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ON FIRST ORDER IMPULSIVE SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. In this paper the Leray-Schauder nonlinear alternative for multivalued maps combined with the semigroup theory is used to investigate the existence of mild solutions for first order impulsive semilinear functional differential inclusions in Banach spaces.

1. Introduction

This paper is concerned with the existence of mild solutions for the impulsive semilinear functional differential inclusion of the form:

$$y' - Ay \in F(t, y_t)\,, \quad t \in J = [0, b]\,, \quad t \neq t_k\,, \quad k = 1, \dots, m, \tag{1.1}$$

$$y(t_k^+) = I_k(y(t_k^-))\,, \qquad k = 1, \dots, m, \tag{1.2}$$

$$y_0 = \phi\,, \tag{1.3}$$

where $F : J \times D \to 2^E$ is a closed, bounded and convex valued multivalued map $D = \{\psi : [-r, 0] \to E \mid \psi$ is continuous everywhere except for a finite number of points s at which $\psi(s^-)$ and $\psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, $\phi \in C([-r, 0], E)$, A is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$ and E a real Banach space with the norm $|\cdot|$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $I_k \in C(E, E)$ $(k = 1, 2, \dots, m)$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively.

For any continuous function y defined on the interval $[-r, b] - \{t_1, \dots, t_m\}$ and any $t \in J$, we denote by y_t the element of $C([-r, 0], E)$ defined by

$$y_t(\theta) = y(t + \theta)\,, \qquad \theta \in [-r, 0]\,.$$

Here $y_t(.)$ represents the history of the state from time $t - r$, up to the present time t.

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Impulsive differential equations have become more important in recent years in some mathematical models of real world phenomena, especially in the biological or medical domain (see the monographs of Lakshmikantham et al. [10], and Samoilenko and Perestyuk [16], and the papers of Agur et al. [1], Erbe et al. [6], Goldbeter et al. [7], Kirane and Rogovchenko [9], Liu et al. [12] and Liu and Zhang [13]).

This paper will be divided into three sections. In Section 2 we will recall briefly some basic definitions and preliminary facts from multivalued analysis which will be used throughout Section 3. In Section 3 we establish an existence theorem for (1.1)-(1.3). Our approach is based on the nonlinear alternative of Leray-Schauder type for multivalued maps combined with the semigroup theory [15].

In our results we do not assume any type of monotonicity condition on $I_k, k = 1, \dots, m$ which is usually the situation in the literature, see for instance, [6], [9] and [12].

2. Preliminaries

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel. Let the Banach space E. $B(E)$ denotes the Banach space of bounded linear operators from E into E.

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. For properties of the Bochner integral, we refer to Yosida [17].

$L^1(J, E)$ denotes the Banach space of functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|\, dt \quad \text{for all} \quad y \in L^1(J, E)\,.$$

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $G : X \to 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x \in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X if for each $x_* \in X$ the set $G(x_*)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x_*)$, there exists an open neighbourhood V of x_* such that $G(V) \subseteq B$.

G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In the following $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G : J \to BCC(E)$ is said to be measurable if for each $x \in E$ the function $Y : J \to \mathbb{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

is measurable. For more details on multivalued maps see the books of Deimling [4] and Hu and Papageorgiou [8].

Definition 2.1. A multivalued map $F : J \times D \to 2^E$ is said to be L^1-Carathéodory if

(i) $t \mapsto F(t, u)$ is measurable for each $u \in D$;
(ii) $u \mapsto F(t, u)$ is upper semicontinuous for almost all $t \in J$;
(iii) For each $\rho > 0$, there exists $\phi_\rho \in L^1(J, \mathbb{R}_+)$ such that $\|F(t, u)\| = \sup\{|v| : v \in F(t, u)\} \leq \phi_\rho(t)$ for all $\|u\| \leq \rho$ and for almost all $t \in J$.

In order to define the mild solution of (1.1)-(1.3) we shall consider the following space

$$\begin{aligned} \Omega = \{y : [-r, b] \to E : y_k \in C(J_k, E), k = 0, \dots, m \quad &\text{and there exist} \\ y(t_k^-) \quad \text{and} \quad y(t_k^+), k = 1, \dots, m \quad &\text{with} \quad y(t_k^-) = y(t_k) \\ \text{with} \quad y(t) = \phi(t), \ \forall t \in [-r, 0]\} \end{aligned}$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_\infty, k = 0, \dots, m\},$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}]$, $k = 0, \dots, m$.

