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ON NEAR-RING IDEALS WITH (σ,τ)-DERIVATION

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Abstract. Let N be a 3-prime left near-ring with multiplicative center Z, a (σ,τ)-derivation D on N is defined to be an additive endomorphism satisfying the product rule $D(xy)=\tau(x)D(y)+D(x)\sigma(y)$ for all $x,y\in N$, where σ and τ are automorphisms of N. A nonempty subset U of N will be called a semigroup right ideal (resp. semigroup left ideal) if $UN\subset U$ (resp. $NU\subset U$) and if U is both a semigroup right ideal and a semigroup left ideal, it be called a semigroup ideal. We prove the following results: Let D be a (σ,τ)-derivation on N such that $\sigma D=D\sigma,\tau D=D\tau$. (i) If U is semigroup right ideal of N and $D(U)\subset Z$ then N is commutative ring. (ii) If U is a semigroup ideal of N and $D^2(U)=0$ then $D=0$. *(iii)* If $a\in N$ and $[D(U),a]_{\sigma,\tau}=0$ then $D(a)=0$ or $a\in Z$.

1. Introduction

H. E. Bell and G. Mason have shown several commutativity theorems for near-rings with derivation in [1]. Bell has proved in [2], that if N be a 3-prime zero symmetric left near-ring and D be a nonzero derivation on N,U is nonzero subset of N such that $UN\subset U$ or $NU\subset U$ and $D(U)\subset Z$ then N is commutative ring. The major purpose of this paper to generalize this result replacing the derivation D by (σ,τ)-derivation.

Throughout this paper, N will denote a zero-symetric left near-ring and usually will be 3-prime, that is, if $aNb=0$ then $a=0$ or $b=0$, with multiplicative center Z. A nonempty subset U of N will be called a semigroup right ideal (resp. semigroup left ideal) if $UN\subset U$ (resp. $NU\subset U$) and if U is both a semigroup right ideal and a semigroup left ideal, it be called a semigroup ideal. For subsets $X,Y\subset N$ the symbol $[X,Y]$ will denote the set $\{xy-yx\mid x\in X,y\in Y\}$. Let σ,τ be two near-ring automorphisms of N. An additive endomorphism of N with the property that $D(xy)=\tau(x)D(y)+D(x)\sigma(y)$ for all $x,y\in N$ is called $(\sigma,\tau)-$derivation of N. Given $x,y\in N$, we write $[x,y]_{\sigma,\tau}=x\sigma(y)-\tau(y)x$; in particular $[x,y]_{1,1}=[x,y]$, in the usual sense. As for terminologies used here without mention, we refer to G. Pilz [4].

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2. Results

Lemma 1 ([3, Lemma 2]). *Let D be a (σ,τ)-derivation on the near-ring N. Then*

$$\big(\tau(x)D(y)+D(x)\sigma(y)\big)\sigma(a)=\tau(x)D(y)\sigma(a)+D(x)\sigma(y)\sigma(a)$$

for all $x,y,a\in N$.

Lemma 2 ([2, Lemma 1.2]). *Let N be a 3-prime near-ring.*

i) *If $z\in Z\backslash\{0\}$ then z is not a zero divisor.*
ii) *If $Z\backslash\{0\}$ contains an element z for which $z+z\in Z$, then $(N,+)$ is abelian.*
iii) *If $z\in Z\backslash\{0\}$ and x is an element of N such that $xz\in Z$ or $zx\in Z$, then $x\in Z$.*

Lemma 3 ([3, Lemma 3]). *Let D be a nonzero (σ,τ)-derivation on prime near-ring N and $a\in N$.*

i) *If $D(N)\sigma(a)=0$ then $a=0$.*
ii) *If $aD(N)=0$ then $a=0$.*

Lemma 4. *Let N be a prime near-ring, D a (σ,τ)-derivation of N and U be nonzero semigroup right ideal (resp. semigroup left ideal). If $D(U)=0$ then $D=0$.*

Proof. U be a nonzero semigroup right ideal of N and $D(U)=0$. For any $x\in N$, $u\in U$, we get

$$0=D(ux)=\tau(u)\,D(x)+D(u)\,\sigma(x)$$

and so,

$$\tau(u)\,D(x)=0\,,\quad \text{for all}\quad u\in U,\ x\in N\,.$$

By Lemma 3(ii), we have $D=0$. If U is a nonzero semigroup left ideal of N, then the proof is similar. □

Lemma 5. *Let N be a 3-prime near-ring, D a nonzero (σ,τ)-derivation of N and U be nonzero semigroup ideal.*

i) *If $x\in N$ and $D(U)\,\sigma(x)=0$ then $x=0$.*
ii) *If $x\in N$ and $xD(U)=0$ then $x=0$.*

Proof. i) Suppose U a nonzero semigroup ideal of N and $D(U)\,\sigma(x)=0$. By Lemma 1 for all $u,v\in U$, we get

$$0=D(uv)\,\sigma(x)=\tau(u)\,D(v)\,\sigma(x)+D(u)\,\sigma(v)\,\sigma(x)\,.$$

Using the hypothesis and σ is an automorphism of N, we have

$$\sigma^{-1}\big(D(u)\big)Ux=0\,,\quad \text{for all}\quad u\in U\,.$$

Since D is a nonzero (σ,τ)-derivation of N, this relation gives us $x=0$ by [2, Lemma 1.4 (i)].

