,u)\varphi \le \int_{Q_T\setminus\omega} \tilde{g}_{1,k}(t,x,u_k)\varphi \le \int_{Q_T\setminus\omega} \bar{g}_1^\varepsilon(t,x,u)\varphi$$

which implies by (19)

$$\int_{Q_T\setminus\omega} \underline{g}_1^\varepsilon(t,x,u)\varphi \le \int_{Q_T\setminus\omega} \varphi_1\varphi \le \int_{Q_T\setminus\omega} \bar{g}_1^\varepsilon(t,x,u)\varphi.$$

Since $u \in L^\infty(Q_T \setminus \omega)$, Lebesgue's dominated convergence theorem implies as $\varepsilon \to 0$

$$\int_{Q_T\setminus\omega} \underline{g}_1(t,x,u)\varphi \le \int_{Q_T\setminus\omega} \varphi_1\varphi \le \int_{Q_T\setminus\omega} \bar{g}_1(t,x,u)\varphi. \tag{28}$$

(28) holds for arbitrary nonnegative $\varphi \in C_0^\infty(Q_T)$, thus we find

$$\underline{g}_1(t,x,u(t,x)) \le \varphi_1(t,x) \le \bar{g}_1(t,x,u(t,x)) \tag{29}$$

for a.e. $(t,x) \in Q_T \setminus \omega$. Inequality (29) holds true for any $a > 0$ and $\omega \subset Q_T$ with $\lambda(\omega) < a$, thus we obtain that (29) is valid a.e. in Q_T.

Remark 3. In certain particular cases (if some Lipschitz conditions are satisfied) one can prove uniqueness of the solution (see also [11]).

It is not difficult to prove an existence theorem for the interval $(0,\infty)$. Denote by X_∞ and $X_\infty^\star$ the set of functions $u : [0,\infty) \to V$, $\quad w : [0,\infty) \to V^\star$, respectively, such that for any finite T $u \in L^p(0,T;V)$, $\quad w \in L^q(0,T;V^\star)$, respectively. Further, define $Q_\infty = (0,\infty) \times \Omega$ and let $L^p_{loc}(Q_\infty)$ be the set of functions $v : Q_\infty \to R$ such that $v \in L^p(Q_T)$ for arbitrary finite T.

Theorem 2. *Assume that functions*

$$f_j : Q_\infty \times R^{n+1} \to R, \quad g, h : Q_\infty \times R \to R$$

satisfy **A** *(i) - (iv),* **B** *(i) - (iii) and* **C** *for any finite* $T > 0$.

Then for arbitrary $F \in X_\infty^\star$ *there exists* $u \in X_\infty$ *such that for any finite* T *the assertion of Theorem 1 holds with some functions* $\varphi_l, \psi \in L^q_{loc}(Q_\infty)$.

Theorem 2 is a consequence of Theorem 1, the proof is based on simple and standard arguments. (Similar arguments can be found e.g. in [12].)

By using arguments of the proof of Theorem 1 we obtain that in the case when Ω is unbounded, the limit (as $k \to \infty$) of certain problems in "large" bounded $\Omega_k \subset \Omega$ is a solution in Ω. Now we give the exact formulation of this statement.

Let $\Omega_k \subset \Omega$ be bounded domains with sufficiently smooth boundary such that $B_k \cap \Omega \subset \Omega_k$ ($B_k = \{x \in R^n : |x| < k\}$) and introduce the notations

$$V_k = W_0^{1,p}(\Omega_k), \quad X_T^k = L^p(0,T;V_k), \quad (X_T^k)^\star = L^q(0,T;V_k^\star)$$

where $W_0^{1,p}(\Omega_k)$ is the completion of $C_0^\infty(\Omega_k)$ with respect to the norm of $W^{1,p}(\Omega_k)$. Further, let $M_k : X_T^k \to X_T$ be the following (extension) operator:

$$M_k v_k(t,x) = v_k(t,x) \text{ for } x \in \Omega_k, \qquad M_k v_k(t,x) = 0 \text{ for } x \in \Omega \setminus \Omega_k$$

Define the restriction F_k of $F \in X_T^\star$ (to Ω_k) by

$$\int_0^T \langle F_k(t), v_k(t)\rangle dt = \int_0^T \langle F(t), (M_k v_k)(t)\rangle dt, \quad v_k \in X_T^k.$$

Finally, let $\varphi \in C_0^\infty(R^n)$ be a function with the properties

$$\varphi(x) = 1 \text{ if } |x| \le 1/2, \quad \varphi(x) = 0 \text{ if } |x| \ge 1$$

and define φ_k by $\varphi_k(x) = \varphi(x/k)$.

Theorem 3. *Assume that the conditions of Theorem 1 are satisfied and the functions $u_k \in X_T^k$ are solutions of the following problems in Ω_k:*

$$u_k(0,\cdot) = \varphi_k u_0 \quad (\in V_k);$$

$D_t u_k \in (X_T^k)^\star$ *and for any* $v_k \in V_k$

$$\begin{aligned}\langle D_t u_k(t,\cdot), v_k\rangle + &\sum_{j=1}^n \int_{\Omega_k} f_j(t,x,u_k(t,x),\nabla u_k(t,x)) D_j v_k(x)dx + \\ &\int_{\Omega_k} f_0(t,x,u_k(t,x),\nabla u_k(t,x)) v_k(x)dx + \\ &\int_{\Omega_k} [\varphi_{1,k}(t,x) + \varphi_{2,k}(t,x) + \psi_k(t,x)] v_k(x)dx = \\ &\langle F_k(t,\cdot), v_k\rangle \text{ for a.e. } t \in [0,T]\end{aligned}$$

with some functions $\varphi_{1,k}, \varphi_{2,k}, \psi_k \in L^q((0,T)\times\Omega_k)$ such that for a.e. $(t,x) \in (0,T)\times\Omega_k$

$$\underline{g}_l(t,x,u_k(t,x)) \le \varphi_{l,k}(t,x) \le \bar{g}_l(t,x,u_k(t,x)), \quad l = 1,2$$

$$\underline{h}(t,x,[H(M_k u_k)](t,x)) \le \psi_k(t,x) \le \bar{h}(t,x,[H(M_k u_k)](t,x)).$$

Then the sequence $(M_k u_k)$ is bounded in X_T and it has a subsequence which is weakly convergent in X_T to a function $u \in X_T$ satisfying (2) – (4).

3. Boundedness and stabilization

Theorem 4. *Assume that the conditions of Theorem 2 are satisfied such that* c_2 *and* k_2 *in* **A** *(iv) are independent of* T*,* $p > 2$*,* $\| F(t) \|_{V^\star}$ *is bounded,*

$$|g(t,x,\eta)|^q \le c_4^\star |\eta|^2 + k_4^\star(x), \quad |h(t,x,\theta)|^q \le c_4^\star |\theta|^2 + k_4^\star(x) \tag{30}$$

with some constant $c_4^\star$ *and a function* $k_4^\star \in L^1(\Omega)$*. Further, for any* $u \in L^p_{loc}(Q_\infty)$

$$\int_\Omega |H(u)|^2(t,x)dx \le const \sup_{\tau\in[0,t]} \int_\Omega |u(\tau,x)|^2 dx. \tag{31}$$

Then for the solution u *the function*

$$y(t) = \int_\Omega |u(t,x)|^2 dx$$

is bounded in $(0,\infty)$ *and there exist constants* $c', c"$ *such that for sufficiently large* T_1, T_2

$$\int_{T_1}^{T_2} \| u(t) \|_V^p \, dt \le c'(T_2 - T_1) + c".$$

Idea of the proof. Apply (3) to $v = u(t,\cdot)$ and integrate over (T_1, T_2). Then one obtains the inequality

$$y(T_2) - y(T_1) + c^\star \int_{T_1}^{T_2} [y(t)]^{p/2} dt \le \text{const} \int_{T_1}^{T_2} [\sup_{[0,t]} |y| + 1] dt$$

with some constant $c^\star > 0$ which implies the assertion of Theorem 4. (See, e.g., the proof of Theorem 2 in [12].)

