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# AN EXTENSION OF THE METHOD OF QUASILINEARIZATION

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Abstract. The method of quasilinearization is a well–known technique for obtaining approximate solutions of nonlinear differential equations. This method has recently been generalized and extended using less restrictive assumptions so as to apply to a larger class of differential equations. In this paper, we use this technique to nonlinear differential problems.

## 1. Introduction

Let $y_0, z_0 \in C^1(J, \mathbb{R})$ with $y_0(t) \le z_0(t)$ on $J$ and define the following sets

$$\bar{\Omega} = \{(t,u) : y_0(t) \le u \le z_0(t)\,,\ t \in J\}\,,$$
$$\Omega = \{(t,u,v) : y_0(t) \le u \le z_0(t)\,,\ y_0(t) \le v \le z_0(t)\,,\ t \in J\}\,.$$

In this paper, we consider the following initial value problem

$$x'(t) = f(t, x(t))\,, \quad t \in J = [0,b],\ x(0) = k_0\,, \tag{1}$$

where $f \in C(\bar{\Omega}, \mathbb{R})$, $k_0 \in \mathbb{R}$ are given. If we replace $f$ by the sum $[f = g_1 + g_2]$ of convex and concave functions, then corresponding monotone sequences converge quadratically to the unique solution of problem (1) ( see [6,8]). In this paper we will generalize this result. Assume that $f$ has the splitting $f(t,x) = F(t,x,x)$, where $F \in C(\Omega, \mathbb{R})$. Then problem (1) takes the form

$$x'(t) = F(t, x(t), x(t))\,, \quad t \in J\,,\ x(0) = k_0\,. \tag{2}$$

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## 2. Main results

A function $v \in C^1(J, \mathbb{R})$ is said to be a lower solution of problem (2) if

$$v'(t) \leq F(t, v(t), v(t)), \quad t \in J, \; v(0) \leq k_0,$$

and an upper solution of (2) if the inequalities are reversed.

**Theorem 1.** *Assume that:*

1° $y_0, z_0 \in C^1(J, \mathbb{R})$ *are lower and upper solutions of problem* (2), *respectively, such that* $y_0(t) \leq z_0(t)$ *on* $J$,

2° $F$, $F_x$, $F_y$, $F_{xx}$, $F_{xy}$, $F_{yx}$, $F_{yy} \in C(\Omega, \mathbb{R})$ *and*

$$F_{xx}(t,x,y) \geq 0, \quad F_{xy}(t,x,y) \leq 0, \quad F_{yy}(t,x,y) \leq 0 \quad \textit{for} \quad (t,x,y) \in \Omega.$$

Then there exist monotone sequences $\{y_n\}, \{z_n\}$ which converge uniformly to the unique solution $x$ of (2) on $J$, and the convergence is quadratic.

**Proof.** The above assumptions guarantee that (2) has exactly one solution on $\Omega$.

Observe that 2° implies that $F_x$ is nondecreasing in the second variable, $F_x$ is nonincreasing in the third variable and $F_y$ is nonincreasing in the last two variables. Denote this property by (A).

Let us construct the elements of sequences $\{y_n\}, \{z_n\}$ by

$$\begin{aligned} y'_{n+1}(t) &= F(t, y_n, y_n) + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][y_{n+1}(t) - y_n(t)], \\ &\qquad y_{n+1}(0) = k_0, \\ z'_{n+1}(t) &= F(t, z_n, z_n) + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][z_{n+1}(t) - z_n(t)], \\ &\qquad z_{n+1}(0) = k_0 \end{aligned}$$

for $n = 0, 1, \cdots$. Note that the above sequences are well defined.

