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A LOGIC OF ORTHOGONALITY

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This paper was inspired by the hard-to-beleive fact that Jiří Rosický is getting sixty. We are happy to dedicate our paper to his birthday.

Abstract. A logic of orthogonality characterizes all "orthogonality consequences" of a given class Σ of morphisms, i.e. those morphisms s such that every object orthogonal to Σ is also orthogonal to s. A simple four-rule deduction system is formulated which is sound in every cocomplete category. In locally presentable categories we prove that the deduction system is also complete (a) for all classes Σ of morphisms such that all members except a set are regular epimorphisms and (b) for all classes Σ, without restriction, under the set-theoretical assumption that Vopěnka's Principle holds. For finitary morphisms, i.e. morphisms with finitely presentable domains and codomains, an appropriate finitary logic is presented, and proved to be sound and complete; here the proof follows immediately from previous joint results of Jiří Rosický and the first two authors.

1. Introduction

The famous "orthogonal subcategory problem" asks whether, given a class Σ of morphisms of a category, the full subcategory $\Sigma^{\perp}$ of all objects orthogonal to Σ is reflective. Recall that an object is orthogonal to Σ iff its hom-functor takes members of Σ to isomorphisms. In the realm of locally presentable categories for the orthogonal subcategory problem

(a) the answer is affirmative whenever Σ is small – more generally, as proved by Peter Freyd and Max Kelly [7], it is affirmative whenever $\Sigma = \Sigma_0 \cup \Sigma_1$ where Σ_0 is small and Σ_1 is a class of epimorphisms,

and

(b) assuming the large-cardinal Vopěnka's Principle, the answer remains affirmative for all classes Σ, as proved by the first author and Jiří Rosický in [3].

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The problem to which the present paper is devoted is "dual": we study the *orthogonality consequences* of classes Σ of morphisms by which we mean morphisms s such that every object of $\Sigma^{\perp}$ is also orthogonal to s. Example: if $\Sigma^{\perp}$ is reflective, then all the reflection maps are orthogonality consequences of Σ. Another important example: given a Gabriel-Zisman category of fractions $C_{\Sigma} : \mathcal{A} \to \mathcal{A}[\Sigma^{-1}]$, then every morphism which C_{Σ} takes to an isomorphism is an orthogonality consequence of Σ. In Section 2 we recall the precise relationship between $\Sigma^{\perp}$ and $\mathcal{A}[\Sigma^{-1}]$.

We formulate a very simple logic for orthogonality consequence (inspired by the calculus of fractions and by the work of Grigore Roçu [12]) and prove that it is sound in every cocomplete category. That is, whenever a morphism s has a formal proof from a class Σ, then s is an orthogonality consequence of Σ. In the realm of locally presentable categories we also prove that our logic is complete, that is, every orthogonality consequence of Σ has a formal proof, provided that

(a) Σ is small – more generally, completeness holds whenever $\Sigma = \Sigma_0 \cup \Sigma_1$ where Σ_0 is small and Σ_1 is a class of regular epimorphisms

or

(b) Vopěnka's Principle is assumed.

(We recall Vopěnka's Principle in Section 4.) In fact the completeness of our logic for all classes of morphisms will be proved to be *equivalent* to Vopěnka's Principle. This is very similar to results of Jiří Rosický and the first author concerning the orthogonal subcategory problem, see 6.24 and 6.25 in [3].

Our logic is quite analogous to the Injectivity Logic of [4] and [1], see also [12]. There a morphism s is called an (*injectivity*) *consequence* of Σ provided that every object injective w.r.t. members of Σ is also injective w.r.t. s. Recall that an object is injective w.r.t. a morphism s iff its hom-functor takes s to an epimorphism. Recall further from [1] that the deduction system for Injectivity Logic has just three deduction rules:

TRANSFINITE COMPOSITION $\quad \dfrac{s_i (i < \alpha)}{t} \quad$ if t is an α-composite of the s_i's

PUSHOUT $\quad \dfrac{s}{t} \quad$ if
$$\begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "t"'] & \cdot \end{tikzcd}$$
is a pushout

CANCELLATION $\quad \dfrac{u \cdot t}{t}$

We recall the concept of α-composite in 3.2 below.

In locally presentable categories the Injectivity Logic is, as proved in [1], complete and sound for all sets Σ of morphisms; but not for classes, in general: a counter-example can be presented, see the end of our paper, independent of set theory. This is quite surprising since under Vopěnka's Principle all injectivity classes are weakly reflective, see [3], 6.27, which seems to indicate that the Injectivity Logic should always be complete - but it is not!

Now both TRANSFINITE COMPOSITION and PUSHOUT are sound rules for orthogonality too. In contrast, CANCELLATION is not sound and has to be substituted by the following weaker form:

WEAK CANCELLATION $\dfrac{u \cdot t \quad v \cdot u}{t}$

Further we have to add a fourth rule in case of orthogonality:

COEQUALIZER $\dfrac{s}{t}$ if $\rightrightarrows^{f}_{g} \xrightarrow{t}$ is a coequalizer such that $f \cdot s = g \cdot s$

We obtain a 4-rule deduction system for which the above completeness results (a) and (b) will be proved.

The above logics are infinitary, in fact, TRANSFINITE COMPOSITION is a scheme of deduction rules, one for every ordinal α. We also study the corresponding finitary logics by restricting ourselves to sets Σ of *finitary morphisms*, meaning morphisms with finitely presentable domain and codomain. Both in the injectivity case and in the orthogonality case one simply replaces TRANSFINITE COMPOSITION by two rules:

IDENTITY $\dfrac{}{\mathrm{id}_A}$

and

COMPOSITION $\dfrac{s_1 \quad s_2}{t}$ if $t = s_2 \cdot s_1$

This finitary logic is proved to be sound and complete for sets of finitary morphisms. In fact, in [10] a description of the category of fractions $\mathcal{A}_\omega[\Sigma^{-1}]$ (see 2.4) as a dual to the theory of the subcategory $\Sigma^\perp$ is presented; our proof of completeness of the finitary logic is an easy consequence.

The result of Peter Freyd and Max Kelly mentioned at the beginning goes beyond locally presentable categories, and also our preceding paper [1] is not restricted to this context. Nonetheless, the present paper studies the orthogonality consequence and its logic in locally presentable categories only.

Throughout the paper we work with categories that are, in general, not locally small. The Axiom of Choice for classes is assumed.

2. Finitary Logic and the Calculus of Fractions

2.1. Assumption. Throughout the paper $\mathcal{A}$ denotes a locally presentable category in the sense of Gabriel and Ulmer; the reader may consult the monograph [3]. Recall that an object is λ-presentable iff its hom-functor preserves λ-filtered colimits. A *locally presentable category* is a cocomplete category $\mathcal{A}$ such that, for some infinite cardinal λ, there exists a set

$$\mathcal{A}_\lambda$$

of objects representing all λ-presentable objects up-to an isomorphism and such that a completion of $\mathcal{A}_\lambda$ under λ-filtered colimits is all of $\mathcal{A}$. The category $\mathcal{A}$ is then said to be *locally λ-presentable.* Recall that a *theory* of a locally λ-presentable category $\mathcal{A}$ is a small category $\mathcal{T}$ with λ-small limits[1] such that $\mathcal{A}$ is equivalent to the category

$$Cont_\lambda(\mathcal{T})$$

of all set-valued functors on $\mathcal{T}$ preserving λ-small limits. For every locally λ-presentable category it follows that the dual $\mathcal{A}_\lambda^{\mathrm{op}}$ of the above full subcategory is a theory of $\mathcal{A}$:

$$\mathcal{A} \cong Cont_\lambda(\mathcal{A}_\lambda^{\mathrm{op}})\,.$$

Morphisms with λ-presentable domain and codomain are called *λ-ary morphisms.*

2.2. Notation. (i) For every class Σ of morphisms of $\mathcal{A}$ we denote by

$$\Sigma^\perp$$

the full subcategory of all objects orthogonal to Σ. If Σ is small, this subcategory is reflective, see e.g. [7].

(ii) We write $\Sigma \models s$ for the statement that s is an orthogonality consequence of s, in other words, $\Sigma^\perp = (\{s\} \cup \Sigma)^\perp$.

(iii) We denote, whenever $\Sigma^\perp$ is reflective, by

$$R_\Sigma : \mathcal{A} \to \Sigma^\perp$$

a reflector functor and by $\eta_A : A \to R_\Sigma A$ the reflection map; without loss of generality we will assume $R_\Sigma \eta_A = \mathrm{id}_{R_\Sigma A} = \eta_{R_\Sigma A}$.

2.3. Observation. If $\Sigma^\perp$ is a reflective subcategory, then orthogonality consequences of Σ are precisely the morphisms s such that $R_\Sigma s$ is an isomorphism.