Now we formulate a theorem on the stabilization of the solution as $t \to \infty$. Assume that the conditions of Theorem 4 are satisfied. Consider a sequence $(t_l) \to +\infty$ and define for a solution u

$$U_l(s,x) = u(t_l + s, x), \quad s \in (-a,b), \quad x \in \Omega$$

with some fixed numbers $a, b > 0$. By Theorem 4 (U_l) is bounded in $L^p(-a,b;V)$.

Theorem 5. *Let the assumptions of Theorem 4 be satisfied; assume that* f_j, g, h *are not depending on* t*, there exists a (finite)* ρ *such that for sufficiently large* $t > 0$*,* $[H(u)](t,x)$ *depends only on the restriction of* u *to* $(t-\rho,t)\times\Omega$ *and it is not depending on* t *if* u *is not depending on* t*. Further, there exists* $F_\infty \in V^\star$ *such that*

$$\lim_{T\to\infty} \int_{T-1}^{T+1} \| F(t) - F_\infty \|_{V^\star} \, dt = 0.$$

Finally,

$$\begin{gathered}\exists u_\infty \in L^p(\Omega) \text{ and } (t_l) \to +\infty \text{ such that } (U_l) \to u_\infty \text{ weakly} \\ \text{in } L^p((-1-\rho,1)\times\Omega).\end{gathered} \tag{32}$$

(u_∞ is not depending on t!)

Then there is a subsequence of (t_l) (again denoted by (t_l)) such that for the sequence (U_l) (defined by the subsequence (t_l))

$$(U_l) \to u_\infty \text{ weakly in } L^p(-1,1;V), \tag{33}$$

$$(U_l) \to u_\infty \text{ in } L^p((-1,1)\times\Omega_0) \tag{34}$$

for each bounded $\Omega_0 \subset \Omega$ and a.e. in $(-1,1)\times\Omega$.

Moreover, u_∞ is a solution of the stationary problem

(35)
$$\sum_{j=1}^n \int_\Omega f_j(x,u_\infty(x),\nabla u_\infty(x))D_j w(x)dx + \int_\Omega f_0(x,u_\infty(x),\nabla u_\infty(x))w(x)dx +$$

$$\int_\Omega [\tilde\varphi_1(x)+\tilde\varphi_2(x)+\tilde\psi(x)]w(x)dx = \langle F_\infty, w\rangle, \quad w\in V$$

with some functions $\tilde\varphi_l, \tilde\psi \in L^q(\Omega)$ satisfying for a.e. $x\in\Omega$

$$\underline{g}_l(x,u_\infty(x)) \le \tilde\varphi_l(x) \le \bar g_l(x,u_\infty(x)), \quad l=1,2 \tag{36}$$

$$\underline{h}(x,[H(u_\infty)](x)) \le \tilde\psi(x) \le \bar h(x,[H(u_\infty)](x)).$$

Remark 4. In (36) u_∞ means the constant function in t, defined in an interval $(t-\rho,t)$. By the assumption of our theorem, $H(u_\infty)$ does not depend on t.

Remark 5. The operators H, defined in Remark 2 satisfy the assumptions of Theorem 5 if

$$\beta_0(s,t,x) = \beta(s-t,x) \text{ for } \max\{t-\rho,0\} \le s \le t,$$

$$\beta_0(s,t,x) = 0 \text{ for } 0 \le s \le \max\{t-\rho,0\}$$

with a function $\beta\in L^\infty((-\rho,0)\times\Omega)$; $t-\rho\le\tau(t)$, respectively.

Remark 6. By Theorem 4 (U_l) is bounded in $L^p((-1-\rho)\times\Omega)$ for any sequence $(t_l)\to+\infty$, hence a subsequence of (U_l) is weakly convergent to a function $U \in L^p((-1-\rho)\times\Omega)$. In (32) we assume that there exists U, not depending on t.

A sufficient condition for (32) is

$$D_t u \in L^2(0,\infty;L^2(\Omega)). \tag{37}$$

For the proof see [11]. In [11] there are given simple sufficient conditions for (37) which imply a stabilization result in the case when g,h are depending on t and Ω is bounded. The formulation and proof of this result for unbounded Ω is similar to the case of bounded Ω.

The sketch of the proof of Theorem 5. By Theorem 4 (U_l) is bounded in $L^p(-2\rho-1,1;V)$ thus $D_t U_l$ is bounded in $L^q(-\rho-1,1;V^\star)$ which implies by (32) that there is a subsequence of (U_l) (again denoted by (U_l)) such that

$$(U_l) \to u_\infty \text{ weakly in } L^p(-\rho-1,1;V) \text{ and strongly in } L^p((-\rho-1,1)\times\Omega_0) \tag{38}$$

for any bounded $\Omega_0 \subset \Omega$;

$$(U_l) \to u_\infty \text{ a.e. in } (-1,1)\times\Omega. \tag{39}$$

Define the functions $\varphi_{1,l}, \varphi_{2,l}, \psi_l$ by

$$\varphi_{1,l}(s,x) = \varphi_1(t_l+s,x), \quad \varphi_{2,l}(s,x) = \varphi_2(t_l+s,x), \quad \psi_l(s,x) = \psi(t_l+s,x).$$

Since $(\varphi_{1,l}), (\varphi_{2,l}), (\psi_l)$ are bounded in $L^q((-1,1)\times\Omega)$, we may assume that

$$(\varphi_{1,l}) \to \varphi_1^\star, \quad (\varphi_{2,l}) \to \varphi_2^\star, \quad (\psi_l) \to \psi^\star \text{ weakly in } L^q((-1,1)\times\Omega). \tag{40}$$

Finally, we may assume that

$$\hat{A}(U_l(t)) \to Y \text{ weakly in } L^q(-1,1;V^\star) \tag{41}$$

with some $Y \in L^q(-1,1;V^\star)$ where the operator $\hat{A}: V \to V^\star$ is defined by

$$\langle \hat{A}(v), w\rangle = \sum_{j=1}^n \int_\Omega f_j(x,v,\nabla v) D_j w + \int_\Omega f_0(x,v,\nabla v) w, \quad v,w \in V.$$

Now we apply arguments of [7]. Let

$$\varphi \in C_0^\infty(-1,1), \quad 1 \geq \varphi \geq 0, \quad \int_{-1}^1 \varphi = 1, \quad w \in V. \tag{42}$$

Since u is a solution of (3), we have (for sufficiently large l)

$$\int_{-1}^1 \int_\Omega U_l w \varphi' dt dx + \int_{-1}^1 \langle \hat{A}(U_l(t)), w\rangle \varphi dt + \tag{43}$$

$$\int_{-1}^1 \int_\Omega (\varphi_{1,l} + \varphi_{2,l} + \psi_l) w \varphi dt dx = \int_{-1}^1 \langle F(t_l+t), w\rangle \varphi dt.$$

By (38), (40) - (42) we obtain from (43) as $l \to \infty$

$$\int_{-1}^1 \langle Y(t), w\rangle \varphi dt + \int_{-1}^1 \int_\Omega (\varphi_1^\star + \varphi_2^\star + \psi^\star) w \varphi dt dx = \langle F_\infty, w\rangle. \tag{44}$$

It is not difficult to costruct fuctions $\varphi = \varphi_j$ satisfying (42) such that

$$\lim_{j\to\infty} (\varphi_j) = 1/2 \text{ in } (-1,1).$$

Applying (44) to $\varphi = \varphi_j$, we obtain as $j \to \infty$

$$\frac{1}{2}\int_{-1}^{1}\langle Y(t), w\rangle dt + \int_{\Omega}(\tilde{\varphi}_1 + \tilde{\varphi}_2 + \tilde{\psi})w dx = \langle F_\infty, w\rangle \tag{45}$$

where

$$\tilde{\varphi}_k = \frac{1}{2}\int_{-1}^{1}\varphi_k^\star dt, \quad \tilde{\psi} = \frac{1}{2}\int_{-1}^{1}\psi^\star dt. \tag{46}$$