So let us start by defining what we mean by a mild solution of problem (1.1)-(1.3).

Definition 2.2. A function $y \in \Omega$ is said to be a mild solution of (1.1)-(1.3) (see [15]) if there exists a function $v \in L^1(J, E)$ such that $v(t) \in F(t, y_t)$ a.e. on J_k, and

$$y(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)v(s)\, ds, & t \in [0, t_1], \\ I_k(y(t_k^-)) + \displaystyle\int_{t_k}^t T(t-s)v(s)\, ds, & t \in J_k, \ k = 1, \dots, m. \end{cases}$$

For the multivalued map F and for each $y \in C([-r, b], E)$ we define $S^1_{F,y}$ by

$$S^1_{F,y} = \{v \in L^1(J, E) : v(t) \in F(t, y_t) \quad \text{for a.e.} \quad t \in J\}.$$

Our main result is based on the following:

Lemma 2.3 [11]. *Let I be a compact real interval and X be a Banach space. Let $F : I \times X \to BCC(X); (t, y) \to F(t, y)$ measurable with respect to t for any $y \in X$ and u.s.c. with respect to y for almost each $t \in I$ and $S^1_{F,y} \neq \emptyset$ for any $y \in C(I, X)$ and let Γ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$. Then the operator*

$$\Gamma \circ S^1_F : C(I, X) \to BCC(C(I, X)),\ y \mapsto (\Gamma \circ S^1_F)(y) := \Gamma(S^1_{F,y})$$

is a closed graph operator in $C(I, X) \times C(I, X)$.

Lemma 2.4 (Nonlinear Alternative [5]). *Let X be a Banach space with $C \subset X$ convex. Assume U is a relatively open subset of C with $0 \in U$ and $G : \overline{U} \to 2^C$ is a compact multivalued map, u.s.c. with convex closed values. Then either,*

(i) *G has a fixed point in $\overline{U}$; or*
(ii) *there is a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u \in \lambda G(u)$.*

Remark 2.5. By $\overline{U}$ and ∂U we denote the closure of U and the boundary of U respectively.

3. Main Result

We are now in a position to state and prove our existence result for the IVP (1.1)-(1.3).

Theorem 3.1. *Let $t_0 = 0,\ t_{m+1} = b$. Suppose:*

(H1) *A is the infinitesimal generator of a linear bounded compact semigroup $T(t), t \geq 0$ and there exists $M \geq 1$ such that $|T(t)|_{B(E)} \leq M$;*
(H2) *$F : J \times D \to BCC(E)$ is an L^1-Carathéodory multivalued map and for each fixed $y \in C([-r, b], E)$ the set*

$$S^1_{F,y} = \left\{v \in L^1(J, E) :\ v(t) \in F(t, y_t) \quad \text{for a.e.} \quad t \in J\right\}$$

is nonempty;

(H3) *there exists a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that*

$$\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \leq p(t)\psi(\|u\|)$$

for a.e. $t \in J$ and each $u \in D$ with

$$\int_{t_{k-1}}^{t_k} p(s)\, ds < \int_{N_{k-1}}^{\infty} \frac{d\tau}{\psi(\tau)}, \qquad k = 1, \dots, m+1,$$

where $N_0 = M\|\phi\|$, and for $k = 2, \dots, m+1$,

$$N_{k-1} = \sup_{y \in [-M_{k-2}, M_{k-2}]} |I_{k-1}(y)|, \quad M_{k-2} = \Gamma^{-1}_{k-1}\left(M \int_{t_{k-2}}^{t_{k-1}} p(s)\, ds\right)$$

with

$$\Gamma_l(z) = \int_{N_{l-1}}^{z} \frac{d\tau}{\psi(\tau)}, \qquad z \geq N_{l-1}, \quad l \in \{1, \dots, m+1\}.$$

Then the problem (1.1)-(1.3) *has at least one mild solution* $y \in \Omega$.

Remark 3.2. (i) If $\dim E < \infty$, then for each $y \in C([-r,b],E)$, $S^1_{F,y} \neq \emptyset$ (see Lasota and Opial [11]).

(ii) If $\dim E = \infty$ and $y \in C([-r,b],E)$ the set $S^1_{F,y}$ is nonempty if and only if the function $Y : J \to \mathbb{R}$ defined by

$$Y(t) := \inf\{|v| : v \in F(t, y_t)\}$$

belongs to $L^1(J,\mathbb{R})$ (see Hu and Papageorgiou [8]).