In fact, if $s : A \to B$ is an orthogonality consequence of Σ, then $R_\Sigma A$ is orthogonal to s, which yields a commutative triangle

$$\begin{tikzcd} A \arrow[rr, "s"] \arrow[dr, "\eta_A"'] & & B \arrow[dl, "u"] \\ & R_\Sigma A & \end{tikzcd}$$

The unique morphism $\bar{u} : R_\Sigma B \to R_\Sigma A$ with $\bar{u} \cdot \eta_B = u$ is inverse to $R_\Sigma s$: this follows from the diagram

$$\begin{tikzcd} A \arrow[rr, "s"] \arrow[dr, "\eta_A"'] & & B \arrow[dl, "u"'] \arrow[dr, "\eta_B"] & \\ & R_\Sigma A \arrow[rr, shift right, "R_\Sigma s"'] & & R_\Sigma B \arrow[ll, shift right, "\bar{u}"'] \end{tikzcd}$$

Conversely, if $s : A \to B$ is turned by R_Σ to an isomorphism, then every object X orthogonal to Σ is orthogonal to s: given $f : A \to X$ we have a unique $\bar{f} : R_\Sigma A \to$

[1] Limits of diagrams of less than λ morphisms are called λ-small limits. Analogously λ-wide pushouts are pushouts of less than λ morphisms.

X with $f = \bar{f} \cdot \eta_A$, and we use $\bar{f} \cdot (R_\Sigma s)^{-1} \cdot \eta_B : B \to X$. It is easy to check that this is the unique factorization of f through s.

2.4. Remark. The above observation shows a connection of the orthogonality logic to the calculus of fractions of Peter Gabriel and Michel Zisman [8], see also Section 5.2 in [5].

Given a class Σ of morphisms in $\mathcal{A}$, its *category of fractions* is a category $\mathcal{A}[\Sigma^{-1}]$ together with a functor

$$C_\Sigma : \mathcal{A} \to \mathcal{A}[\Sigma^{-1}]$$

universal w.r.t. the property that C_Σ takes members of Σ to isomorphisms. (That is, if a functor $F : \mathcal{A} \to \mathcal{B}$ takes members of Σ to isomorphisms, then there exists a unique functor $\bar{F} : \mathcal{A}[\Sigma^{-1}] \to \mathcal{B}$ with $F = \bar{F} \cdot C_\Sigma$.)

The category of fractions is unique up-to isomorphism of categories. If $\mathcal{A}$ is locally small, the category of fractions is also locally small if Σ is small, see [5], 5.2.2.

2.5. Example (see [5], 5.3.1)**.** For every reflective subcategory $\mathcal{B}$ of $\mathcal{A}$, $R : \mathcal{A} \to \mathcal{B}$ the reflector, put $\Sigma = \{s \mid Rs \text{ is an isomorphism}\}$. Then $\mathcal{B} = \Sigma^\perp \simeq \mathcal{A}[\Sigma^{-1}]$. More precisely, there exists an equivalence $E : \mathcal{A}[\Sigma^{-1}] \to \Sigma^\perp$ such that $E \cdot C_\Sigma = R = R_\Sigma$.

2.6. Example (see [6])**.** In the category $\mathbf{Ab}$ of abelian groups consider the single morphism

$$\Sigma = \{\mathbb{Z} \to 0\}$$

where $\mathbb{Z}$ is the group of integers. Then clearly

$$\Sigma^\perp = \{0\}\,.$$

Observe that

$$\mathbf{Ab}[\Sigma^{-1}] \not\cong \{0\}$$

because the coreflector $F : \mathbf{Ab} \to \mathbf{Ab}_t$ of the full subcategory $\mathbf{Ab}_t$ of all torsion groups takes $\mathbb{Z} \to 0$ to an isomorphism, but F is the identity functor on $\mathbf{Ab}_t$. This of course implies that $C_\Sigma : \mathbf{Ab} \to \mathbf{Ab}[\Sigma^{-1}]$ is monic on $\mathbf{Ab}_t$.

2.7. Definition (see [8])**.** A class Σ of morphisms is said to *admit a left calculus of fractions* provided that

(i) Σ contains all identity morphisms,

(ii) Σ is closed under composition,

(iii) for every span

$$\begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, "f"'] & \cdot \\ \cdot & \end{tikzcd} \qquad \text{with } s \in \Sigma$$

there exists a commutative square

$$\begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, "f"'] & \cdot \arrow[d, "f'"] \\ \cdot \arrow[r, "s'"'] & \cdot \end{tikzcd} \quad \text{with } s' \in \Sigma$$

and

(iv) for every parallel pair f, g equalized by a member s of Σ there exists a member s' of Σ coequalizing the pair:

$$\xrightarrow{s} \;\underset{g}{\overset{f}{\rightrightarrows}}\; \xrightarrow{s'}$$

2.8. Theorem (see [10], IV.2)**.** *Let Σ be a set of finitary morphisms of a locally finitely presentable category $\mathcal{A}$. If Σ admits a left calculus of fractions in the subcategory $\mathcal{A}_\omega$, then $\Sigma^\perp$ is a locally finitely presentable category whose theory is dual to $\mathcal{A}_\omega[\Sigma^{-1}]$.*

More precisely: Let $C_\Sigma : \mathcal{A}_\omega \to \mathcal{A}_\omega[\Sigma^{-1}]$ be the canonical functor from $\mathcal{A}_\omega$ into the category of fractions of Σ in $\mathcal{A}_\omega$, see 2.4. Then there exists an equivalence functor

$$J : Cont_\omega(\mathcal{A}_\omega[\Sigma^{-1}]^{\mathrm{op}}) \to \Sigma^\perp$$

such that for the inclusion functor $I : \mathcal{A}_\omega \to \mathcal{A}$ and the Yoneda embedding $Y : \mathcal{A}_\omega[\Sigma^{-1}] \to Cont_\omega(\mathcal{A}_\omega[\Sigma^{-1}]^{\mathrm{op}})$ the following diagram

(2.1)
$$\begin{tikzcd}
\mathcal{A}_\omega \arrow[r, "C_\Sigma"] \arrow[dd, hook, "I"'] & \mathcal{A}_\omega[\Sigma^{-1}] \arrow[d, hook, "Y"] \\
& Cont_\omega(\mathcal{A}_\omega[\Sigma^{-1}]^{\mathrm{op}}) \arrow[d, "J"] \\
\mathcal{A} \arrow[r, "R_\Sigma"] & \Sigma^\perp
\end{tikzcd}$$

commutes.

2.9. Corollary. *Let Σ admit a left calculus of fractions in $\mathcal{A}_\omega$. Then the orthogonality consequences of Σ in $\mathcal{A}_\omega$ are precisely the finitary morphisms s such that $C_\Sigma s$ is an isomorphism.*

In fact, since $J \cdot Y$ is a full embedding, we know that $C_\Sigma s$ is an isomorphism iff $(J \cdot Y \cdot C_\Sigma)s$ is one, thus, this follows from Observation 2.3.

2.10. Example (refer to 2.6)**.** For $\Sigma = \{\mathbb{Z} \to 0\}$, the smallest class Σ_0 in $\mathbf{Ab}$ (resp., in $\mathbf{Ab}_\omega$) containing Σ and admitting a left calculus of fractions is the class of all (resp., all finitary) morphisms which are identities or have codomain 0. One sees easily that $\mathbf{Ab}[\Sigma_0^{-1}] = \{0\} = \mathbf{Ab}_\omega[\Sigma_0^{-1}] = \Sigma_0^\perp = \Sigma^\perp$.

2.11. Remark. In a finitely cocomplete category $\mathcal{A}$ for every set Σ of finitary morphisms there is a canonical extension of Σ to a set Σ' admitting a left calculus of fractions in $\mathcal{A}_\omega$: let Σ' be the closure in $\mathcal{A}_\omega$ of

$$\Sigma \cup \{\mathrm{id}_A\}_{A \in \mathcal{A}_\omega}$$

under

(a) composition
(b) pushout
and
(c) "weak coequalizers" in the sense that Σ' contains, for every pair $f, g : A \to B$, a coequalizer of f, g whenever $f \cdot s = g \cdot s$ for some member s of Σ'.
We will see in Observation 2.16 below that Σ and Σ' have the same orthogonality consequences.

2.12. Theorem (see [5], 5.9.3)**.** *If a set Σ admits a left calculus of fractions, then the class of all morphisms taken by C_Σ to isomorphisms is the smallest class Σ' containing Σ and such that given three composable morphisms*

$$\xrightarrow{t} \xrightarrow{u} \xrightarrow{v}$$

with $u \cdot t$ and $v \cdot u$ both in Σ', then t lies in Σ'.