Now we show $Y = \hat{A}(u_\infty)$. Let $\Omega_0 \subset \Omega$ be any bounded domain and $\zeta \in C_0^\infty(\Omega)$ with the properties: $\zeta \geq 0$, $\zeta(x) = 1$ for $x \in \Omega_0$ and denote by K the support of ζ. By (38) (for a suitable subsequence)

$$(U_l(t)) \to u_\infty \text{ in } L^2(K) \text{ for a.e. } t \in (-1,1),$$

hence there exist $\delta_l, \varepsilon_l > 0$ such that (for a suitable subsequence of (U_l))

$$\begin{gathered}\lim_{l\to\infty}(\delta_l) = 0, \quad \lim_{l\to\infty}(\varepsilon_l) = 0, \text{ and } U_l(-1+\delta_l) \to u_\infty, \\ U_l(1-\varepsilon_l) \to u_\infty \text{ in } L^2(K).\end{gathered} \tag{47}$$

By (3) we find

(48)

$$\frac{1}{2}\int_{\Omega}|U_l(1-\varepsilon_l)|^2\zeta dx - \frac{1}{2}\int_{\Omega}|U_l(-1+\delta_l)|^2\zeta dx + \int_{-1+\delta_l}^{1-\varepsilon_l}\langle \hat{A}(U_l(t)), U_l(t)\zeta\rangle dt +$$
$$\int_{-1+\delta_l}^{1-\varepsilon_l}\int_{\Omega}(\varphi_{1,l} + \varphi_{2,l} + \psi_l)U_l\zeta dt dx = \int_{-1+\delta_l}^{1-\varepsilon_l}\langle F(t_l + t), U_l(t)\zeta\rangle dt,$$

hence by (38), (40), (45) - (47)

$$\lim_{l\to\infty}\int_{-1+\delta_l}^{1-\varepsilon_l}\langle \hat{A}(U_l(t)), U_l(t)\zeta\rangle dt = \tag{49}$$

$$2\langle F_\infty, u_\infty\zeta\rangle - \int_{-1}^{1}\int_{\Omega}(\varphi_1^\star + \varphi_2^\star + \psi^\star)u_\infty\zeta dt dx = \int_{-1}^{1}\langle Y(t), u_\infty\zeta\rangle dt.$$

By using arguments of [5] we obtain from (49)

$$\nabla U_l \to u_\infty \text{ a.e. in } (-1,1)\times\Omega_0$$

which implies by (39)

$$(\hat{A}(U_l)) \to \hat{A}(u_\infty) \text{ weakly in } L^q(-1,1;V^\star),$$

i.e. $Y = \hat{A}(u_\infty)$.

Finally, by (39), (40) we get (similarly to the proof of (4))

$$\underline{g}_l(x, u_\infty(x)) \leq \varphi_l^\star(t,x) \leq \bar{g}_l(x, u_\infty(x)), \quad l = 1,2$$

$$\underline{h}(x, [H(u_\infty)](x)) \leq \psi^\star(t,x) \leq \bar{h}(x, [H(u_\infty)](x))$$

Integrating these inequalities over $(-1,1)$, we obtain (36).

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SOME REMARKS ABOUT THE NONOSCILLATORY SOLUTIONS

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Abstract. The paper investigates the relation between a linear homogeneous differential equation and its nonhomogeneous variant concerning the nonoscillatory property.

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The aim of the paper is to investigate the relation between a linear homogeneous differential equation and its nonhomogeneous variant concerning the nonoscillatory property. More precisely, we formulate the problem as follows.

Problem. If the homogeneous linear differential equation is nonoscillatory and $f(x)$ is a continuous one-signed function (i. e. $f(x) \geq 0$ or $f(x) \leq 0$) which is not identically zero for large x, we ask which other properties has the homogeneous differential equation to have so that also the nonhomogeneous differential equation will have the nonoscillatory property.

For the simplicity we will consider the selfadjoint differential equation

$$y^{(4)} + p(x)y = 0 \tag{1}$$

and

$$z^{(4)} + p(x)y = f(x) \tag{2}$$

We assume that $p(x) \in C([a,\infty))$ is nonnegative function defined on $J = [a,\infty)$ and $f(x) \in C([a,\infty))$ is a one-signed function on J not identically zero for large x.

It follows from the assumptions about $p(x)$ that either all solutions of (1) are oscillatory or all are nonoscillatory [1].

Definition 1. A solution of (1) or (2) is oscillatory if it has an upper unbounded set of zeros. A solution is nonoscillatory if it is not oscillatory.

Definition 2. Equation (1) or (2) is oscillatory if it has at least one oscillatory solution. Otherwise the equation is nonoscillatory.

Definition 3. Equation (1) is said to be disconjugate (on an interval I) if no nontrivial solution of (1) has more than 3 zeros (on I).

The above problem was solved for the linear differential equations of the second order in paper [2].

Theorem 1. ([2]). *Let the equation*

$$y'' + p(x)y = 0$$

be a nonoscillatory equation and let $f(x)$ be a one-signed function not identically zero for large x. Then the equation

$$z'' + p(x)z = f(x)$$

is also nonoscillatory.

For the equation of higher order our problem was solved in the paper [3], where the condition for the nonoscillatory behaviour of the homogeneous differential equation was substituted by the condition of disconjugacy of the homogeneous differential equation. It has to be mentioned that the disconjugacy doesn't follow from the nonoscillatory property.

Our problem was discussed in the paper [4] for the linear differential equations of the n-th order, where the condition of disconjugacy is assumed for the so-called reduced operator $\hat{L}_{n-1}$ associated to the operator L_n.

Definition 4. Equation

$$L_n y = y^{(n)} + a_1 y^{n-1} + ... + a_n y = 0, \tag{3}$$

where $a_i \in C([a,\infty))$, $i = 1,2,...,n$, is said to be disconjugate (on an interval I), if no nontrivial solution of (3) has more than $n-1$ zeros (on I).

Assume that the equation (3) is nonoscillatory and that $\Phi(x)$ is a nonoscillatory solution of (3). If we set $y = \Phi z$, then for sufficiently large x we get

$$L_n y = zL_n\Phi + \Phi\left[z^{(n)} + \sum_{i=1}^{n-1} \hat{a}_i(x) z^{(n-i)}\right] = zL_n\Phi + \Phi\hat{L}_{n-1}z',$$

where $\hat{a}_i(x)$ depend on $\Phi(x)$. Operator $\hat{L}_{n-1}$ is called the reduced operator for L_n associated with Φ.

Our problem is partially solved in the paper [4].

Lemma 1. ([4]). *Let the equation* (3) *be nonoscillatory and let for solution* Φ *of* (3) *be* $\hat{L}_{n-1}z = 0$ *disconjugate for large* x*. Let* $f(x)$ *be a one-signed continuous function on* $[a, \infty)$ *not identically zero for large* x*. Then the equation*

$$L_n y = f(x) \tag{4}$$

is also nonscillatory.

In the following we will consider our problem for the equations (1) and (2). Instead of the disconjugacy we will use the condition of selfadjointness of (1) and the property that each solution $y(x)$ of (1) can have at most one double zero.

We know that all solutions of (1) are of the same oscillatory character. We will assume that all solutions of (1) are nonoscillatory.