Proof. The proof is given in several steps.

Step 1. Consider the problem (1.1)-(1.3) on $[-r,t_1]$

$$y' - Ay \in F(t, y_t)\,, \quad \text{a.e.} \quad t \in J_0\,, \tag{3.1}$$

$$y_0 = \phi\,. \tag{3.2}$$

We shall show that the possible mild solutions of (3.1)-(3.2) are *a priori* bounded, that is, there exists a constant b_0 such that, if $y \in \Omega$ is a mild solution of (3.1)-(3.2), then

$$\sup\{|y(t)| : t \in [-r,0] \cup J_0\} \le b_0\,.$$

Let y be a (possible) mild solution to (3.1)-(3.2). Then for each $t \in [0,t_1]$

$$y(t) - T(t)\phi(0) \in \int_0^t T(t-s)F(s,y_s)\,ds\,.$$

From (H3) we get

$$|y(t)| \le M\|\phi\| + M\int_0^t p(s)\psi(\|y_s\|)\,ds\,, \qquad t \in [0,t_1]\,.$$

We consider the function μ_0 defined by

$$\mu_0(t) = \sup\{|y(s)| : -r \le s \le t\}\,, \qquad 0 \le t \le t_1\,.$$

Let $t^* \in [-r,t]$ be such that $\mu_0(t) = |y(t^*)|$. If $t^* \in [0,t_1]$, by the previous inequality we have for $t \in [0,t_1]$

$$\mu_0(t) \le M\|\phi\| + M\int_0^t p(s)\psi(\mu_0(s))\,ds\,.$$

If $t^* \in [-r,0]$ then $\mu_0(t) = \|\phi\|$ and the previous inequality holds since $M \ge 1$.

Let us take the right-hand side of the above inequality as $v_0(t)$, then we have

$$v_0(0) = M\|\phi\| = N_0\,, \qquad \mu_0(t) \le v_0(t)\,, \quad t \in [0,t_1]$$

and

$$v_0'(t) = Mp(t)\psi(\mu_0(t)), \qquad t \in [0, t_1].$$

Using the nondecreasing character of ψ we get

$$v_0'(t) \le Mp(t)\psi(v_0(t)), \qquad t \in [0, t_1].$$

This implies for each $t \in [0, t_1]$ that

$$\int_{N_0}^{v_0(t)} \frac{d\tau}{\psi(\tau)} \le M \int_0^{t_1} p(s)\, ds.$$

In view of (H3), we obtain

$$|v_0(t^*)| \le \Gamma_1^{-1}\left(M \int_0^{t_1} p(s)\, ds\right) := M_0.$$

Since for every $t \in [0, t_1]$, $\|y_t\| \le \mu_0(t)$, we have

$$\sup_{t \in [-r, t_1]} |y(t)| \le \max(\|\phi\|, M_0) = b_0.$$

We transform this problem into a fixed point problem. A mild solution to (3.1)-(3.2) is a fixed point of the operator $G : C([-r, t_1], E) \to 2^{C([-r,t_1],E)}$ defined by

$$G(y) := \left\{ h \in C([-r, t_1], E) : h(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ T(t)\phi(0) & \\ \quad + \displaystyle\int_0^t T(t-s)v(s)\, ds, & t \in J_0 \end{cases} \right\}$$

where $v \in S^1_{F,y}$. We shall show that G satisfies the assumptions of Lemma 2.2.

Claim 1: *$G(y)$ is convex for each $y \in C(J_0, E)$.*

Indeed, if h, $\overline{h}$ belong to $G(y)$, then there exist $v \in S^1_{F,y}$ and $\overline{v} \in S^1_{F,y}$ such that

$$h(t) = T(t)\phi(0) + \int_0^t T(t-s)v(s)\, ds, \qquad t \in J_0$$

and

$$\overline{h}(t) = T(t)\phi(0) + \int_0^t T(t-s)\overline{v}(s)\, ds, \qquad t \in J_0.$$

Let $0 \le l \le 1$. Then for each $t \in J_0$ we have

$$[lh + (1-l)\overline{h}](t) = T(t)\phi(0) + \int_0^t T(t-s)[lv(s) + (1-l)\overline{v}(s)]\, ds.$$