2.13. Remark. Apply the above theorem to Σ' of Remark 2.11: if Σ'' denotes the closure of Σ' under "weak cancellation" in the sense that from $u \cdot t \in \Sigma''$ and $v \cdot u \in \Sigma''$ we derive $t \in \Sigma''$, then Σ'' is precisely the class taken by C_Σ to isomorphisms. This leads us to the following

2.14. Definition. The *Finitary Orthogonality Deduction System* consists of the following deduction rules:

IDENTITY $\qquad \dfrac{}{\mathrm{id}_A}$

COMPOSITION $\qquad \dfrac{s_1 \quad s_2}{t}$ $\quad$ if $t = s_2 \cdot s_1$

PUSHOUT $\qquad \dfrac{s}{t}$ $\quad$ if $\begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "t"'] & \cdot \end{tikzcd}$ is a pushout

COEQUALIZER $\qquad \dfrac{s}{t}$ $\quad$ if $\begin{tikzcd} \cdot \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \cdot \arrow[r, "t"] & \cdot \end{tikzcd}$ is a coequalizer and $f \cdot s = g \cdot s$

WEAK CANCELLATION $\qquad \dfrac{u \cdot t \quad v \cdot u}{t}$

We say that a morphism s *can be proved* from a set Σ of morphisms using the Finitary Orthogonality Logic, in symbols

$$\Sigma \vdash s$$

provided that there exists a formal proof of s from Σ using the above five deduction rules (in $\mathcal{A}_\omega$).

2.15. Remark. A formal proof of s is a finite list

$$t_1, t_2, \ldots, t_k$$

of finitary morphisms such that $s = t_k$ and for every $i = 1, \ldots, k$ either $t_i \in \Sigma$, or t_i is the conclusion of one of the deduction rules whose assumptions lie in the set $\{t_1, \ldots, t_{i-1}\}$.

For a locally presentable category the *Finitary Orthogonality Logic* is the application of the relations $\vdash$ and $\models$ to finitary morphisms of $\mathcal{A}$.

2.16. Observation. In every finitely cocomplete category the Finitary Orthogonality Logic is sound: if a finitary morphism s has a proof from a set Σ of finitary morphisms then s is an orthogonality consequence of Σ. Shortly:

$$\Sigma \vdash s \quad \text{implies} \quad \Sigma \models s\,.$$

It is sufficient to check individually the soundness of the five deduction rules. Every object X is clearly orthogonal to id_A; and it is orthogonal to $s_2 \cdot s_1$ whenever X is orthogonal to s_1 and s_2. The soundness of the pushout rule is also elementary:

$$\begin{tikzcd}
{} \arrow[r, "s"] \arrow[d] & {} \arrow[d] \arrow[ddr, "\exists!"] & \\
{} \arrow[r, "t"] \arrow[drr, "\forall"'] & {} \arrow[dr, dashed] & \\
& & X
\end{tikzcd}$$

Suppose t is a coequalizer of f, $g : A \to B$ and let $f \cdot s = g \cdot s$. Whenever X is orthogonal to s, it is orthogonal to t. In fact, given a morphism $p : B \to X$,

$$\begin{tikzcd}
A' \arrow[r, "s"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "t"] \arrow[d, "p"'] & B' \arrow[dl, dashed] \\
& & X &
\end{tikzcd}$$

then from $p \cdot f \cdot s = p \cdot g \cdot s$ it follows that $p \cdot f = p \cdot g$ (due to $X \perp s$) and thus p uniquely factors through $t = \mathrm{coeq}(f, g)$.

Finally, let X be orthogonal to $u \cdot t$ and $v \cdot u$,

$$\begin{tikzcd}
A \arrow[r, "t"] \arrow[dr, "p"'] & B \arrow[r, "u"] \arrow[d, shift right, "r"'] \arrow[d, shift left, dashed, "r'"] & C \arrow[r, "v"] \arrow[dl, "q"] & D \arrow[dll, "w"] \arrow[dll, dashed, bend left, "w'"] \\
& X & &
\end{tikzcd}$$

then we show $X \perp t$. Given $p : A \to X$ there exists $q : C \to X$ with $p = q \cdot (u \cdot t)$. Then $r = q \cdot u$ fulfils $p = r \cdot t$. Suppose r' fulfils $p = r' \cdot t$. We have, since $X \perp v \cdot u$, a unique $w : D \to X$ with $r = w \cdot v \cdot u$ and a unique w' with $r' = w' \cdot v \cdot u$. The equality $w \cdot v \cdot u \cdot t = w' \cdot v \cdot u \cdot t$ implies $w \cdot v = w' \cdot v$, thus,

$$r = w \cdot v \cdot u = w' \cdot v \cdot u = r'\,.$$

2.17. Theorem. *In locally finitely presentable categories the Finitary Orthogonality Logic is complete:*

$$\Sigma \models s \quad \textit{implies} \quad \Sigma \vdash s\,.$$

for all sets $\Sigma \cup \{s\}$ *of finitary morphisms.*

Proof. Let s be an orthogonality consequence of Σ in $\mathcal{A}_\omega$ and let $\bar{\Sigma}$ be the set of all finitary morphisms that can be proved from Σ; we have to verify that $s \in \bar{\Sigma}$. Due to the first four deduction rules, $\bar{\Sigma}$ clearly admits a left calculus of fractions in $\mathcal{A}_\omega$. Hence $C_{\bar{\Sigma}}s$ is, by Corollary 2.9, an isomorphism. Theorem 2.12 implies (due to WEAK CANCELLATION) that $s \in \bar{\Sigma}$.

2.18. Example demonstrating that we cannot, for the finitary orthogonality logic, work entirely within the full subcategory $\mathcal{A}_\omega$: let us denote by

$$\Sigma \models_\omega s$$

the statement that every finitely presentable object $X \in \Sigma^\perp$ is orthogonal to s. Then it is in general *not* true that, given a set of finitary morphisms Σ, then $\Sigma \models_\omega s$ implies $\Sigma \vdash s$.

Let $\mathcal{A} = \mathbf{Rel(2, 2)}$ be the category of relational structures on two binary relations α and β. We denote by

$\emptyset$ the initial (empty) object,

1 a terminal object (a single node which is a loop of α and β),

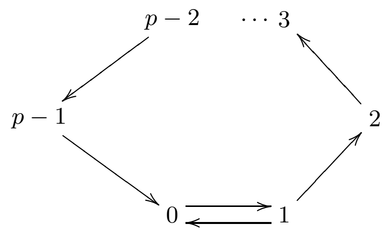
T a one-element object with $\alpha = \emptyset$ and β a loop

and, for every prime $p \geq 3$, by

A_p the object on $\{0, 1, \ldots, p-1\}$ whose relation β is a *clique* (that is, two elements

are related by β iff they are distinct) and the relation α is a cycle of length p with

one additional edge from 1 to 0:



Consider the set Σ of finitary morphisms given by

$$\Sigma = \{u, v\} \cup \{\emptyset \to A_p;\ p \geq 3 \text{ a prime}\}$$

where $u : T \to 1$ and $v : 1 + 1 \to 1$ are the unique morphisms. Orthogonality of a relational structure X to Σ implies that every loop of the relation β is a joint loop of both relations (due to u) and such a loop is unique (due to v). Moreover, the given object X has a unique morphism from each A_p. If X is finitely presentable (i.e., in this case, finite), then one of these morphisms $f : A_p \to X$ is not monic; given $i \neq j$ with $f(i) = x = f(j)$, then x is a loop of β in X (recall that β is a clique in A_p), thus, X has a unique joint loop of α and β, in other words, a unique morphism $1 \to X$. Consequently, X is orthogonal to $\emptyset \to 1$. This proves

$$\Sigma \models_\omega (\emptyset \to 1)\,.$$

However $\emptyset \to 1$ cannot be deduced from Σ in the Finitary Deduction System because the object

$$Y = \coprod_{\substack{p \geq 3 \\ p \text{ prime}}} A_p$$

is orthogonal to Σ but not to $\emptyset \to 1$. In fact, Y has no loop of β, thus, Y is orthogonal to u and v. Furthermore for every prime $p \geq 3$ the coproduct injection $i_p : A_p \to Y$ is the only morphism in $\hom(A_p, Y)$. In fact, due to the added edge $1 \to 0$ a morphism $f : A_p \to Y$ necessarily takes $\{0,1\} \subseteq A_p$ onto $\{0,1\} \subseteq A_q$ for some q. Since p and q are primes and f restricts to a mapping of a p-cycle into a q-cycle, it is obvious that $p = q$. And it is also obvious that A_p has no endomorphisms mapping $\{0,1\}$ into itself except the identity – consequently, $f = i_p$. □

3. General Orthogonality Logic

3.1. Remark. (i) Recall our standing assumption that $\mathcal{A}$ is a locally presentable category. We will now present a (non-finitary) logic for orthogonality and prove that it is always sound, and that for sets of morphisms it is also complete. We will actually prove the completeness not only for sets, but also for classes Σ of morphisms which are presentable, i.e., for which there exists a cardinal λ such that every member $s : A \to B$ of Σ is a λ-presentable object of the slice category $A \downarrow \mathcal{A}$. The completeness of our logic for *all* classes Σ of morphisms is the topic of the next section.