Let be $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$ nonoscillatory solutions of (1) on J given by the initial conditions in $x_0 \in [a, \infty)$

$$y_i^{(j)}(x_0) = \begin{cases} 1 & , \quad for \quad j = i-1 \\ 0 & , \quad for \quad j \neq i-1 \end{cases} , i = 1, 2, 3, 4; \quad j = 0, 1, 2, 3. \tag{5}$$

These solutions form a fundamental system. Their wronskian is

$$W(y_1, y_2, y_3, y_4)(x) = 1. \tag{6}$$

From the fact that (1) is selfadjoint it follows ([5], Chap. II,5) that the wronskians

$$\begin{array}{c} W_1 = W(y_2, y_3, y_4)(x), \;\; W_2 = W(y_1, y_3, y_4)(x) \\ W_3 = W(y_1, y_2, y_4)(x), \;\; W_4 = W(y_1, y_2, y_3)(x) \end{array} \tag{7}$$

are solutions of (1) on J. It is easy to see that

$$\left. \begin{array}{rcl} W_k^{(j)}(x_0) & = & 0, \\ W_k^{k-1}(x_0) & = & 1. \end{array} \right\} k = 1, 2, 3, 4, \; j \neq k-1, \tag{8}$$

Thus

$$W_1 = y_4(x), W_2 = y_3(x), W_3 = y_2(x), W_4 = y_1(x). \tag{9}$$

Using the method of variation of constants we get for the general solution $z(x)$ of (2) the expression

$$z(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) + c_4 y(x) + \int_{x_0}^{x} A(t, x) f(t) dt, \tag{10}$$

where

$$A(t,x)=\begin{vmatrix} y_1(t), & y_2(t), & y_3(t), & y_4(t)\\ y_1'(t), & y_2'(t), & y_3'(t), & y_4'(t)\\ y_1''(t), & y_2''(t), & y_3''(t), & y_4''(t)\\ y_1(x), & y_2(x), & y_3(x), & y_4(x)\end{vmatrix}, x_0\le t\le x. \tag{11}$$

Respecting (7) and (8) we get

$$A(t,x)=-y_1(x)y_4(t)+y_2(x)y_3(t)-y_3(x)y_2(t)+y_4(x)y_1(t),\quad x_0\le t\le x. \tag{12}$$

It is evident that $A(t,x)$ as the function of t is a solution of (1). It is easy to see that $t=x$ is a triple zero of the solution $A(t,x)$. Using the expression (12) we get from (10)

$$z(x)=\sum_{i=1}^{4} y_i(x)\left[c_i+(-1)^i\int_{x_0}^{x} y_{5-i}(t)f(t)dt\right]. \tag{13}$$

We remark that evidently $\int_{x_0}^{x} y_{5-i}(t)f(t)dt,\ i=1,2,3,4$, is a monotone function in a neighbourhood of $+\infty$.

Lemma 2. *Let $p(x)$ be continuous and nonnegative on $[a,\infty)$. Let all solutions of the equation (1) be nonoscillatory. Then not all solutions of the equation (1) are bounded.*

Proof. Let be all solutions of the equation (1) nonoscillatory and bounded. Thus, the solutions $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$ satisfying (5) are nonoscillatory and bounded on $[x_0,\infty)$. From this it folows that $\lim_{x\to\infty} y_i^{(j)}=0$, $i=1,2,3,4$, $j=1,2,3$ and $\lim_{x\to\infty} y_i(x)$ is finite. Therefore, $\lim_{x\to\infty} W(y_1,y_2,y_3,y_4)(x)=0$, which contradicts the fact that $W(y_1,y_2,y_3,y_4)(x)=1$ for all $x\in[a,\infty)$.

Lemma 3. *Let be $p(x)\in C([a,\infty))$ nonnegative and not identically zero on some subinterval of $[a,\infty)$. Then every nontrivial solution $y(x)$ of (1) has at most one double (triple) zero point on $[a,\infty)$.*

Proof. Multiplying (1) by $y(x)$ we get $y^{(4)}y+p(x)y^2=0$ or after modification $(y'''y-y'y'')'=-{y''}^2-p(x)y^2\le 0$. It means that the function $F(y(x))=y'''(x)y(x)-y'(x)y''(x)$ is a nonincreasing one. From this the assertion of Lemma 3 follows.

Lemma 4. *Let $y_i(x)$, $i=1,2,3,4$ be the nonoscillatory solutions of (1) satisfying (5). Then there exists $\bar{x}\in[a,\infty)$ such that for $x\ge\bar{x}$ $y_i(x)\ne 0$, $i=1,2,3,4$,*

$$\begin{gathered} W(y_4,y_3,y_2,y_1)(x)\ne 0,\quad W(y_4,y_3,y_2)(x)\ne 0,\\ W(y_4,y_3)(x)\ne 0,\quad W(y_4)(x)=y_4\ne 0. \end{gathered} \tag{14}$$

Proof. It follows from the assumption of nonoscillatority of $y_i(x),\ i = 1,2,3,4$ that there exists $\bar{x} > x_0$ such that $y_i(x) \neq 0$ for $x \geq \bar{x}$ and $i = 1,2,3,4$. Moreover, we know that $W(y_4, y_3, y_2, y_1)(x) = const \neq 0$ for all $x \in [a, \infty)$ and $W(y_4, y_3, y_2)(x) = -y_4(x) \neq 0$ for $x \geq \bar{x}$. Consider the solution $u(x) = c_1 y_4(x) + c_2 y_3(x)$. Evidently, $u(x_0) = u'(x_0) = 0$, $u''(x_0) = c_2$. Thus $u(x)$ has no double zero for $x > x_0$ and therefore there doesn't exist $t > x_0$ such that

$$u(t) = c_1 y_4(t) + c_2 y_3(t) = 0$$

$$u'(t) = c_1 y_4'(t) + c_2 y_3'(t) = 0.$$

From this we have that $W(y_4, y_3)(t) \neq 0$ for all $t > x_0$ and therefore also for $t = x \geq \bar{x}$. Evidently, $W(y_4)(x) = y_4(x) \neq 0$ for $x \geq \bar{x}$. This ends the proof of Lemma 4.

Lemma 5. *Let be $p(x),\ f(x) \in C([a, \infty))$, $p(x)$ nonnegative and not identically zero on some subinterval of $[a, \infty)$ and $f(x)$ a one-signed function not identically zero for large x. Then the equation (2) allows the Frobenius factorization ([6], Chap. IV, §8, IX.)*

$$a_4(a_3(a_2(a_1(a_0 z)')')')' = f(x),\ x \geq \bar{x}, \tag{15}$$

where

$$a_j(x) = \frac{W_j^2(x)}{W_{j-1}(x) W_{j+1}(x)},\ j = 0,1,2,3,4, \tag{16}$$

$$W_0(x) = W_{-1}(x) = W_5(x) = 1,$$

$$W_j(x) = W(y_4, ..., y_{5-j})(x),\ j = 1,2,3,4. \tag{17}$$

Proof. From Lemma 4 we have that for $x \geq \bar{x}$

$$W_1(x) = W(y_4)(x) = y_4(x) \neq 0, \quad W_2(x) = W(y_4, y_3)(x) \neq 0,$$

$$W_3(x) = W(y_4, y_3, y_2)(x) = -y_4(x) \neq 0, \quad W(y_4, y_3, y_2, y_1)(x) = 1.$$

Thus

$$a_0(x) = \frac{1}{y_4(x)} \neq 0, \quad a_1(x) = \frac{y_4^2(x)}{W(y_4, y_3)(x)} \neq 0, \quad a_2(x) = \frac{W^2(y_4, y_3)(x)}{y_4^2(x)} \neq 0,$$

$$a_3(x) = \frac{y_4^2(x)}{W(y_4, y_3)(x)} \neq 0, \quad a_4(x) = \frac{1}{y_4(x)} \neq 0,$$

and (2) or (15) will have the form

(18)

$$\frac{1}{y_4(x)}\left[\frac{y_4^2(x)}{W(y_4,y_3)(x)}\left[\frac{W^2(y_4,y_3)(x)}{y_4^2(x)}\left[\frac{y_4^2(x)}{W(y_4,y_3)(x)}\left[\frac{z(x)}{y_4(x)}\right]'\right]'\right]'\right]' = f(x).$$

Theorem 2. *Let $p(x)$, $f(x) \in C([a,\infty))$, $p(x)$ nonnegative and not identically zero on some subinterval of $[a,\infty)$ and $f(x)$ a one-signed function in a neighbourhood of $+\infty$ not identically zero for large x. Let be equation (1) nonoscillatory. Then the equation (2) is also nonoscilatory.*

Proof. Under the given conditions on $p(x)$ and $f(x)$ the equation (2) can be transformed to the equivalent equation (15) and also (18), where the functions $a_i(x) \neq 0$, $i = 0,1,2,3,4$ on some neighbourhood of $+\infty$. The nonoscillatory character of solutions of (15) and (18) is evident.