Indeed, $y_0(t) \leq z_0(t)$ on $J$, by 1°. We shall show that

$$y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t) \quad \text{on} \quad J. \tag{3}$$

Put $p = y_0 - y_1$ on $J$. Then

$$\begin{aligned} p'(t) &\leq F(t, y_0, y_0) - F(t, y_0, y_0) - [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)][y_1(t) - y_0(t)] \\ &= [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)]p(t). \end{aligned}$$

Hence $p(t) \leq 0$ on $J$, since $p(0) \leq 0$, showing that $y_0(t) \leq y_1(t)$ on $J$. Note that if we put $p = z_1 - z_0$ on $J$, then

$$\begin{aligned} p'(t) &\leq F(t, z_0, z_0) + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)][z_1(t) - z_0(t)] - F(t, z_0, z_0) \\ &= [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)]p(t), \quad \text{and} \quad p(0) \leq 0, \end{aligned}$$

so $z_1(t) \le z_0(t)$ on $J$. Next, we let $p = y_1 - z_1$ on $J$, so $p(0) = 0$. By the mean value theorem and property (A), we have

$$\begin{aligned} p'(t) &= F(t, y_0, y_0) - F(t, z_0, y_0) + F(t, z_0, y_0) - F(t, z_0, z_0) \\ &\quad + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)][y_1(t) - y_0(t) - z_1(t) + z_0(t)] \\ &= [F_x(t, \xi, y_0) + F_y(t, z_0, \sigma)][y_0(t) - z_0(t)] \\ &\quad + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)][y_1(t) - y_0(t) - z_1(t) + z_0(t)] \\ &\le [F_x(t, y_0, z_0) - F_x(t, y_0, y_0)][z_0(t) - y_0(t)] \\ &\quad + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)]p(t) \\ &\le [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)]p(t)\,, \end{aligned}$$

where $y_0(t) < \xi(t)$, $\sigma(t) < z_0(t)$ on $J$. As the result we get $p(t) \le 0$ on $J$, so $y_1(y) \le z_1(t)$ on $J$. It proves that (3) holds.

Now we prove that $y_1, z_1$ are lower and upper solutions of (2), respectively. The mean value theorem and property (A) yield

$$\begin{aligned} y_1'(t) &= F(t, y_0, y_0) - F(t, y_1, y_0) + F(t, y_1, y_0) - F(t, y_1, y_1) + F(t, y_1, y_1) \\ &\quad + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)][y_1(t) - y_0(t)] \\ &= [F_x(t, \xi_1, y_0) + F_y(t, y_1, \sigma_1)][y_0(t) - y_1(t)] + F(t, y_1, y_1) \\ &\quad + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)][y_1(t) - y_0(t)] \\ &\le [F_x(t, y_0, z_0) - F_x(t, y_0, y_0) + F_y(t, z_0, z_0) - F_y(t, y_1, y_1)][y_1(t) - y_0(t)] \\ &\quad + F(t, y_1, y_1) \le F(t, y_1, y_1)\,, \end{aligned}$$

where $y_0(t) < \xi_1(t)$, $\sigma_1(t) < y_1(t)$ on $J$. Similarly, we get

$$\begin{aligned} z_1'(t) &= F(t, z_1, z_1) + F(t, z_0, z_0) - F(t, z_1, z_0) + F(t, z_1, z_0) - F(t, z_1, z_1) \\ &\quad + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)][z_1(t) - z_0(t)] \\ &= F(t, z_1, z_1) + [F_x(t, \xi_2, z_0) + F_y(t, z_1, \sigma_2)][z_0(t) - z_1(t)] \\ &\quad + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)][z_1(t) - z_0(t)] \\ &\ge F(t, z_1, z_1) + [F_x(t, z_1, z_0) - F_x(t, y_0, z_0) + F_y(t, z_1, z_0) \\ &\quad - F_y(t, z_0, z_0)][z_0(t) - z_1(t)] \ge F(t, z_1, z_1)\,, \end{aligned}$$

where $z_1(t) < \xi_2(t)$, $\sigma_2(t) < z_0(t)$ on $J$. The above proves that $y_1, z_1$ are lower and upper solutions of (2).

Let us assume that

$$y_0(t) \le y_1(t) \le \cdots \le y_{k-1}(t) \le y_k(t) \le z_k(t) \le z_{k-1}(t) \le \cdots \le z_1(t) \le z_0(t)\,, \quad t \in J\,,$$

and let $y_k, z_k$ be lower and upper solutions of problem (2) for some $k \ge 1$. We shall prove that:

$$y_k(t) \le y_{k+1}(t) \le z_{k+1}(t) \le z_k(t)\,, \qquad t \in J\,. \tag{4}$$

Let $p = y_k - y_{k+1}$ on $J$, so $p(0) = 0$. Using the mean value theorem, property (A) and the fact that $y_k$ is a lower solution of problem (2), we obtain

$$\begin{aligned} p'(t) &\le F(t, y_k, y_k) - F(t, y_k, y_k) - [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)][y_{k+1}(t) - y_k(t)] \\ &= [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)]p(t) . \end{aligned}$$

Hence $p(t) \le 0$, so $y_k(t) \le y_{k+1}(t)$ on $J$. Similarly, we can show that $z_{k+1}(t) \le z_k(t)$ on $J$.