(ii) We recall the concept of a *transfinite composition* of morphisms as used in homotopy theory. Given an ordinal α (considered, as usual, as the chain of all smaller ordinals), an α-chain in $\mathcal{A}$ is simply a functor C from α to $\mathcal{A}$. It is called *smooth* provided that C preserves directed colimits, i.e., if $i < \alpha$ is a limit ordinal then $C_i = \operatorname{colim}_{j<i} C_j$.

3.2. Definition. Let α be an ordinal. A morphism h is called an α-*composite* of morphisms $h_i (i < \alpha)$, provided that there exists a smooth $(\alpha+1)$-chain $C_i (i \leq \alpha)$ such that h is the connecting morphism $C_0 \to C_\alpha$ and each h_i is the connecting morphism $C_i \to C_{i+1}$ $(i < \alpha)$.

3.3. Examples. (1) An ω-composite of a chain

$$A_0 \xrightarrow{h_0} A_1 \xrightarrow{h_1} A_2 \xrightarrow{h_2} \cdots$$

is, for any colimit cocone $c_i : A_i \to C\,(i < \omega)$ of the chain, the morphism $c_0 : A_0 \to C$.

(2) A 2-composite is the usual concept of a composite of two morphisms.

(3) Any identity morphism is the 0-composite of a 0-chain.

3.4. Definition. The *Orthogonality Deduction System* consists of the following deduction rules.

TRANSFINITE COMPOSITION $\dfrac{s_i\,(i < \alpha)}{t}$ if t is an α-composite of the s_i's

PUSHOUT $\dfrac{s}{t}$ if $\begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ is a pushout

COEQUALIZER $\dfrac{s}{t}$ if $\begin{tikzcd} \cdot \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \cdot \arrow[r, "t"'] & \cdot \end{tikzcd}$ is a coequalizer and $f \cdot s = g \cdot s$

WEAK CANCELLATION $\dfrac{u \cdot t \quad v \cdot u}{t}$

We say that a morphism s *can be proved* from a class Σ of morphisms in the Orthogonality Logic, in symbols

$$\Sigma \vdash s$$

provided that there exists a formal proof of s from Σ using the above deduction rules.

3.5. Remark. (1) The deduction rule TRANSFINITE COMPOSITION is, in fact, a scheme of deduction rules: one for every ordinal α.

(2) A *proof* of s from Σ is a collection of morphisms t_i $(i \le \alpha)$ for some ordinal α such that $s = t_\alpha$ and for every $i \le \alpha$ either $t_i \in \Sigma$, or t_i is the conclusion of one of the deduction rules above whose assumptions lie in the set $\{t_j\}_{j<i}$.

(3) The λ-*ary Orthogonality Deduction System* is the deduction system obtained from 3.4 by restricting TRANSFINITE COMPOSITION to all ordinals $\alpha < \lambda$. We obtain the λ-*ary Orthogonality Logic* by applying this deduction system to λ-ary morphisms, see 2.1. In the λ-ary Orthogonality Logic the proofs are also restricted to those of length $\alpha < \lambda$.

Example: if $\lambda = \omega$ we get precisely the Finitary Orthogonality Logic of Section 2.

3.6. Examples. Other useful sound rules for orthogonality consequence can be derived from the above deduction system. Here are some examples:

(i) The 2-out-of-3 rule: in a commutative triangle

$$\begin{tikzcd} & A \arrow[dl, "t"'] \arrow[dr, "s"] & \\ B \arrow[rr, "u"'] & & C \end{tikzcd}$$

any morphism can be derived from the remaining two. In fact

$\{t, u\} \vdash s$ by COMPOSITION,
$\{u, s\} \vdash t$ by WEAK CANCELLATION (put $v = \mathrm{id}$),

and to prove

$$\{t, s\} \vdash u$$

form a pushout of t and s:

$$\begin{tikzcd}
A \arrow[r, "t"] \arrow[d, "s"'] & B \arrow[d, "\bar{s}"'] \arrow[ddr, dashed, "u"] & \\
C \arrow[r, "\bar{t}"] \arrow[drr, dashed, "\mathrm{id}_C"'] & D \arrow[dr, dashed, "r"] & \\
& & C
\end{tikzcd}$$

We obtain a unique morphism r as indicated. Observe that due to $r \cdot \bar{t} = \mathrm{id}$ the diagram

$$\begin{tikzcd}
D \arrow[r, shift left, "\bar{t} \cdot r"] \arrow[r, shift right, "\mathrm{id}"'] & D \arrow[r, "r"] & C
\end{tikzcd}$$

is a coequalizer with the parallel pair equalized by $\bar{t}$. Thus we have

$$\begin{array}{cl}
t \quad s & \\
\hline
\bar{t} \quad \bar{s} & \text{PUSHOUT} \\
\hline
r & \text{COEQUALIZER} \\
\hline
u = r \cdot \bar{s} & \text{COMPOSITION}
\end{array}$$

(ii) A coproduct $t + t' : A + B \to A' + B'$ can be derived from t and t'. This follows from the pushouts along coproduct injections (denoted by $\rightarrowtail$):

$$\begin{tikzcd}
A \arrow[r, "t"] \arrow[d, tail] & A' \arrow[d, tail] & & B \arrow[r, "t'"] \arrow[d, tail] & B' \arrow[d, tail] \\
A + B \arrow[r, "t + \mathrm{id}_B"'] & A' + B & & A' + B \arrow[r, "\mathrm{id}_{A'} + t'"'] & A' + B'
\end{tikzcd}$$

Thus we have

$$\begin{array}{cl}
t \qquad\qquad t' & \\
\hline
t + \mathrm{id}_B \qquad \mathrm{id}_{A'} + t' & \text{PUSHOUT} \\
\hline
t + t' = (\mathrm{id}_{A'} + t') \cdot (t + \mathrm{id}_B) & \text{COMPOSITION}
\end{array}$$

(iii) More generally: $\coprod_{i \in I} t_i$ can be derived from $\{t_i\}_{i \in I}$. This follows easily from (ii) and TRANSFINITE COMPOSITION.

(iv) Given two parallel pairs, a natural transformation with components s_1, s_2 between them and a colimit t of that natural transformation between their coequalizers:

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "s_1"'] & A_2 \arrow[r, "c"] \arrow[d, "s_2"'] & C \arrow[d, "t"] \\
A_1' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & A_2' \arrow[r, "c'"] & C'
\end{tikzcd}$$

(where $c = \mathrm{coeq}(f, g)$ and $c' = \mathrm{coeq}(f', g')$), then t can be deduced from the components of the natural transformation,

$$\{s_1,\, s_2\} \vdash t\,.$$

In fact, form a pushout P of s_2 and c and denote by $u : P \to C'$ the obvious factorization morphism:

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[ddd, "s_1"'] & A_2 \arrow[rr, "c"] \arrow[ddd, "s_2"'] & & C \arrow[dl, "\bar{s}_2"'] \arrow[ddd, "t"] \\
& & P \arrow[d, dashed, "q"'] \arrow[ddr, "u"] & \\
& & Q & \\
A_1' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & A_2' \arrow[rr, "c'"'] \arrow[uur, "\bar{c}"] & & C' \arrow[ul, dashed, "v"]
\end{tikzcd}$$

Then u is a coequalizer of $\bar{c} \cdot f'$ and $\bar{c} \cdot g'$. (In fact, given $q : P \to Q$ merging that pair, then $q \cdot \bar{c}$ merges f', g', thus, there exists v with $q \cdot \bar{c} = v \cdot c'$. Since $\bar{c}$ is an epimorphism, this implies $q = v \cdot u$. The uniqueness of v is clear: suppose $q = w \cdot u$, then $w \cdot c' = w \cdot u \cdot \bar{c} = q \cdot \bar{c} = v \cdot c'$, thus, $w = v$.) The above diagram shows that s_1 equalizes $\bar{c} \cdot f'$ and $\bar{c} \cdot g'$:

$$(\bar{c} \cdot f') \cdot s_1 = \bar{c} \cdot s_2 \cdot f = \bar{s}_2 \cdot c \cdot f = \bar{s}_2 \cdot c \cdot g = \bar{c} \cdot s_2 \cdot g = (\bar{c} \cdot g') \cdot s_1\,.$$

Consequently we have

$$\dfrac{\dfrac{s_1 \quad s_2}{u \quad \bar{s}_2}\quad \text{\textsc{coequalizer} and \textsc{pushout}}}{t}\quad \text{\textsc{composition}}$$

(v) More generally: For any small category $\mathcal{D}$, given diagrams $D_1, D_2 : \mathcal{D} \to \mathcal{A}$ and given a natural transformation between them

$$s_X : D_1 X \to D_2 X \quad \text{for } X \in \mathrm{obj}\mathcal{D}$$

then its colimit $t : \operatorname{colim} D_1 \to \operatorname{colim} D_2$ can be derived from its components:

$$\{s_X\}_{X \in \mathrm{obj}\mathcal{D}} \vdash t\,.$$

This follows easily from (iii) and (iv) by applying the standard construction of colimits by means of coproducts and coequalizers ([11]).