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POSITIVE SOLUTIONS AND OSCILLATION OF HIGHER ORDER NEUTRAL DIFFERENCE EQUATIONS

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Abstract. Sufficient conditions are established for the existence of positive solutions and oscillation of bounded solutions of p-th order neutral difference equations of the form

$$\Delta^p[x_n + a_n x_{\tau(n)}] + \delta\, q_n f(x_{\sigma(n)}) = h_n, \quad n \in \mathbb{N}(n_0),$$

where $\delta = \pm 1$, $\mathbb{N}(n_0) = \{n_0, n_0+1, \dots\}$, n_0 is fixed in $\mathbb{N} = \{1, 2, \dots\}$, $a, q, h : \mathbb{N}(n_0) \to \mathbb{R}$, $\tau, \sigma \in \mathbb{N}(n_0) \to \mathbb{N}$ with $\tau(n) < n$ and $\lim_{n\to\infty} \tau(n) = \lim_{n\to\infty} \sigma(n) = \infty$. Combining the sufficient conditions we are able to give necessary and sufficient conditions for every bounded solution of the above equation to be oscillatory or almost oscillatory. Our results improve and generalize several oscillation criteria obtained previously.

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Keywords. Higher order neutral equations, positive solution, oscillation

1. Introduction

In this paper we consider p-th order neutral difference equations of the form

$$\Delta^p[x_n + a_n x_{\tau(n)}] + \delta\, q_n f(x_{\sigma(n)}) = h_n, \quad n \in \mathbb{N}(n_0), \tag{1}$$

where $\delta = \pm 1$, $\mathbb{N}(n_0) = \{n_0, n_0+1, \dots\}$, n_0 is fixed in $\mathbb{N} = \{1, 2, \dots\}$, $a, q, h : \mathbb{N}(n_0) \to \mathbb{R}$, $\tau, \sigma \in \mathbb{N}(n_0) \to \mathbb{N}$ with $\tau(n) < n$ and $\lim_{n\to\infty} \tau(n) = \lim_{n\to\infty} \sigma(n) = \infty$. Throughout this paper it is assumed that $f \in C(\mathbb{R}, \mathbb{R})$.

In what follows $n^{(s)}$ denotes the factorial function; that is, $n^{(0)} = 1$ and $n^{(s)} = n(n-1)\cdots(n-s+1)$ for any integer $s \geq 1$.

As usual a solution $\{x_n\}$ of equation (1) is called oscillatory if for a given $M \geq 0$, there exists $n \geq M$ such that $x_n x_{n+1} \leq 0$, and it is said to be almost oscillatory if $\{x_n\}$ is either oscillatory or satisfies $\lim_{n\to\infty} x_n = 0$.

The oscillatory behavior of solutions of first and second order difference equations has been extensively studied by many authors However, much less has been done for higher order equations. For some results regarding the oscillation and asymptotic behavior of higher order difference equation, we refer in particular to [2]-[10] and the references cited therein. In [8], the first author of the present article considered a special case of (1), namely, the difference equation

$$\Delta^p[x_n + c\, x_{n-l}] + \delta\, q_n f(x_{n-k}) = h_n, \quad n \in \mathbb{N}(n_0), \tag{2}$$

where l and k are integers with $l > 0$, and proved that if

- (C_1) $c \neq \pm 1$,
- (C_2) f satisfies Lipschitz conditions on an interval $[a, b]$, where a and b depend upon the range of $c \neq 0$,
- (C_3) $\sum^{\infty} n^{(p-1)} |q_n| < \infty$,
- (C_4) $\sum^{\infty} n^{(p-1)} |h_n| < \infty$,

then (2) has a positive solution, and if

- (H_1) $xf(x) > 0$ for all $x \neq 0$,
- (H_2) $q_n \geq 0$ with infinitely many positive terms,
- (H_3) there exists an oscillatory function ρ on $\mathbb{N}$ such that $\Delta^p \rho_n = h_n$ and $\lim_{n\to\infty} \Delta^j \rho_n = 0$ for $j = 0, 1, \dots, p-1$,
- (H_4) $\sum^{\infty} n^{(p-1)} q_n = \infty$,

then every bounded solution $\{x_n\}$ of (2) is oscillatory when $(-1)^p \delta = 1$, and almost oscillatory when $(-1)^p \delta = -1$.

Later the same author [9] gave a necessary and sufficient condition for the oscillation of bounded solutions of (1) when $\tau(n) = n - l$, $\sigma(n) = n - k$, and $-b_0 \leq c_n \leq -b_1 < -1$, where b_0 and b_1 are fixed real numbers. The dependence mentioned in (C_2) was obtained as $a/b < (b_1 - 1)/b_0$.

A similar result was also established in [7] for equation (1) when p is even, $\tau(n) = n - l$, $\sigma(n) = n - k$, $h_n \equiv 0$, and $0 \leq c_n < b_2 < 1$. Instead of (H_4), they had imposed the condition that

$$\sum^{\infty} q_n f\Big(\big(\frac{n-k}{2^{p-1}}\big)^{p-1} \Big) = \infty.$$

Our purpose here in this paper is to find sufficient conditions for the existence of positive solutions and oscillation of bounded solutions of equation (1), and thereby establish necessary and sufficient conditions for oscillation or almost oscillation of bounded solutions of equation (1).

For simplicity we first consider the difference equation

$$\Delta^p[x_n + c\, x_{\tau(n)}] + \delta\, q_n f(x_{\sigma(n)}) = 0, \quad n \in \mathbb{N}(n_0) \tag{3}$$

in sections 2 and 3, and next extend the results obtained to equation (1) in section 4.

2. Existence of positive solutions

In this section we are concerned with the existence of positive solutions of neutral type difference equations of the form (3). It will be proved that (3) has a positive solution when $|c| \neq 1$ provided that the function f satisfies a Lipschitz condition on an interval $[a, b]$, where a and b are arbitrary positive real numbers.

Theorem 1. *If (C_1) and (C_3) hold and*

($\bar{C}_2$) for some positive numbers a and b, the function f satisfies the Lipschitz condition with a constant L on the interval $[a, b]$,

then equation (3) has a positive solution.

Proof. Let $K = \max\{|f(x)|/|x| : a \leq x \leq b\}$ and $M = \max\{K, L\}$.

We first consider the case $|c| < 1$. Because of (C_3), there exists a sufficiently large integer $n_1 \geq n_0$ such that

$$\sum_{s=n_1}^{\infty} s^{(p-1)}|q_s| < \frac{(p-1)!}{Mb}\beta, \quad \beta = \frac{(b-a)(1-|c|)}{2}, \tag{4}$$

and such that $\tau(n) \geq n_0$ and $\sigma(n) \geq n_0$ for all $n \in \mathbb{N}(n_1)$.

We introduce the Banach Space

$$Y = \Big\{ x : \sup_{n \geq N_0} |x_n| < \infty \Big\}$$

with the norm

$$||x|| = \sup_{n \geq N_0} |x_n|,$$

where $N_0 = \inf_{n\geq n_1}\{\tau(n), \sigma(n)\}$.