Now, if $p = y_{k+1} - z_{k+1}$ on $J$, then

$$\begin{aligned} p'(t) &= F(t, y_k, y_k) - F(t, z_k, y_k) + F(t, z_k, y_k) - F(t, z_k, z_k) \\ &\quad + [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)][y_{k+1}(t) - y_k(t) - z_{k+1}(t) + z_k(t)] \\ &= [F_x(t, \bar{\xi}, y_k) + F_y(t, z_k, \bar{\sigma})][y_k(t) - z_k(t)] \\ &\quad + [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)][y_{k+1}(t) - y_k(t) - z_{k+1}(t) + z_k(t)] \\ &\le [F_x(t, y_k, z_k) - F_x(t, y_k, y_k)][z_k(t) - y_k(t)] \\ &\quad + [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)]p(t) \\ &\le [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)]p(t) \end{aligned}$$

with $y_k(t) < \bar{\xi}(t)$, $\bar{\sigma}(t) < z_k(t)$. It proves that $y_{k+1}(t) \le z_{k+1}(t)$ on $J$, so relation (4) holds.

Hence, by induction, we have

$$y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t) , \qquad t \in J ,$$

for all $n$. Employing standard techniques [5], it can be shown that the sequences $\{y_n\}$, $\{z_n\}$ converge uniformly and monotonically to the unique solution $x$ of problem (2).

We shall next show the convergence of $y_n$, $z_n$ to the unique solution $x$ of problem (2) is quadratic. For this purpose, we consider

$$p_{n+1} = x - y_{n+1} \ge 0 , \qquad q_{n+1} = z_{n+1} - x \ge 0 \quad \text{on} \quad J ,$$

and note that $p_{n+1}(0) = q_{n+1}(0) = 0$ for $n \ge 0$. Using the mean value theorem and property (A), we get

$$\begin{aligned} p'_{n+1}(t) &= F(t, x, x) - F(t, y_n, x) + F(t, y_n, x) - F(t, y_n, y_n) \\ &\quad - [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][y_{n+1}(t) - x(t) + x(t) - y_n(t)] \\ &= [F_x(t, \bar{\xi}_1, x) + F_y(t, y_n, \bar{\sigma}_1)]p_n(t) \\ &\quad + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][p_{n+1}(t) - p_n(t)] \\ &\le [F_x(t, x, x) - F_x(t, y_n, x) + F_x(t, y_n, x) - F_x(t, y_n, z_n) \\ &\quad + F_y(t, y_n, y_n) - F_y(t, z_n, y_n) + F_y(t, z_n, y_n) - F_y(t, z_n, z_n)]p_n(t) \\ &\quad + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)]p_{n+1}(t) \\ &= \{F_{xx}(t, \bar{\xi}_2, x)p_n(t) - F_{xy}(t, y_n, \bar{\sigma}_2)q_n(t) - F_{yx}(t, \bar{\xi}_3, y_n)[z_n(t) - y_n(t)] \\ &\quad - F_{yy}(t, z_n, \bar{\sigma}_3)[z_n(t) - y_n(t)]\}p_n(t) \\ &\quad + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)]p_{n+1}(t) , \end{aligned}$$

where $y_n(t) < \bar{\xi}_1(t)$, $\bar{\xi}_2(t), \bar{\sigma}_1(t) < x(t)$, $x(t) < \bar{\sigma}_2(t) < z_n(t)$, $y_n(t) < \bar{\xi}_3(t)$, $\bar{\sigma}_3(t) < z_n(t)$ on $J$. Thus we obtain