(vi) In a commutative diagram

$$\begin{tikzcd}
 & & A_1 \arrow[ddll, "f"'] \arrow[d, "s_1"] \arrow[ddrr, "g"] & & \\
 & & A_1' \arrow[dl, "p"'] \arrow[dr, "q"] & & \\
A_2 \arrow[r, "s_2"] \arrow[ddrr, "\bar{g}"'] & A_2' \arrow[dr, "\bar{q}"'] & & A_3' \arrow[dl, "\bar{p}"] & A_3 \arrow[l, "s_3"'] \arrow[ddll, "\bar{f}"] \\
 & & P' & & \\
 & & P \arrow[u, "t"'] & &
\end{tikzcd}$$

where the outer and inner squares are pushouts, the morphism t (a colimit of the natural transformation with components $s_1,\ s_2,\ s_3$) can be derived from $\{s_1, s_2, s_3\}$. This is (v) for the obvious $\mathcal{D}$.

(vii) The following (strong) cancellation property

$$\frac{u \cdot t}{t}$$

holds for all epimorphisms t. In fact, the square

$$\begin{tikzcd}
{} \arrow[r, "u \cdot t"] \arrow[d, "t"'] & {} \arrow[d, "\mathrm{id}"] \\
{} \arrow[r, "u"'] & {}
\end{tikzcd}$$

is a pushout, thus, from $u \cdot t$ we derive u via PUSHOUT, and then we use (i).

(viii) A wide pushout $t = \bar{s}_i \cdot s_i$ of morphisms $s_i\ (i \in I)$

$$\begin{tikzcd}
 & A \arrow[dl, "s_i"'] \arrow[dd, "t"] \\
A_i \arrow[dr, "\bar{s}_i"'] & \\
 & B
\end{tikzcd}$$

can be derived from those morphisms :

$$\{s_i\}_{i \in I} \vdash t$$

If I is finite, this follows easily from PUSHOUT, IDENTITY and COMPOSITION. For I infinite use TRANSFINITE COMPOSITION.

(viii) COEQUALIZER has the following generalization: given parallel morphisms $g_j : A \to B\,(j \in J)$ such that a morphism $s : A' \to A$ equalizes the whole collection, then the joint coequalizer $t : B \to B'$ of the collection fulfils

$$s \vdash t\,.$$

In fact, for every $(j, j') \in J \times J$ a coequalizer $t_{jj'}$ of g_j and $g_{j'}$ fulfils $s \vdash t_{jj'}$. By (viii), we have $s \vdash t$ since t is a wide pushout of all $t_{jj'}$.

3.7. Observation. In every cocomplete (not necessarily locally presentable) category the Orthogonality Logic is sound: for every class Σ of morphisms a morphism s which has a proof from Σ is an orthogonality consequence of Σ:

$$\Sigma \vdash s \;\text{ implies }\; \Sigma \models s$$

The verification that TRANSFINITE COMPOSITION is sound is trivial: given a smooth chain $C : \alpha \to \mathcal{A}$ and an object X orthogonal to $h_i : C_i \to C_{i+1}$ for every $i < \alpha$, then X is orthogonal to the composite $h : C_0 \to C_\alpha$ of the h_i's. In fact, for every morphism $u : C_0 \to X$ there exists a unique cocone $u_i : C_i \to X$ of the chain C with $u_0 = u$: the isolated steps are determined by $X \perp h_i$ and the limit steps follow from the smoothness of C. Consequently $u_\alpha : C_\alpha \to X$ is the unique morphism with $u = u_\alpha \cdot h$.

3.8. Definition (see [9]). A morphism $t : A \to B$ of $\mathcal{A}$ is called λ-*presentable* if, as an object of the slice category $A \downarrow \mathcal{A}$, it is λ-presentable.

3.9. Remark. (i) This is closely related to a λ-ary morphism: t is λ-ary (i.e., A and B are λ-presentable objects of $\mathcal{A}$) iff t is a λ-presentable object of the arrow category $\mathcal{A}^{\to}$, see [3].

(ii) Unlike the λ-ary morphisms (which are the morphisms of the small category $\mathcal{A}_\lambda$, see 2.1) the λ-presentable morphisms form a proper class: for example all identity morphisms are λ-presentable.

(iii) A simple characterization of λ-presentable morphisms was proved in [9]:

f is λ-presentable $\;\Leftrightarrow\;$ f is a pushout of a λ-ary morphism

(along an arbitrary morphism).

(iv) The λ-ary morphisms are precisely the λ-presentable ones with λ-presentable domain (see [9]). That is, given $f : A \to B$ λ-presentable, then

$$A \;\lambda\text{-presentable} \;\Rightarrow\; B \;\lambda\text{-presentable}.$$

(v) For every object A the cone of all λ-presentable morphisms with domain A is essentially small. This follows from (iii), or directly: since $A \downarrow \mathcal{A}$ is a locally presentable category, it has up to isomorphism only a set of λ-presentable objects.

3.10. Example. A regular epimorphism which is the coequalizer of a pair of morphisms with λ-presentable domain is λ-presentable. That is, given a coequalizer diagram

$$\begin{tikzcd} K \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, "t"] & B \end{tikzcd}$$

then

$$K \text{ is } \lambda\text{-presentable} \;\Rightarrow\; t \text{ is } \lambda\text{-presentable.}$$

In fact, given a λ-filtered diagram in $A \downarrow \mathcal{A}$ with objects $d_i : A \to D_i$ and with a colimit cocone $c_i : (d_i, D_i) \to (d, D) = \mathrm{colim}_{i \in I}(d_i, D_i)$, then for every morphism $h : (t, B) \to (d, D)$ of $A \downarrow \mathcal{A}$ we find an essentially unique factorization through the cocone as follows:

$$\begin{tikzcd}
K \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, "t"] \arrow[dr, "d"] \arrow[ddr, "d_i"'] & B \arrow[d, "h"] \\
& & D \\
& & D_i \arrow[u, "c_i"']
\end{tikzcd}$$

The morphism $d = h \cdot t$ merges f and g. Observe that c_i merges $d_i \cdot f$ and $d_i \cdot g$ for any $i \in I$. Since K is λ-presentable and $D = \mathrm{colim}\, D_i$ is a λ-filtered colimit in $\mathcal{A}$, it follows that some connecting map $d_{ij} : (d_i, D_i) \to (d_j, D_j)$ of our diagram merges $d_i \cdot f$ and $d_i \cdot g$. This implies $d_j \cdot f = d_j \cdot g$, hence, d_j factors through t:

$$d_j = k \cdot t \;\text{ for some }\; k : B \to D_j\,.$$

Then $k : (t, B) \to (d_j, D_j)$ is the desired factorization. It is unique because t is an epimorphism.

3.11. Definition. A class Σ of morphisms is called *presentable* provided that there exists a cardinal λ such that every member of Σ is a λ-presentable morphism.

3.12. Example. Every small class is presentable. In this case there even exists λ such that all members are λ-ary morphisms. This follows from the fact that every object of a locally presentable category is λ-presentable for some λ, see [3].

3.13. Remark. We will prove that the Orthogonality Logic is complete for presentable classes of morphisms. This sharply contrasts with the following: if $\mathcal{A}$ is a locally finitely presentable category and Σ is a class of finitely presentable morphisms, the Finitary Orthogonality Logic needs not be complete:

3.14. Example (see [4]). Let $\mathcal{A}$ be the category of algebras on countably many nullary operations (constants) $a_0, a_1, a_2, \ldots$ Denote by $I = \{a_n\}_{n \in \mathbb{N}}$ an initial algebra, by 1 a terminal algebra, and by $\sim_k$ the congruence on I merging just a_k and a_{k+1}. The corresponding quotient morphism

$$e_k : I \to I/\sim_k$$

is clearly finitely presentable, and so is the quotient morphism

$$f : C \to 1$$

where $C = \{0, 1\}$ is the algebra with $a_0 = 0$ and $a_i = 1$ for all $i \geq 1$. It is obvious that

$$\{e_1, e_2, e_3, \ldots\} \cup \{f\} \models e_0\,.$$

Nevertheless, as proved in [4], e_0 cannot be proved from $\{e_1, e_2, e_3, \dots\} \cup \{f\}$ in the Finitary Orthogonality Logic. Observe that this does not contradict Theorem 2.17: the morphism f above is not finitary.

3.15. Construction of a Reflection. Let Σ be a class of λ-presentable morphisms in a locally λ-presentable category $\mathcal{A}$. For every object A of $\mathcal{A}$ a reflection

$$r_A : A \to \bar{A}$$

of A in the orthogonal subcategory $\Sigma^{\perp}$ is constructed as follows:

We form the diagram $D_A : \mathcal{D}_A \to \mathcal{A}$ of all λ-presentable morphisms $s : A \to A_s$ provable from Σ with domain A. Let $\bar{A}$ be a colimit of D_A with the colimit cocone $\bar{s} : A_s \to \bar{A}$. We show that the morphism

$$r_A = \bar{s} \cdot s : A \to \bar{A} \qquad \text{(independent of } s\text{)}$$

is the desired reflection.