Set $X = \{x \in Y : a \leq x \leq b\}$. It is clear that X is a bounded, convex and closed subset of Y. Define an operator $S : X \to Y$ by

$$\begin{aligned} Sx_n &= \alpha - c\,x_{\tau(n)} + \frac{(-1)^p}{(p-1)!}\sum_{s=n}^{\infty}(s+p-1-n)^{(p-1)}q_s f(x_{\sigma(s)}), \quad n \geq n_1 \\ &= Sx_{n_1}, \quad N_0 \leq n \leq n_1, \end{aligned}$$

where

$$\alpha = \frac{(b+a)(1+c)}{2}.$$

We shall show that S is a contraction mapping on X. We prove this when $0 \leq c < 1$, the case $-1 < c < 0$ is similar. It is easy to see that S maps X into itself. In fact, for $x \in X$, $n \geq n_1$, using (4) it follows that

$$Sx_n \geq \alpha - c\,b - \beta = a$$

and

$$Sx_n \leq \alpha - c\,a + \beta = b,$$

and hence $Sx \in X$. To show that S is a contraction, let $x, y \in X$. It is easy to see that

$$\begin{aligned} |Sx_n - Sy_n| &\leq c\,|x_{\tau(n)} - y_{\tau(n)}| \\ &\quad + \frac{M}{(p-1)!}\sum_{s=n}^{\infty}(s+p-1-n)^{(p-1)}|q_s||x_{\sigma(s)} - y_{\sigma(s)}| \\ &\leq c||x-y|| + \frac{\beta}{b}||x-y||, \end{aligned}$$

and so

$$||Sx - Sy|| \leq (c + \frac{\beta}{b})||x-y||.$$

Since $c + \beta/b < 1$, S is a contraction on X. It follows that S has a fixed point $x \in X$, that is, $Sx = x$. It is easy to check that x is a positive solution of equation (3).

Suppose that $|c| > 1$. In this case we fix

$$\beta = \frac{(b-a)(|c|-1)}{2|c|}$$

and let n_1 be so large that

$$\sum_{s=\tau^{-1}(n_1)}^{\infty} s^{(p-1)}|q_s| < \frac{(p-1)!}{Mb}|c|\beta. \tag{5}$$

Define an operator $S : X \to Y$ as follows:

$$
\begin{aligned}
Sx_n &= \frac{1}{c}[\alpha - x_{\tau^{-1}(n)} + \frac{(-1)^p}{(p-1)!} \sum_{s=\tau^{-1}(n)}^{\infty} (s+p-1-\tau^{-1}(n))^{(p-1)} q_s f(x_{\sigma(s)})], \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \geq n_1 \\
&= Sx_{n_1}, \quad N_0 \leq n \leq n_1
\end{aligned}
$$

where

$$\alpha = \frac{(b+a)(1+c)}{2}.$$

We may claim that S is contraction on X. We shall prove our claim when $c > 1$, the case $c < -1$ is similar. In view of (5) we see that

$$Sx_n \geq \frac{\alpha}{c} - \frac{b}{c} - \beta = a$$

and

$$Sx_n \leq \frac{\alpha}{c} - \frac{a}{c} + \beta = b.$$

Thus we have $Sx \in X$. It is not also difficult to see that if $x, y \in X$ then

$$
\begin{aligned}
|Sx_n - Sy_n| &\leq \frac{1}{c}|x_{\tau^{-1}(n)} - y_{\tau^{-1}(n)}| \\
&\quad + \frac{1}{c}\frac{M}{(p-1)!} \sum_{s=\tau^{-1}(n)}^{\infty} (s+p-1-\tau^{-1}(n))^{(p-1)} |q_s||x_{\sigma(s)} - y_{\sigma(s)}| \\
&\leq (\frac{1}{c} + \frac{\beta}{b})||x-y||.
\end{aligned}
$$

Since $1/c + \beta/b < 1$, S is a contraction on X. This completes the proof.

3. Oscillation of bounded solutions

In this section we investigate the oscillation behavior of bounded solutions of (3) and establish necessary and sufficient conditions under which every solution $\{x_n\}$ of (3) is either oscillatory or almost oscillatory.

The following lemmas will be needed in the proof of our theorems. The first three of them can be found in [1]. The last one is essentially new and may be of interest for other studies as well.

Lemma 1. *Let $\{y_n\}$ and $\{\Delta^p y_n\}$ be sequences defined on $\mathbb{N}(n_0)$ with $y_n \Delta^p y_n < 0$ on $\mathbb{N}(n_0)$. Then there exists an integer l, $0 \leq l \leq p-1$, with $p-l$ odd such that for $n \in \mathbb{N}(n_0)$,*

$$
\begin{aligned}
y_n \Delta^j y_n &> 0, \quad j = 0, 1, \dots, l, \\
(-1)^{j-l} y_n \Delta^j y_n &> 0, \quad j = l+1, \dots, p-1.
\end{aligned}
$$

Lemma 2. *Let $\{y_n\}$ and $\{\Delta^p y_n\}$ be sequences defined on $\mathbb{N}(n_0)$ with $y_n \Delta^p y_n > 0$ on $\mathbb{N}(n_0)$. Then for $n \in \mathbb{N}(n_0)$, either*

$$y_n \Delta^j y_n > 0, \quad j = 1, \dots, p$$

or there exists an integer l, $0 \le l \le p-2$, with $p-l$ even such that for $n \in \mathbb{N}(n_0)$,

$$\begin{aligned} y_n \Delta^j y_n &> 0, \quad j = 0, 1, \dots, l, \\ (-1)^{j-l} y_n \Delta^j y_n &> 0, \quad j = l+1, \dots, p-1. \end{aligned}$$

Lemma 3. *If $\{y_n\}$ is a sequence defined on $\mathbb{N}(n_0)$, then*

$$\sum_{s=n_1}^{n-1} s^{(p-1)} \Delta^p y_s = \sum_{k=1}^{p} (-1)^{k+1} \Delta^{k-1} s^{(p-1)} \Delta^{p-k} y_{s+k-1} |_{s=n_1}^{n}.$$

Lemma 4. *Let g be a continuous monotone function such that $\lim_{n\to\infty} g(n) = \infty$. Set*

$$z_n = x_n + a_n x_{g(n)}. \tag{6}$$

If x_n is eventually positive, $\liminf_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} z_n = \ell \in \mathbb{R}$ exists, then $\ell = 0$ provided that for some real numbers b_1, b_2, b_3 and b_4 the sequence $\{a_n\}$ satisfies one of the following:

$$(a) \quad b_1 \le a_n \le 0, \qquad (b) \quad 0 \le a_n \le b_2 < 1, \qquad (c) \quad 1 < b_3 \le a_n \le b_4.$$

Proof. We see from (6) that

$$z_{g^{-1}(n)} - z_n = x_{g^{-1}(n)} + a_{g^{-1}(n)} x_n - x_n - a_n x_{g(n)}$$

and so

$$\lim_{n\to\infty} \left\{ x_{g^{-1}(n)} + a_{g^{-1}(n)} x_n - x_n - a_n x_{g(n)} \right\} = 0. \tag{7}$$

Let $\{n_k\}$ be a sequence of real numbers such that

$$\lim_{k\to\infty} x_{n_k} = 0. \tag{8}$$

Assume that (a) holds. It follows from (7) and (8) that

$$\lim_{k\to\infty} \left\{ x_{g^{-1}(n_k)} - a_{n_k} x_{g(n_k)} \right\} = 0.$$

As $x_{g^{-1}(n_k)} > 0$ and $-a_{n_k} x_{g(n_k)} \ge 0$, we see that

$$\lim_{k\to\infty} x_{g^{-1}(n_k)} = 0$$

and so from (6) we get

$$\ell = \lim_{k\to\infty} z_{g^{-1}(n_k)} = \lim_{k\to\infty} \left\{ x_{g^{-1}(n_k)} + a_{g^{-1}(n_k)} x_{n_k} \right\} = 0.$$

Assume that (b) holds. By replacing n by $g(n)$ in (7) and using (8) we have

$$\lim_{n\to\infty} \left\{ x_n + a_n x_{g(n)} - x_{g(n)} - a_{g(n)} x_{g(g(n))} \right\} = 0. \tag{9}$$

It is clear from (8) and (9) that

$$\lim_{k\to\infty} \left\{ [a_{n_k} - 1] x_{g(n_k)} - a_{g(n_k)} x_{g(g(n_k))} \right\} = 0$$

and so

$$\lim_{k\to\infty} x_{g(n_k)} = 0.$$

Thus,

$$\ell = \lim_{k\to\infty} z_{g(n_k)} = \lim_{k\to\infty} \left\{ x_{g(n_k)} + a_{g(n_k)} x_{g(g(n_k))} \right\} = 0.$$

Finally, let (c) be satisfied. Replacing n by $g^{-1}(n)$ in (7) and using (8) leads to

$$\lim_{k\to\infty} \left\{ x_{g^{-1}(g^{-1}(n_k))} + [a_{g^{-1}(g^{-1}(n_k))} - 1] x_{g^{-1}(n_k)} \right\} = 0$$

and hence

$$\lim_{k\to\infty} x_{g^{-1}(n_k)} = 0. \tag{10}$$

In view of (6) and (10), it follows that

$$\ell \;=\; \lim_{k\to\infty} z_{g^{-1}(n_k)} = 0.$$

This completes the proof.