Since $S^1_{F,y}$ is convex (because F has convex values) then

$$lh + (1-l)\overline{h} \in G(y)\,.$$

Claim 2: *G sends bounded sets into bounded sets in $C(J_0, E)$.*

Let $B_q := \{y \in C(J_0, E) : \|y\|_\infty = \sup_{t \in J_0} |y(t)| \le q\}$ be a bounded set in $C(J_0, E)$ and $y \in B_q$, then for each $h \in G(y)$ there exists $v \in S^1_{F,y}$ such that

$$h(t) = T(t)\phi(0) + \int_0^t T(t-s)v(s)\,ds\,, \qquad t \in J_0\,.$$

Thus for each $t \in J_0$ we get

$$\begin{aligned}|h(t)| &\le M\|\phi\| + M\int_0^t |v(s)|\,ds\\ &\le M\|\phi\| + M\|\phi_q\|_{L^1}\,.\end{aligned}$$

Claim 3: *G sends bounded sets in $C(J_0, E)$ into equicontinuous sets.*

Let $r_1, r_2 \in J_0,\ r_1 < r_2,\ B_q := \{y \in C(J_0, E) : \|y\|_\infty \le q\}$ be a bounded set in $C(J_0, E)$ as in Claim 2 and $y \in B_q$. For each $h \in G(y)$ there exists $v \in S^1_{F,y}$ such that

$$h(t) = T(t)\phi(0) + \int_0^t T(t-s)v(s)\,ds\,, \qquad t \in J_0\,.$$

Hence,

$$\begin{aligned}|h(r_2) - h(r_1)| &\le |(T(r_2)\phi(0) - T(r_1)\phi(0)| + \left|\int_0^{r_2} [T(r_2-s) - T(r_1-s)]v(s)\,ds\right|\\ &\quad + \left|\int_{r_1}^{r_2} T(r_1-s)v(s)\,ds\right|\\ &\le |(T(r_2)\phi(0) - T(r_1)\phi(0)| + \left|\int_0^{r_2} [T(r_2-s) - T(r_1-s)]v(s)\,ds\right|\\ &\quad + M\int_{r_1}^{r_2} |v(s)|\,ds\\ &\le |(T(r_2)\phi(0) - T(r_1)\phi(0)| + \left|\int_0^{r_2} [T(r_2-s) - T(r_1-s)]\phi_r(s)\,ds\right|\\ &\quad + M\int_{r_1}^{r_2} |\phi_r(s)|\,ds.\end{aligned}$$

The equicontinuity for the cases $r_1 < r_2 \le 0$ and $r_1 \le 0 \le r_2$ are obvious.

Set
$$U = \{y \in C([-r, t_1], E) : \|y\|_\infty < b_0 + 1\}\,.$$
As a consequence of Claim 2 and Claim 3 together with the Arzela-Ascoli theorem we can conclude that $G : \overline{U} \to 2^{C([-r,t_1],E)}$ is a compact multivalued map.

Claim 4: *G has a closed graph.*

Let $y_n \to y_*$, $h_n \in G(y_n)$ and $h_n \to h_*$. We shall prove that $h_* \in G(y_*)$. $h_n \in G(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that
$$h_n(t) = T(t)\phi(0) + \int_0^t T(t-s)v_n(s)\,ds\,, \qquad t \in J_0\,.$$
We must prove that there exists $v_* \in S^1_{F,y_*}$ such that
$$h_*(t) = T(t)\phi(0) + \int_0^t T(t-s)v_*(s)\,ds\,, \qquad t \in J_0\,.$$
Consider the linear continuous operator $\Gamma : L^1(J_0, E) \to C(J_0, E)$ defined by
$$(\Gamma v)(t) = \int_0^t T(t-s)v(s)\,ds\,.$$
We have
$$\|(h_n - T(t)\phi(0)) - (h_* - T(t)\phi(0))\|_\infty \to 0 \quad \text{as} \quad n \to \infty\,.$$
From Lemma 2.1 it follows that $\Gamma \circ S^1_F$ is a closed graph operator.

Also from the definition of Γ we have that
$$h_n(t) - T(t)\phi(0) \in \Gamma(S^1_{F,y_n})\,.$$
Since $y_n \to y_*$, it follows from Lemma 2.1 that
$$h_*(t) = T(t)\phi(0) + \int_0^t T(t-s)v_*(s)\,ds\,, \qquad t \in J_0$$
for some $v_* \in S^1_{F,y_*}$.