The precise definition of D_A is as follows: we denote by $\bar{\Sigma}_\lambda$ the class of all λ-presentable morphisms s with $\Sigma \vdash s$. Let $\mathcal{D}_A$ be the full subcategory of the slice category $A \downarrow \mathcal{A}$ on all objects lying in $\bar{\Sigma}_\lambda$. By 3.9 (v) the diagram

$$D_A : \mathcal{D}_A \to \mathcal{A}, \;\; D_A(\, A \xrightarrow{s} A_s \,) = A_s$$

is essentially small.

3.16. Proposition. *For every object A the diagram D_A is λ-filtered and $r_A : A \to \bar{A}$ is a reflection of A in $\Sigma^{\perp}$; moreover, $\Sigma \vdash r_A$.*

Proof. (1) The diagram D_A is λ-filtered: From COEQUALIZER and 3.6(viii), $\bar{\Sigma}_\lambda$ is closed under weak coequalizers in the sense of 2.11(c) and under λ-wide pushouts. This assures that $A \downarrow \bar{\Sigma}_\lambda$ is closed under λ-small colimits in $A \downarrow \mathcal{A}$, thus the category $\mathcal{D}_A$ is λ-filtered.

(2) We prove

$$\Sigma \vdash r_A$$

and

$$\Sigma \vdash \bar{s} \quad \text{for all } s \text{ in } \mathcal{D}_A\,.$$

This follows from 3.6(v) applied to the natural transformation from the constant diagram of value A to D_A with components $s : A \to A_s$: Its colimit is r_A.

Now observe that the rule 2-out-of-3, 3.6(i), also yields that $\Sigma \vdash \bar{s}$ for all s in $\mathcal{D}_A$.

(3) Given a morphism $t : R \to Q$ in Σ we prove that every morphism $f : R \to \bar{A}$ has a factorization through t.

$$
\begin{tikzcd}
 & R^* \arrow[r, "t^*"] \arrow[d, "u"] \arrow[ddl, "g"'] & Q^* \arrow[d, "v"] \arrow[dddr, "\tilde{g}"] & \\
 & R \arrow[r, "t"] \arrow[d, "f"'] & Q \arrow[d, "\bar{f}"] & \\
A_s \arrow[r, "\bar{s}"] \arrow[drrr, "\tilde{t}"'] & \bar{A} \arrow[r, "\hat{t}"] & \hat{P} & \\
 & & & \tilde{P} \arrow[ul, "q"']
\end{tikzcd}
$$

By 3.9(iii) there exists a λ-ary morphism $t^* : R^* \to Q^*$ such that t is a pushout of t^* (along a morphism u). Due to (1) and since R^* is a λ-presentable object, the morphism

$$f \cdot u : R^* \to \bar{A} = \operatorname{colim} A_s$$

factors through one of the colimit morphisms:

$$f \cdot u = \bar{s} \cdot g \ \text{ for some } s : A \to A_s \text{ in } D_A \text{ and some } g : R^* \to A_s.$$

We denote by $\hat{t}$ a pushout of t^* along $f \cdot u$, and by $\tilde{t}$ a pushout of t^* along g. This leads to the unique morphism

$$q : \tilde{P} \to \hat{P} \quad \text{with } q \cdot \tilde{t} = \hat{t} \cdot \bar{s} \text{ and } q \cdot \tilde{g} = \bar{f} \cdot v\,.$$

By (2) we know that $\Sigma \vdash \bar{s}$. Consequently, COMPOSITION yields

$$\Sigma \vdash q \cdot \tilde{t}$$

since $q \cdot \tilde{t} = \hat{t} \cdot \bar{s}$, and $\Sigma \vdash \hat{t}$ by PUSHOUT. Next, we observe that

$$\Sigma \vdash q$$

by 3.6(vi): apply it to the pushouts $\tilde{P}$ and $\hat{P}$ and the natural transformation with components id_{R^*}, $\bar{s}$ and id_{Q^*}. Now the 2-out-of-3 rule yields

$$\Sigma \vdash \tilde{t}\,.$$

Moreover, $\tilde{t}$ is λ-presentable since t^* is λ-ary, see 3.9(iii). Therefore, the morphism

$$p = \tilde{t} \cdot s : A \to \tilde{P}$$

is also λ-presentable, and $\Sigma \vdash p$ by COMPOSITION. Thus,

$$p : A \to \tilde{P} \ \text{ is an object of } \mathcal{D}_A.$$

The corresponding colimit morphism $\bar{p} : \tilde{P} \to \bar{A}$ fulfils

$$r_A = \bar{p} \cdot p\,.$$

Further, since $\tilde{t}$ is a connecting morphism of the diagram D_A from s to p, it follows that

$$\bar{s} = \bar{p} \cdot \tilde{t}\,.$$

Consequently,

$$(\bar{p} \cdot \tilde{g}) \cdot t^* = \bar{p} \cdot \tilde{t} \cdot g = \bar{s} \cdot g = f \cdot u$$

and the universal property of the pushout Q of t^* and u yields a unique

$$h : Q \to \bar{A} \text{ with } f = h \cdot t \text{ and } \bar{p} \cdot \tilde{g} = h \cdot v\,.$$

This is the desired factorization of f through t.

(4) $\bar{A}$ lies in $\Sigma^{\perp}$: Given h, $k : Q \to \bar{A}$ equalized by t, we prove $h = k$.

$$\begin{tikzcd}
R^* \arrow[r, "t^*"] \arrow[d, "u"'] & Q^* \arrow[d, "v"] \arrow[dr, shift left, "h^*"] \arrow[dr, shift right, "k^*"'] & & \\
R \arrow[r, "t"] & Q \arrow[d, shift right, "h"'] \arrow[d, shift left, "k"] & A_s \arrow[r, "c^*"] \arrow[dl, "\bar{s}"'] & C^* \arrow[dll, dashed, "\bar{w}"'] \arrow[dl, dashed] \\
& \bar{A} \arrow[r, "c"] & C &
\end{tikzcd}$$

Since Q^* is λ-presentable, the morphisms $h \cdot v$, $k \cdot v : Q^* \to \bar{A}$ both factor through some of the colimit morphisms of the λ-filtered colimit $\bar{A} = \operatorname{colim} D_A$:

$$h \cdot v = \bar{s} \cdot h^* \text{ and } k \cdot v = \bar{s} \cdot k^* \text{ for some } h^*, k^* : Q^* \to A_s\,.$$

Form coequalizers

$$c = \operatorname{coeq}(h, k) \quad \text{and} \quad c^* = \operatorname{coeq}(h^*, k^*)\,.$$

From $h \cdot t = k \cdot t$ COEQUALIZER yields

$$\Sigma \vdash c$$

and then (2) above and COMPOSITION yields

$$\Sigma \vdash c \cdot \bar{s}\,.$$

From the equality $(c \cdot \bar{s}) \cdot h^* = (c \cdot \bar{s}) \cdot k^*$ we conclude that $c \cdot \bar{s}$ factors through c^*. Since c^* is an epimorphism, 3.6(vii) yields

$$\Sigma \vdash c^*\,.$$

Moreover, c^* is a λ-presentable morphism since $c^* = \operatorname{coeq}(h^*, k^*)$ and Q^* is λ-presentable, see Example 3.10. The morphism

$$w = c^* \cdot s : A \to C^*$$

is thus also a λ-presentable morphism with $\Sigma \vdash w$, in other words (w, C^*) is an object of $\mathcal{D}_A$, and

$$c^* : (s, A_S) \to (w, C^*) \quad \text{is a morphism of} \quad \mathcal{D}_A\,.$$

This implies that the colimit maps fulfil

$$\bar{s} = \bar{w} \cdot c^*\,.$$

We are ready to prove $h = k$: by the universal property of the pushout Q we only need showing $h \cdot v = k \cdot v$:

$$h \cdot v = \bar{s} \cdot h^* = \bar{w} \cdot c^* \cdot h^*$$

and analogously $k \cdot v = \bar{w} \cdot c^* \cdot k^*$, thus $c^* \cdot h^* = c^* \cdot k^*$ finishes the proof.