$$\begin{aligned} p'_{n+1}(t) &\le \{A_1 p_n(t) + A_2 q_n(t) + [A_2 + A_3][q_n(t) + p_n(t)]\} p_n(t) + M p_{n+1}(t) \\ &\le M p_{n+1}(t) + B_1 p_n^2(t) + B_2 q_n^2(t)\,, \end{aligned}$$

where

$$|F_{xx}(t,u,v)| \le A_1\,, \quad |F_{xy}(t,u,v)| \le A_2\,, \quad |F_{yy}(t,u,v)| \le A_3\,, \quad |F_x(t,u,v)| \le M_1\,,$$

$$|F_y(t,u,v)| \le M_2 \ \text{ on } \ \Omega \ \text{ with } M = M_1 + M_2\,, \quad B_1 = A_1 + 2A_2 + \frac{3}{2}A_3\,,$$

$$B_2 = A_2 + \frac{1}{2}A_3\,.$$

Now, the differential inequality implies

$$0 \le p_{n+1}(t) \le \int_0^t [B_1 p_n^2(s) + B_2 q_n^2(s)] \exp[M(t-s)]\, ds\,.$$

This yields the following relation

$$\max_{t\in J} |x(t) - y_{n+1}(t)| \le a_1 \max_{t\in J} |x(t) - y_n(t)|^2 + a_2 \max_{t\in J} |x(t) - z_n(t)|^2\,,$$

where $a_i = B_i S$, $i = 1, 2$ with

$$S = \begin{cases} b & \text{if} \quad M = 0\,, \\ \frac{1}{M}[\exp(Mb) - 1] & \text{if} \quad M > 0\,. \end{cases}$$

Similarly, we find that

$$\begin{aligned} q'_{n+1}(t) &= F(t, z_n, z_n) - F(t, x, z_n) + F(t, x, z_n) - F(t, x, x) \\ &\quad + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][z_{n+1}(t) - x(t) + x(t) - z_n(t)] \\ &= [F_x(t, \bar{\xi}_4, z_n) + F_y(t, x, \bar{\sigma}_4)] q_n(t) \\ &\quad + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][q_{n+1}(t) - q_n(t)] \\ &\le [F_x(t, z_n, z_n) - F_x(t, y_n, z_n) + F_y(t, x, x) - F_y(t, z_n, x) \\ &\quad + F_y(t, z_n, x) - F_y(t, z_n, z_n)] q_n(t) + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)] q_{n+1}(t) \\ &= \{F_{xx}(t, \bar{\xi}_5, z_n)[z_n(t) - y_n(t)] \\ &\quad - F_{yx}(t, \bar{\xi}_6, x) q_n(t) - F_{yy}(t, z_n, \bar{\sigma}_5) q_n(t)\} q_n(t) \\ &\quad + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)] q_{n+1}(t)\,, \end{aligned}$$

where $x(t) < \bar{\xi}_4(t)$, $\bar{\xi}_6(t), \bar{\sigma}_4(t), \bar{\sigma}_5(t) < z_n(t)$, $y_n(t) < \bar{\xi}_5(t) < z_n(t)$ on $J$. Hence, we get

$$\begin{aligned} q'_{n+1}(t) &\le \{A_1[q_n(t) + p_n(t)] + A_2 q_n(t) + A_3 q_n(t)\} q_n(t) + M q_{n+1}(t)\,, \\ &\le M q_{n+1}(t) + \bar{B}_1 p_n^2(t) + \bar{B}_2 q_n^2(t)\,, \end{aligned}$$

where

$$\bar{B}_1 = \frac{1}{2}A_1\,, \quad \bar{B}_2 = \frac{3}{2}A_1 + A_2 + A_3\,.$$

Now, the last differential inequality implies

$$q_{n+1}(t) \le [\bar{B}_1 \max_{s\in J} p_n^2(s) + \bar{B}_2 \max_{s\in J} q_n^2(s)]S\,, \quad t \in J$$

or

$$\max_{t\in J} |x(t) - z_{n+1}(t)| \le \bar{a}_1 \max_{t\in J} |x(t) - y_n(t)|^2 + \bar{a}_2 \max_{t\in J} |x(t) - z_n(t)|^2$$

with $\bar{a}_i = \bar{B}_i S$, $i = 1, 2$.