Theorem 2. *Suppose that (H_1), (H_2) and (H_4) hold.*

(i) *If $c \geq 0$ and $c \neq 1$, then every bounded solution $\{x_n\}$ of (3) is oscillatory when $(-1)^p\delta = 1$, and is almost oscillatory when $(-1)^p\delta = -1$.*

and

(ii) *If $c < -1$ and $\inf_{n\geq 0}[n - \tau(n)] > 0$, then every bounded solution $\{x_n\}$ of (3) is oscillatory when $(-1)^p\delta = -1$, and is almost oscillatory when $(-1)^p\delta = 1$.*

Proof. Suppose on the contrary that $\{x_n\}$ is a nonoscillatory bounded solution of (3). Without loss of generality we may assume that $\{x_n\}$ is eventually positive. Set

$$z_n = x_n + c\, x_{\tau(n)}.$$

Clearly, $\{z_n\}$ is bounded and

$$\delta\Delta^p z_n = -q_n f(x_{\sigma(n)}) < 0. \tag{11}$$

Let $c \geq 0$ and $c \neq 1$. It is obvious that $\{z_n\}$ is eventually positive and $\delta z_n \Delta^p z_n < 0$. Applying Lemma 1 and Lemma 2 we see that there exist n_1 and integer $l \in \{0, 1\}$ with $(-1)^{p-l}\delta = -1$ such that

$$\begin{aligned} \Delta^k z_n &> 0, \quad k = 0, 1, \dots, l \\ (-1)^{k-l}\Delta^k z_n &> 0, \quad k = l, l+1, \dots, p-1 \end{aligned} \tag{12}$$

for all $n \geq n_1$. Multiplying (3) by $s^{(p-1)}$ and summing from n_1 to $n-1$ we obtain

$$\sum_{s=n_1}^{n-1} s^{(p-1)}\delta\Delta^p z_s + \sum_{s=n_1}^{n-1} s^{(p-1)} q_s f(x_{\sigma(s)}) = 0. \tag{13}$$

Applying Lemma 3 to the first term in the left side of (13) we have

$$\begin{aligned} \sum_{s=n_1}^{n-1} s^{(p-1)}\delta\Delta^p z_s &= \sum_{k=1}^{p-1}(-1)^{k+1}\delta\Delta^{k-1}s^{(p-1)}\Delta^{p-k}z_{s+k-1}|_{s=n_1}^{n} \\ &\quad + (-1)^{p+1}\delta\Delta^{p-1}s^{(p-1)}\delta\Delta^{p-p}z_{s+p-1}|_{s=n_1}^{n} \\ &= \sum_{k=1}^{p-1}(-1)^{k+1}\delta\Delta^{k-1}n^{(p-1)}\Delta^{p-k}z_{n+k-1} \\ &\quad + (-1)^{p+1}\delta(p-1)![z_{n+p-1} - z_{n_1+p-1}] - K \end{aligned} \tag{14}$$

where in view of (12)

$$K = \sum_{k=1}^{p-1}(-1)^{k+1}\delta\Delta^{k-1}n_1^{(p-1)}\Delta^{p-k}z_{n_1+k-1} \geq 0.$$

Using (14) in (13) leads to

$$\sum_{s=n_1}^{n-1} s^{(p-1)} q_s f(x_{\sigma(s)}) \leq K + (-1)^p\delta(p-1)![z_{n+p-1} - z_{n_1+p-1}]. \tag{15}$$

Since $\{z_n\}$ is bounded and (H_4) holds, we obtain from (15) that

$$\liminf_{n\to\infty} f(x_n) = 0$$

or

$$\liminf_{n\to\infty} x_n = 0.$$

It follows from Lemma 4 that $\ell = \lim_{n\to\infty} z_n = 0$. But $\ell = 0$ is possible only when $l = 0$, since in the case $l = 1$, $\{z_n\}$ being positive and increasing cannot approach zero. This means that bounded solutions of (3) must be oscillatory when $(-1)^p\delta = 1$. It is clear that if $\ell = 0$, then in view of $0 < x_n \le z_n$ we have

$$\lim_{n\to\infty} x_n = 0.$$

Suppose that $c < -1$. We claim that $\{z_n\}$ is eventually negative. Otherwise, for sufficiently large values of n, $x_n > -c\,x_{\tau(n)}$. Replacing n by $\tau^{-1}(n)$, using mathematical induction one can see that

$$x_{r_m(n)} > (-c)^m x_n, \tag{16}$$

where

$$r_1(n) = \tau^{-1}(n) \text{ and } r_m(n) = \tau^{-1}(r_{m-1}(n)) \text{ for } m \ge 2.$$

We shall show that $\lim_{m\to\infty} r_m = \infty$. In that case since $\{x_n\}$ is bounded we get a contradiction. We first notice that $\tau(n) < n$ and so $r_1(n) > n$. In view of $\inf_{n\ge 0}[n - \tau(n)] > 0$ there exists $\varepsilon > 0$ such that $r_1(n) > n + \varepsilon$. By mathematical induction we obtain

$$r_m(n) > n + m\varepsilon$$

and hence $\lim_{m\to\infty} r_m = \infty$. Therefore $\{z_n\}$ is eventually negative. Since

$$\delta\Delta^p z_n = -q_n f(x_{\sigma(s)}) < 0$$

we have $\delta z_n \Delta^p z_n > 0$. Applying Lemma 1 and Lemma 2 it follows that there are n_1 and $l \in \{0,1\}$ with $(-1)^{p-l}\delta = 1$ such that

$$\begin{aligned} \Delta^j z_n < 0, &\quad j = 0, 1, \ldots .l, \\ (-1)^{j-l}\Delta^j z_n < 0, &\quad j = l+1, \ldots, p-1. \end{aligned}$$

Using the arguments of the previous case we see that

$$\liminf_{n\to\infty} x_n = 0$$

and hence by Lemma 4, $\ell = \lim_{n\to\infty} z_n = 0$. Moreover, we observe as in the previous case that $\ell = 0$ is possible only when $l = 0$. In this case since $z_n < 0$ it follows that for a given $\epsilon > 0$ there exists an n_2 so large that

$$z_n > -\epsilon \quad \text{for } n \ge n_2.$$

This means that

$$x_n > -\epsilon - c\,x_{\tau(n)} \quad \text{for } n \ge n_2. \tag{17}$$

If we define $\tilde{c} = -1/c$, then we see from (17) that

$$x_n < \tilde{c}\,\epsilon + x_{r_1(n)}.$$

It follows that

$$x_n < (\tilde{c} + \tilde{c}^2 + \cdots \tilde{c}^m)\epsilon + \tilde{c}^m\, x_{r_m(n)}$$

and therefore

$$x_n < \frac{\tilde{c}}{1-\tilde{c}}\,\epsilon + \tilde{c}^m\, x_{r_m(n)}. \tag{18}$$

In view of $0 < \tilde{c} < 1$ we easily deduce from (18) that $\lim_{n\to\infty} x_n = 0$. This completes the proof.

In view of Theorem 1 and Theorem 2, we obtain a necessary and sufficient condition for oscillation of bounded solutions of (3), which gives an improvement of the theorem given in Section 1.

Theorem 3. *Let (H_1), (H_2) and $(\bar{C}_2)$ be satisfied. Then the conclusion of Theorem 2 holds if and only if (H_4) is satisfied.*

4. Some generalizations

In this section we extend the results obtained for equation (3) to equation (1). Since the proofs are similar, we will omit the details.