(5) The universal property of r_A: Let $f : A \to B$ be a morphism with B orthogonal to Σ. Thus B is orthogonal to all morphisms s with $\Sigma \vdash s$, see 3.7.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "s"'] \arrow[dd, bend right=60, "r_A"'] & B \\
A_s \arrow[ur, "f_s"] \arrow[d, "\bar{s}"'] & \\
\bar{A} \arrow[uur, "g"'] &
\end{tikzcd}$$

For every object $s : A \to A_s$ of $\mathcal{D}_A$ let $f_s : A_s \to B$ be the unique factorization of f through s. These morphisms clearly form a compatible cocone of D_A, and the unique factorization $g : \bar{A} \to B$ fulfils, for any object s of $\mathcal{D}_A$,

$$f = f_s \cdot s = g \cdot \bar{s} \cdot s = g \cdot r_A \,.$$

Conversely, suppose $g' \cdot r_A = f$, then $g = g'$ because for every object s of $\mathcal{D}_A$ we have

$$g' \cdot \bar{s} = f_s = g \cdot \bar{s};$$

this follows from $B \perp s$ due to $(g' \cdot \bar{s}) \cdot s = f = f_s \cdot s$. □

3.17. Theorem. *The Orthogonality Logic is complete for all presentable classes Σ of morphisms: every orthogonality consequence of Σ has a proof from Σ in the Orthogonality Deduction System. Shortly,*

$$\Sigma \models t \;\; \textit{implies} \;\; \Sigma \vdash t \,.$$

Proof. Given an orthogonality consequence $t : A \to B$ of Σ, form a reflection $r_A : A \to \bar{A}$ of A in $\Sigma^{\perp}$ as in 3.15. Then $\Sigma \models t$ implies that $\bar{A}$ is orthogonal to t, thus we have $u : B \to \bar{A}$ with $r_A = u \cdot t$. From 3.16 we know that

$$\Sigma \vdash u \cdot t \,.$$

Now we have that $\Sigma \models u \cdot t \,(= r_A)$ and $\Sigma \models t$, and this trivially implies that $\Sigma \models u$. Thus by the same argument with t replaced by u there exists a morphism v such that

$$\Sigma \vdash v \cdot u \,.$$

The last step is WEAK CANCELLATION :

$$\frac{u \cdot t \quad v \cdot u}{t}$$

□

3.18. Corollary. *The Orthogonality Logic is complete for classes Σ of morphisms of the form*

$$\Sigma = \Sigma_0 \cup \Sigma_1, \quad \Sigma_0 \;\textit{small and}\; \Sigma_1 \subseteq \textit{RegEpi} \,.$$

Proof. Let λ be a regular cardinal such that $\mathcal{A}$ is locally λ-presentable, and all morphisms of Σ_0 are λ-presentable. We will substitute Σ_1 with a class $\tilde{\Sigma}_1$ of λ-presentable morphisms as follows: for every member $s : A \to B$ of Σ_1 choose a pair $f, g : A' \to A$ with $s = \mathrm{coeq}(f, g)$. Express A' as a λ-filtered colimit of λ-presentable objects A_i with a colimit cocone

$$a_i : A_i \to A' \ (i \in I_s)\,.$$

Form a coequalizer $s_i : A \to B_i$ of $f \cdot a_i$, $g \cdot a_i : A_i \to B$ for every $i \in I_s$. Then we obtain a filtered diagram with the objects B_i $(i \in I_s)$ and the obvious connecting morphisms. The unique $b_i : B_i \to B$ with $s = b_i \cdot s_i$ form a colimit of that diagram. Moreover, an object X is orthogonal to s iff it is orthogonal to s_i for every $i \in I_s$:

$$\begin{tikzcd}
A_i \arrow[r, "a_i"] & A' \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, "s"] \arrow[d, "s_i"'] \arrow[dd, bend right=60, "u"'] & B \arrow[ddl, dashed, "v"] \\
& & B_i \arrow[ur, "b_i"] \arrow[d, "v_i"] & \\
& & X &
\end{tikzcd}$$

Let $\tilde{\Sigma}_1$ be the class of all morphisms s_i for all $s \in \Sigma_1$ and $i \in I_s$. Then the class

$$\tilde{\Sigma} = \Sigma_0 \cup \tilde{\Sigma}_1$$

consists of λ-presentable morphisms, see Example 3.10, and $\Sigma^\perp = \tilde{\Sigma}^\perp$. Given an orthogonality consequence t of Σ, we thus have a proof of t from $\tilde{\Sigma}$, see Theorem 3.17. It remains to prove

$$s \vdash s_i \quad \text{for every } s \in \Sigma \text{ and } i \in I_s;$$

then $\tilde{\Sigma} \vdash t$ implies $\Sigma \vdash t$. In fact, since s_i is an epimorphism, apply 3.6(vii) to $s = b_i \cdot s_i$. $\square$

3.19. Remark. Since all λ-ary morphisms form essentially a set (since $\mathcal{A}_\lambda$ is small), the λ-ary Orthogonality Logic (see 3.5) is complete for classes of λ-ary morphisms – the proof is analogous to that of Theorem 2.17.

4. Vopěnka's Principle

4.1. Remark. The aim of the present section is to prove that the Orthogonality Logic is complete (for all classes of morphisms) in all locally presentable categories iff the following large-cardinal Vopěnka's principle holds. Throughout this section we assume that the set theory we work with satisfies the Axiom of Choice for classes.

4.2. Definition. Vopěnka's Principle states that the category **Rel(2)** of graphs (or binary relational structures) does not have a large discrete full subcategory.

4.3. Remark. (1) The following facts can be found in [3]:

(i) Vopěnka's Principle is a large-cardinal principle: it implies the existence of measurable cardinals. Conversely, the existence of huge cardinals implies that Vopěnka's Principle is consistent.

(ii) An equivalent formulation of Vopěnka's Principle is: the category **Ord** of ordinals cannot be fully embedded into any locally presentable category.

(2) The following proof is analogous to the proof of Theorem 6.22 in [3].

4.4. Theorem. *Assuming Vopěnka's Principle, the Orthogonality Logic is complete for all classes of morphisms (of a locally presentable category).*

Proof. (1) Every class Σ can be expressed as the union of a chain

$$\Sigma = \bigcup_{i \in \mathbf{Ord}} \Sigma_i \quad (\Sigma_i \subseteq \Sigma_j \text{ if } i \leq j)$$

of small subclasses – this follows from the Axiom of Choice. We prove that every object A has a reflection in $\Sigma^{\perp}$ by forming reflections

$$r_i(A) : A \to A_i$$

in $\Sigma_i^{\perp}$ for every $i \in \mathbf{Ord}$, see 2.2. These reflections form a transfinite chain in the slice category $A \downarrow \mathcal{A}$: for $i \leq j$ the fact that $\Sigma_i \subseteq \Sigma_j$ implies the existence of a unique $a_{ij} : A_i \to A_j$ forming a commutative triangle

$$\begin{tikzcd} & A \arrow[dl, "r_i(A)"'] \arrow[dr, "r_j(A)"] & \\ A_i \arrow[rr, "a_{ij}"'] & & A_j \end{tikzcd}$$

We prove that this chain is stationary, i.e., there exists an ordinal i_0 such that $a_{i_0 j}$ is an isomorphism for all $j \geq i_0$ – it will follow immediately that $r_A = r_{i_0}(A)$ is a reflection of A in $\Sigma^{\perp}$.

(2) Assuming the contrary, we have an object A and ordinals $i(k)$ for $k \in \mathbf{Ord}$ with $i(k) < i(l)$ for $k < l$ such that none of the morphisms

$$a_{i(k),i(l)} \quad \text{with} \quad k < l$$

is an isomorphism. We derive a contradiction to Vopěnka's Principle: the slice category $A \downarrow \mathcal{A}$ is locally presentable, and we prove that the functor

$$E : \mathbf{Ord} \to A \downarrow \mathcal{A},\ k \mapsto r_{i(k)}(A)$$

is a full embedding. In fact, for every morphism u such that the diagram

$$\begin{tikzcd} & A \arrow[dl, "r_{i(k)}(A)"'] \arrow[dr, "r_{i(l)}(A)"] & \\ A_{i(k)} \arrow[rr, "u"'] & & A_{i(l)} \end{tikzcd}$$

commutes, we have $k \leq l$ and $u = a_{k,l}$. The latter follows from the universal property of $r_{i(k)}(A)$. Thus, it is sufficient to prove the former: assuming $k \geq l$ we show $k = l$. In fact, the morphism u is inverse to $a_{i(l),i(k)}$ because

$$(u \cdot a_{i(l),i(k)}) \cdot r_{i(l)}(A) = r_{i(l)}(A) \quad \text{implies} \quad u \cdot a_{i(l),i(k)} = \mathrm{id}$$

and analogously for the other composite. Our choice of the ordinals $i(k)$ is such that whenever $a_{i(l),i(k)}$ is an isomorphism, then $k = l$.