The proof is complete. □

**Remark 1.** Let $f = h + g$, and $h, h_x, h_{xx}, g, g_x, g_{xx} \in C(\Omega_1, \mathbb{R})$ for $\Omega_1 = \{(t,u) : t \in J,\ y_0(t) \le u \le z_0(t)\}$. Put $F(t,x,y) = h(t,x) + g(t,y)$. Indeed, $F(t,x,x) = f(t,x)$ and $F_{xx}(t,x,y) = h_{xx}(t,x)$, $F_{xy}(t,x,y) = F_{yx}(t,x,y) = 0$, $F_{yy}(t,x,y) = g_{yy}(t,y)$. In this case Theorem 1 reduces to Theorem 1.3.1 of [8].

**Remark 2.** Let $f, h, g$ be as in Remark 1 and moreover let $\Phi, \Phi_x, \Phi_{xx}, \Psi, \Psi_x, \Psi_{xx} \in C(\Omega_1, \mathbb{R})$. Put $F(t,x,y) = H(t,x) + G(t,y) - \Phi(t,y) - \Psi(t,x)$ for $H = h + \Phi$, $G = g + \Psi$. Indeed, $F(t,x,x) = f(t,x)$ and $F_{xx}(t,x,y) = H_{xx}(t,x) - \Psi_{xx}(t,x)$, $F_{xy}(t,x,y) = F_{yx}(t,x,y) = 0$, $F_{yy}(t,x,y) = G_{yy}(t,y) - \Phi_{yy}(t,y)$. If assumptions of Theorem 1.4.3[8] hold ($H_{xx} \ge 0$, $\Psi_{xx} \le 0$, $G_{yy} \le 0$, $\Phi_{yy} \ge 0$) then Theorem 1 is satisfied ( see also a result of [6] for $g = \Psi = 0$, $\Phi(t,x) = Mx^2$, $M > 0$).

**Theorem 2.** *Assume that*

(i) *condition* $1^\circ$ *of Theorem 1 holds,*

(ii) $F, F_x, F_y, F_{xx}, F_{xy}, F_{yx}, F_{yy} \in C(\Omega, \mathbb{R})$ *and*

$$F_{xx}(t,x,y) \ge 0\,, \quad F_{xy}(t,x,y) \ge 0\,, \quad F_{yy}(t,x,y) \le 0 \quad \text{for} \quad (t,x,y) \in \Omega\,.$$

Then the conclusion of Theorem 1 remains valid.

**Proof.** Note that, in view of (ii), $F_x$ is nondecreasing in the last two variables, $F_y$ is nondecreasing in the second variable, and $F_y$ is nonincreasing in the third one. Denote this property by (B).

We construct the monotone sequences $\{y_n\}, \{z_n\}$ by formulas:

$$\begin{aligned} y'_{n+1}(t) &= F(t, y_n, y_n) + [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)][y_{n+1}(t) - y_n(t)]\,, \\ &\qquad y_{n+1}(0) = k_0\,, \\ z'_{n+1}(t) &= F(t, z_n, z_n) + [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)][z_{n+1}(t) - z_n(t)]\,, \\ &\qquad z_{n+1}(0) = k_0 \end{aligned}$$

for $n = 0, 1, \ldots$.

Let $p = y_0 - y_1$ on $J$. Then

$$\begin{aligned} p'(t) &\leq F(t, y_0, y_0) - F(t, y_0, y_0) - [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)][y_1(t) - y_0(t)] \\ &= [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)]p(t)\,, \quad \text{and} \quad p(0) \leq 0\,. \end{aligned}$$

Hence $p(t) \leq 0$ on $J$, showing that $y_0(t) \leq y_1(t)$ on $J$. Similarly, we can show that $z_1(t) \leq z_0(t)$ on $J$. If we now put $p = y_1 - z_1$ on $J$, then the mean value theorem and property (B), we have

$$\begin{aligned} p'(t) = {} & F(t, y_0, y_0) - F(t, z_0, y_0) + F(t, z_0, y_0) - F(t, z_0, z_0) \\ & + [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)][y_1(t) - y_0(t) - z_1(t) + z_0(t)] \\ = {} & [F_x(t, \xi, y_0) + F_y(t, z_0, \sigma)][y_0(t) - z_0(t)] \\ & + [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)][p(t) - z_1(t) + z_0(t)] \\ \leq {} & [F_y(t, y_0, z_0) - F_y(t, z_0, z_0)][z_0(t) - y_0(t)] \\ & + [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)]p(t) \\ \leq {} & [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)]p(t)\,, \quad p(0) = 0 \end{aligned}$$

with $y_0(t) < \xi(t)$, $\sigma(t) < z_0(t)$ on $J$. Hence $y_1(t) \leq z_1(t)$ on $J$, and as a result, we obtain

$$y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t) \quad \text{on} \quad J\,.$$