Theorem 4. *Suppose that (C_3) and (C_4) are satisfied, and (C_2) holds with positive real numbers a and b satisfying the following:*

(A) $a/b < (b_2+1)/(b_1+1)$, when $b_1 \le a_n \le b_2 < -1$,
(B) $a/b < (b_1+1)/(b_2+1)$, when $-1 < b_1 \le a_n \le b_2 \le 0$,
(C) $a/b < (1-b_2)/(1-b_1)$, when $0 \le b_1 \le a_n \le b_2 < 1$,
(D) $a/b < (b_1-1)/(b_2-1)$, when $1 < b_1 \le a_n \le b_2$,

where b_1 and b_2 are real numbers.

Then equation (1) has a positive solution.

Proof. Let $K = \max\{|f(x)|/|x| : a \le x \le b\}$ and $M = \max\{K, L\}$.

We first consider case (A). Let

$$\beta = \frac{b(b_2+1) - a(b_1+1)}{2b_2}.$$

In view of (C_3) and (C_4) we can find sufficiently large $n_1 \ge n_0$ such that if $n \ge n_1$ then

$$\sum_{s=\tau^{-1}(n_1)}^{\infty} s^{(p-1)}|q_s| < \frac{(p-1)!\beta}{2Mb}(-b_2) \tag{19}$$

and

$$\sum_{s=\tau^{-1}(n_1)}^{\infty} s^{(p-1)}|h_s| < \frac{(p-1)!\beta}{2}(-b_2). \tag{20}$$

We may assume that $\tau(n) \geq n_0$ and $\sigma(n) \geq n_0$ for all $n \geq n_1$.

We introduce the Banach Space

$$Y = \left\{ x : \sup_{n \geq N_0} |x_n| < \infty \right\}$$

with the supremum norm

$$||x|| = \sup_{n \geq N_0} |x_n|,$$

where $N_0 = \inf_{n \geq n_1} \{\tau(n), \sigma(n)\}$. Let

$$X = \{x \in Y : a \leq x \leq b\}.$$

It is clear that X is a bounded, convex and closed subset of Y.

Define an operator $S : X \to Y$ by

$$\begin{aligned} Sx_n = & \frac{1}{a_{\tau^{-1}(n)}}[\alpha - x_{\tau^{-1}(n)} + \frac{(-1)^p}{(p-1)!} \sum_{s=\tau^{-1}(n)}^{\infty} (s+p-1-\tau^{-1}(n))^{(p-1)} q_s f(x_{\sigma(s)}) \\ & + \frac{(-1)^{p-1}}{(p-1)!} \sum_{s=\tau^{-1}(n)}^{\infty} (s+p-1-\tau^{-1}(n))^{(p-1)} h_s], \qquad n \geq n_1 \\ = & Sx_{n_1}, \quad N_0 \leq n \leq n_1, \end{aligned}$$

where

$$\alpha = \frac{b(b_2+1) + a(b_1+1)}{2}.$$

We shall show that S is a contraction mapping on X. It is easy to show that S maps X into itself. In fact if $x \in X$ then, because of (19) and (20), it follows that

$$Sx_n \leq \frac{-1}{b_2}[-\alpha + b - b_2\beta] = b$$

and

$$Sx_n \geq \frac{-1}{b_1}[-\alpha + a + b_2\beta] = a.$$

Therefore $SX \subseteq X$.

To show that S is a contraction, we take $x, y \in X$. Obviously,

$$\begin{aligned}|Sx_n - Sy_n| &\leq \frac{-1}{b_2}|x_{\tau^{-1}(n)} - y_{\tau^{-1}(n)}| \\ &+ \frac{M}{(-b_2)(p-1)!}\sum_{s=\tau^{-1}(n)}^{\infty} s^{(p-1)}|q_s||x_{\sigma(s)} - y_{\sigma(s)}| \\ &\leq (\frac{-1}{b_2} + \frac{\beta}{2b})||x-y||. \end{aligned} \tag{21}$$

Since $\dfrac{-1}{b_2} + \dfrac{\beta}{2b} < 1$, S is a contraction on X, and therefore there exists a fixed point $x \in X$ such that $Sx = x$. It can easily be verified that x is a positive solution of equation (1). This completes the proof in the case when (A) is satisfied.

To prove the theorem for the cases (B), (C), and (D) we need only to make the following modifications on β, α and S in each case:

Case (B) :

$$\beta = \frac{b(b_1+1) - a(b_2+1)}{2}, \quad \alpha = \frac{b(b_1+1) + a(b_2+1)}{2},$$

$$\begin{aligned} Sx_n &= \alpha - a_n x_{\tau(n)} + \frac{(-1)^p}{(p-1)!}\sum_{s=n}^{\infty}(s+p-1-n)^{(p-1)} q_s f(x_{\sigma(s)}) \\ &+ \frac{(-1)^{p-1}}{(p-1)!}\sum_{s=n}^{\infty}(s+p-1-n)^{(p-1)} h_s, \qquad n \geq n_1 \\ &= Sx_{n_1}, \quad N_0 \leq n \leq n_1, \end{aligned}$$

where n_1 is chosen so large that

$$\sum_{s=n_1}^{\infty} s^{(p-1)}|q_s| < \frac{(p-1)!}{2Mb}\beta, \tag{22}$$

$$\sum_{s=n_1}^{\infty} s^{(p-1)}|h_s| < \frac{(p-1)!}{2}\beta \tag{23}$$

for all $n \geq n_1$.

Case(C) :

$$\beta = \frac{b(1-b_2) - a(1-b_1)}{2}, \quad \alpha = \frac{b(b_2+1) + a(b_1+1)}{2},$$

S is defined as in the case (B), and (22) and (23) are satisfied for all $n \geq n_1$.

Case(D):

$$\beta = \frac{b(b_1-1) - a(b_2-1)}{2b_1}, \quad \alpha = \frac{b(b_1+1) + a(b_2+1)}{2},$$

S is defined as in the case (i), and

$$\sum_{s=\tau^{-1}(n_1)}^{\infty} s^{(p-1)}|q_s| < \frac{(p-1)!}{2Mb}\beta b_1$$

$$\sum_{s=\tau^{-1}(n_1)}^{\infty} s^{(p-1)}|h_s| < \frac{(p-1)!}{2}\beta b_1$$

for all $n \geq n_1$.

The next theorem is a generalization of the results given in Theorem 2 to equation (1). For a similar result and especially the technique about handling the difficulty of having a forcing term, we refer the reader to [8,9].

Theorem 5. *Suppose that (H_1) - (H_4) hold.*

(i) *If* $0 \leq a_n \leq b_2 < 1$ *or* $1 < b_1 \leq a_n \leq b_2$, *then every bounded solution* $\{x_n\}$ *of* (1) *is oscillatory when* $(-1)^p\delta = 1$, *and is almost oscillatory when* $(-1)^p\delta = -1$.

(ii) *If* $b_1 \leq a_n \leq b_2 < -1$ *and* $\inf_{n\geq 0}[n-\tau(n)] > 0$, *then every bounded solution* $\{x_n\}$ *of* (1) *is oscillatory when* $(-1)^p\delta = -1$, *and is almost oscillatory when* $(-1)^p\delta = 1$.

Finally, by combining Theorem 4 and Theorem 5 we obtain the following necessary and sufficient condition for oscillation of bounded solutions of (1).

Theorem 6. *Suppose that (C_4), (H_1) - (H_3) hold, and that (C_2) is fulfilled on* $[a,b]$, *where* a *and* b *are as in* (A), (C), *and* (D). *Then the conclusion of Theorem 5 holds if and only if (H_4) is satisfied.*

Remark 1. In this paper we have assumed that $\{a_n\}$ is bounded away from ± 1. It is not difficult to provide specific examples showing that this assumption cannot be dropped. Therefore, finding similar results concerning (1) when $\{a_n\}$ is not bounded away from ± 1 seems to be very interesting.

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Author Index to CDDE issue

Subject Index to CDDE issue

CONTENTS

CONTENTS CONTINUATION

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