(3) Every orthogonality consequence $t : A \to B$ of Σ has a proof from Σ. The argument is now precisely as in Theorem 3.17: we use the above reflections r_A and the fact that $\Sigma \vdash r_A$ (see Proposition 3.16 and the above fact that $r_A = r_{i_0}(A)$ for some i_0). □

4.5. Example (under the assumption of the negation of Vopěnka's Principle)**.** In the category

$$\mathbf{Rel}(\mathbf{2},\mathbf{2})$$

of relational structures on two binary relations α, β we present a class Σ of morphisms together with an orthogonality consequence t which cannot be proved from Σ:

$$\Sigma \models t \ \text{ but } \ \Sigma \not\vdash t\,.$$

We use the notation of Example 2.18. The negation of the Vopěnka's Principle yields graphs

$$(X_i, R_i) \quad \text{in} \quad \mathbf{Rel}(\mathbf{2})$$

for $i \in \mathbf{Ord}$, forming a discrete category. For every i let A_i be the object of $\mathbf{Rel}(\mathbf{2},\mathbf{2})$ on X_i whose relation α is R_i and β is a clique (see 2.18). Our class Σ consists of the morphisms u, v of 2.18 and

$$\emptyset \to A_i \quad \text{for all} \ \ i \in \mathbf{Ord}\,.$$

We claim that the morphism

$$t : \emptyset \to 1$$

is an orthogonality consequence of Σ. In fact, let B be an object orthogonal to Σ and let i be an ordinal such that A_i has cardinality larger than B. We have a (unique) morphism $h : A_i \to B$, and since h cannot be monic, the relation β of B contains a loop (recall that β is a clique in A_i). This implies that B has a unique joint loop of α and β, therefore, $B \perp t$.

To prove

$$\Sigma \not\vdash t$$

it is sufficient to find a category $\mathcal{A}$ in which

(i) $\mathbf{Rel}(\mathbf{2},\mathbf{2})$ is a full subcategory closed under colimits

and

(ii) some object K of $\mathcal{A}$ is orthogonal to Σ but not to t.

From (ii) we deduce that t cannot be proved from Σ in the category $\mathcal{A}$, see Observation 3.7. However, (i) implies that every formal proof using the Orthogonality Deduction System 3.4 in the category $\mathbf{Rel}(\mathbf{2},\mathbf{2})$ is also a valid proof in $\mathcal{A}$. Together, this implies $\Sigma \not\vdash t$ in $\mathbf{Rel}(\mathbf{2},\mathbf{2})$.

The simplest approach is to choose $\mathcal{A} = \mathbf{REL}(\mathbf{2},\mathbf{2})$, the category of all possibly large relational systems on two binary relations, i.e., triples (X, α, β) where X is a class and α, β are subclasses of $X \times X$. Morphisms are class functions preserving the binary relations in the expected sense. This category contains $\mathbf{Rel}(\mathbf{2},\mathbf{2})$ as a full subcategory closed under small colimits, and the object

$$K = \coprod_{i \in \mathbf{Ord}} A_i$$

is not orthogonal to $t : \emptyset \to 1$ since none of A_i contains a joint loop of α and β. However, it is easy to verify that K is orthogonal to Σ.

A more "economical" approach is to use as $\mathcal{A}$ just the category $\mathbf{Rel}(\mathbf{2},\mathbf{2})$ with the unique object K added to it, i.e., the full subcategory of $\mathbf{REL}(\mathbf{2},\mathbf{2})$ on $\{K\} \cup \mathbf{Rel}(\mathbf{2},\mathbf{2})$.

4.6. Corollary. *Vopěnka's Principle is equivalent to the statement that the Orthogonality Logic is complete for classes of morphisms of locally presentable categories.*

5. A counterexample

The Orthogonality Logic can be formulated in every cocomplete category, and we know that it is always sound, see 3.7. But outside of the realm of locally presentable categories the completeness can fail (even for finite sets Σ):

5.1. Example. We start with the category $\mathbf{CPO}_\perp$ of strict CPO's: objects are posets with a least element $\perp$ and with directed joins, morphisms are strict continuous functions (preserving $\perp$ and directed joins). This category is well-known to be cocomplete. We form the category

$$\mathbf{CPO}_\perp(\mathbf{1})$$

of all unary algebras on strict CPO's: objects are triples $(X, \leq, \alpha)$, where $(X, \leq)$ is a strict CPO and $\alpha : X \to X$ is an endofunction of X, morphisms are the strict continuous algebra homomorphisms. It is easy to verify that the forgetful functor $\mathbf{CPO}_\perp(\mathbf{1}) \to \mathbf{CPO}_\perp$ is monotopological, thus, by 21.42 and 21.16 in [2] the category $\mathbf{CPO}_\perp(\mathbf{1})$ is cocomplete.

We present morphisms s_1, s_2 and t of $\mathbf{CPO}_\perp(\mathbf{1})$ such that an algebra A is orthogonal to

(a) s_1 iff its operation α has at most one fixed point

(b) s_2 iff its operation α fulfils $x \leq \alpha x$ for all x

and

(c) t iff α has precisely one fixed point.

We then have

$$\{s_1, s_2\} \models t$$

In fact, if an algebra A fulfils (b), we can define a transfinite chain a_i ($i \in Ord$) of its elements by

$$a_o = \perp$$
$$a_{i+1} = \alpha a_i,$$

and

$$a_j = \bigvee_{i<j} a_i \quad \text{for all limit ordinals } j.$$

This chain cannot be 1–1, thus, there exist $i < j$ with $a_i = a_j$ and we conclude that a_i is a fixed point of α. The fixed point is unique due to (a), thus, A is orthogonal to t. On the other hand

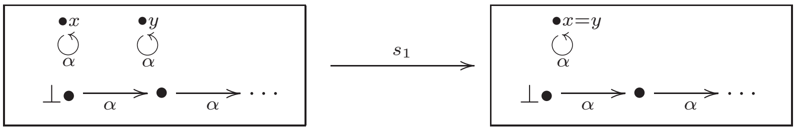
$$\{s_1, s_2\} \not\vdash t$$

The argument is analogous to that in Example 4.5: The category $\mathcal{A}$ of possibly large CPO's with a unary operation contains $\mathbf{CPO}_\perp(\mathbf{1})$ as a full subcategory closed under small colimits. And the following object K is orthogonal to s_1 and s_2 but not to t:

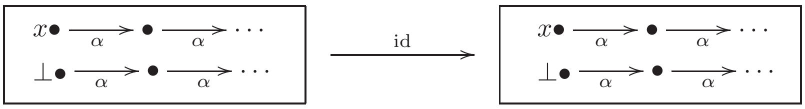
$$K = (Ord, \leq, \mathrm{succ})$$

where $\leq$ is the usual ordering of the class of all ordinalds, and $\mathrm{succ}\, i = i + 1$ for all ordinals i.

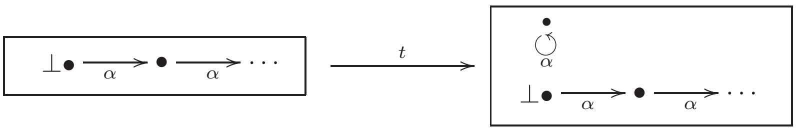
Thus, it remains to produce the desired morphisms s_1, s_2 and t. The morphism s_1 is the following quotient



where both the domain and codomain are flat CPO's (all elements except $\perp$ are pairwise incomparable). The morphism s_2 is carried by the identity homomorphism



where the domain is flat and the codomain is flat except for the unique comparable pair not involving $\perp$ being $x < \alpha x$. Finally, t is the embedding



with both the domain and the codomain flat.

6. Injectivity Logic

As mentioned in the Introduction, for the injectivity logic the deduction system consisting of TRANSFINITE COMPOSITION, PUSHOUT and CANCELLATION is sound and complete for sets Σ of morphisms. In contrast to Theorem 4.4 this deduction system fails to be complete for classes of morphisms in general, independently of set theory:

6.1. Example. Let $\mathbf{Rel}(\mathbf{2})$ be the category of graphs. For every cardinal n let C_n denote a clique (2.18) on n nodes. Then the morphism

$$t : \emptyset \to 1$$

is an injectivity consequence of the class

$$\Sigma = \{\emptyset \to C_n;\ n \in Card\}\,.$$

In fact, given a graph X injective w.r.t. Σ, choose a cardinal $n > \mathrm{card}X$. We have a morphism $f : C_n \to X$ which cannot be monomorphic. Consequently, X has a loop. This proves that X is injective w.r.t. t.

The argument to show that t cannot be proved from Σ is completely analogous to 5.1: the category $\mathbf{REL}(\mathbf{2})$ of potentially large graphs contains $\mathbf{Rel}(\mathbf{2})$ as a full subcategory closed under small colimits. The object $K = \coprod_{n \in Card} C_n$ is injective w.r.t. Σ but not injective w.r.t. t. Therefore, t does not have a formal proof from Σ in the Injectivity Deduction System above applied in $\mathbf{REL}(\mathbf{2})$. Consequently, no such formal proof exists in $\mathbf{Rel}(\mathbf{2})$.

Instead of $\mathbf{REL}(\mathbf{2})$ we can, again, use the full subcategory on $\mathbf{Rel}(\mathbf{2}) \cup \{K\}$ for our argument.

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