Continuing this process successively, by induction, we get

$$y_0(t) \leq y_1(t) \leq \cdots \leq y_n(t) \leq z_n(t) \leq \cdots \leq z_1(t) \leq z_0(t)\,, \quad t \in J\,,$$

for all $n$. Indeed, the sequences $\{y_n\}$, $\{z_n\}$ converge uniformly and monotonically to the unique solution $x$ of problem (2). Now, we are in a position to show that this convergence is quadratic.

Let

$$p_{n+1} = x - y_{n+1} \geq 0\,, \quad q_{n+1} = z_{n+1} - x \geq 0 \quad \text{on} \quad J\,.$$

Hence $p_{n+1}(0) = q_{n+1}(0) = 0$. The mean value theorem and property (B) yield

$$\begin{aligned} p'_{n+1}(t) = {} & F(t, x, x) - F(t, y_n, x) + F(t, y_n, x) - F(t, y_n, y_n) \\ & - [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)][y_{n+1}(t) - x(t) + x(t) - y_n(t)] \\ = {} & [F_x(t, \xi_1, x) + F_y(t, y_n, \sigma_1)]p_n(t) \\ & + [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)][p_{n+1}(t) - p_n(t)] \\ \leq {} & [F_x(t, x, x) - F_x(t, y_n, x) + F_x(t, y_n, x) - F_x(t, y_n, y_n) \\ & + F_y(t, y_n, y_n) - F_y(t, y_n, z_n)]p_n(t) \\ & + [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)]p_{n+1}(t) \\ = {} & \{F_{xx}(t, \xi_2, x)p_n(t) + F_{xy}(t, y_n, \sigma_2)p_n(t) \\ & - F_{yy}(t, y_n, \sigma_3)[z_n(t) - y_n(t)]\}p_n(t) \\ & + [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)]p_{n+1}(t)\,, \end{aligned}$$

where $y_n(t) < \xi_1(t),\ \xi_2(t), \sigma_1(t), \sigma_2(t) < x(t),\ y_n(t) < \sigma_3(t) < z_n(t)$ on $J$. Thus we obtain

$$\begin{aligned} p'_{n+1}(t) &\le \{(A_1 + A_2)p_n(t) + A_3[q_n(t) + p_n(t)]\}p_n(t) + Mp_{n+1}(t) \\ &\le Mp_{n+1}(t) + D_1 p_n^2(t) + D_2 q_n^2(t)\,, \end{aligned}$$

where $D_1 = A_1 + A_2 + \frac{3}{2}A_3$, $D_2 = \frac{1}{2}A_3$. Hence, we get

$$0 \le p_{n+1}(t) \le \int_0^t [D_1 p_n^2(s) + D_2 q_n^2(s)] \exp[M(t-s)]\, ds\,,$$

and it yields the relation

$$\max_{t\in J} |x(t) - y_{n+1}(t)| \le d_1 \max_{t\in J} |x(t) - y_n(t)|^2 + d_2 \max_{t\in J} |x(t) - z_n(t)|^2\,,$$

where $d_i = D_i S$, $i = 1, 2$.

By the similar argument, we can show that

$$\max_{t\in J} |x(t) - z_{n+1}(t)| \le \bar{d}_1 \max_{t\in J} |x(t) - y_n(t)|^2 + \bar{d}_2 \max_{t\in J} |x(t) - z_n(t)|^2\,,$$

with $\bar{d}_i = \bar{D}_i S$, $i = 1, 2$, for $\bar{D}_1 = \frac{1}{2}A_1 + A_2$, $\bar{D}_2 = \frac{3}{2}A_1 + 2A_2 + A_3$.

This ends the proof. □

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