From the choice of U there is no $y \in \partial U$ such that $y \in \lambda G(y)$ for any $\lambda \in (0,1)$.

As a consequence of Lemma 2.2 we deduce that G has a fixed point $y_0 \in \overline{U}$ which is a mild solution of (3.1)–(3.2).

Step 2. Consider now the following problem on $J_1 := [t_1, t_2]$

$$y' - Ay \in F(t, y_t)\,, \quad \text{a.e.} \quad t \in J_1\,, \tag{3.3}$$
$$y(t_1^+) = I_1(y(t_1^-))\,. \tag{3.4}$$

Let y be a (possible) mild solution to (3.3)–(3.4). Then for each $t \in [t_1, t_2]$

$$y(t) - I_1(y(t_1^-)) \in \int_{t_1}^{t} T(t-s)F(s, y_s)\, ds\,.$$

From (H3) we get

$$|y(t)| \le \sup_{t \in [-r, t_1]} |I_1(y_0(t^-))| + M \int_{t_1}^{t} p(s)\psi(\|y_s\|)\, ds\,, \qquad t \in [t_1, t_2]\,.$$

We consider the function μ_1 defined by

$$\mu_1(t) = \sup\{|y(s)| : t_1 \le s \le t\}\,, \qquad t_1 \le t \le t_2\,.$$

Let $t^* \in [t_1, t]$ be such that $\mu_1(t) = |y(t^*)|$. Then we have for $t \in [t_1, t_2]$

$$\mu_1(t) \le N_1 + M \int_{t_1}^{t} p(s)\psi(\mu_1(s))\, ds\,.$$

Let us take the right-hand side of the above inequality as $v_1(t)$, then we have

$$v_1(t_1) = N_1\,, \quad \mu_1(t) \le v_1(t)\,, \qquad t \in [t_1, t_2]$$

and

$$v_1'(t) = Mp(t)\psi(\mu_1(t))\,, \qquad t \in [t_1, t_2]\,.$$

Using the nondecreasing character of ψ we get

$$v_1'(t) \le Mp(t)\psi(v_1(t))\,, \qquad t \in [t_1, t_2]\,.$$

This implies for each $t \in [t_1, t_2]$ that

$$\int_{N_1}^{v_1(t)} \frac{d\tau}{\psi(\tau)} \le M \int_{t_1}^{t_2} p(s)\, ds\,.$$

In view of (H3), we obtain

$$|v_1(t^*)| \le \Gamma_2^{-1}\left(M \int_{t_1}^{t_2} p(s) ds\right) := M_1\,.$$

Since for every $t \in [t_1, t_2]$, $\|y_t\| \le \mu_1(t)$, we have

$$\sup_{t \in [t_1, t_2]} |y(t)| \le M_1\,.$$

A mild solution to (3.3)-(3.4) is a fixed point of the operator $G : C(J_1, E) \to 2^{C(J_1,E)}$ defined by

$$G(y) := \left\{ h \in C(J_1, E) : h(t) = I_1(y(t_1^-)) + \int_{t_1}^{t} T(t-s)v(s)\,ds : \ v \in S^1_{F,y} \right\}.$$

Set

$$U = \{y \in C(J_1, E) : \|y\|_\infty < M_1 + 1\}.$$

As in Step 1 we can show that $G : \overline{U} \to 2^{C(J_1,E)}$ is a compact multivalued map and u.s.c. From the choice of U there is no $y \in \partial U$ such that $y \in \lambda G(y)$ for any $\lambda \in (0, 1)$.

As a consequence of Lemma 2.2 we deduce that G has a fixed point $y_1 \in \overline{U}$ which is a mild solution of (3.3)-(3.4).

Step 3. Continue this process and construct solutions $y_k \in C(J_k, E), k = 2, \dots, m$ to

$$y' - Ay \in F(t, y_t), \quad \text{a.e.} \quad t \in J_k, \tag{3.5}$$

$$y(t_k^+) = I_k(y(t_k^-)). \tag{3.6}$$

Then

$$y(t) = \begin{cases} y_0(t), & t \in [-r, t_1] \\ y_1(t), & t \in (t_1, t_2] \\ \vdots & \\ y_{m-1}(t), & t \in (t_{m-1}, t_m] \\ y_m(t), & t \in (t_m, b] \end{cases}$$

is a mild solution of (1.1)-(1.3). □

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