ii) A similar argument works if $xD(U)=0$. □

Theorem 1. *Let N be a 3-prime near-ring, D be a nonzero (σ,τ)-derivation of N and U a nonzero semigroup right ideal. If $D(U)\subset Z$, then N is commutative ring.*

Proof. For all $u, v \in U$, we get

$$D(uv) = \tau(u)\,D(v) + D(u)\,\sigma(v) \in Z$$

and commuting this element with $\sigma(v)$ gives us

$$\big(\tau(u)\,D(v) + D(u)\,\sigma(v)\big)\sigma(v) = \sigma(v)\big(\tau(u)\,D(v) + D(u)\,\sigma(v)\big)\,.$$

Using Lemma 1 and $D(u) \in Z$, we have

$$\tau(u)\,D(v)\,\sigma(v) + D(u)\,\sigma(v)\,\sigma(v) = \sigma(v)\,\tau(u)\,D(v) + D(u)\,\sigma(v)\,\sigma(v)$$

and so,

$$D(v)\big[\tau(u), \sigma(v)\big] = 0\,, \quad \text{for all} \quad u, v \in U\,.$$

Since $D(v) \in Z$, we obtain

$$D(v) = 0 \quad \text{or} \quad \big[\tau(u), \sigma(v)\big] = 0\,, \quad \text{for all} \quad u, v \in U\,.$$

Suppose that $D(v) = 0$, then $D(uv) = \tau(u)\,D(v) + D(u)\,\sigma(v) = D(u)\,\sigma(v) \in Z$. This gives us $v \in Z$ from Lemma 2(iii). For any cases we conclude that

$$\big[\tau(u), \sigma(v)\big] = 0\,, \text{for all} \quad u, v \in U\,. \tag{2.1}$$

Since τ is an automorphism of N, the relation (2.1) yields

$$\big[U, \tau^{-1}\big(\sigma(v)\big)\big] = 0\,, \quad \text{for all} \quad v \in U$$

and so, $U \subset Z$ from [2, Lemma 1.3 (iii)]. Thus, we obtain that N is commutative ring by [2, Lemma 1.5]. □

Theorem 2. *Let N be a 3-prime near-ring, D a nonzero $(\sigma,\tau)-$derivation of N and U be nonzero semigroup left ideal. If $D(U) \subset Z$, then N is commutative ring.*

Proof. If we use the same argument in the proof of Theorem 1, we conclude that

$$D(v) = 0 \quad \text{or} \quad \big[\tau(u), \sigma(v)\big] = 0\,, \quad \text{for all} \quad u, v \in U\,. \tag{2.2}$$

Suppose that $D(v) = 0$, then $D(uv) = \tau(u)\,D(v) + D(u)\,\sigma(v) = D(u)\,\sigma(v) \in Z$, for all $u \in U$, so that

$$D(u)\,\sigma(v)x = xD(u)\,\sigma(v) = D(u)\,x\sigma(v)\,, \quad \text{for all} \quad u \in U,\ x \in N\,.$$

Thus, $D(u)\big[\sigma(v), x\big] = 0$, for all $u \in U, x \in N$. By Lemma 2 (i) and Lemma 4, we have show that

$$\text{If} \quad v \in U \quad \text{and} \quad D(v) = 0\,, \quad \text{then} \quad v \in Z\,. \tag{2.3}$$

Thus, the relation (2.2) yields

$$\big[\tau(u), \sigma(v)\big] = 0\,, \quad \text{for all} \quad u, v \in U\,.$$

By the hypothesis, we get $D(wu) \in Z$, for all $u, w \in U$. That is,

$$D(wu) = \tau(w)\,D(u) + D(w)\,\sigma(u) \in Z\,.$$

Commuting this element with $\sigma(v)$, one can obtain

$$\big(\tau(w)\,D(u) + D(w)\,\sigma(u)\big)\sigma(v) = \sigma(v)\big(\tau(w)\,D(u) + D(w)\,\sigma(u)\big)\,.$$

Applying Lemma 1 and using $\sigma(v)\,\tau(w) = \tau(w)\,\sigma(v),\ D(u) \in Z$, we have

$$D(u)\,\tau(w)\,\sigma(v) + D(w)\,\sigma(u)\,\sigma(v) = D(u)\,\tau(w)\,\sigma(v) + D(w)\,\sigma(v)\,\sigma(u)$$

and so,

$$D(w)\,\sigma\big([u,v]\big) = 0\,, \quad \text{for all} \quad u,v,w \in U\,.$$

Since $D(w) \in Z$, we have

$$D(w) = 0 \quad \text{or} \quad [u,v] = 0\,, \quad \text{for all} \quad u,v,w \in U\,.$$

In the first case, we find that $D = 0$, a contradiction. So, we must have U is commutative.

Now, we assume that $U \cap Z \neq \{0\}$. Then U contains a nonzero central element w, we have

$$(wx)u = (xw)u = u(xw) = u(wx) = (uw)x = (wu)x$$

that is,

$$w[x,u] = 0\,, \quad \text{for all} \quad u \in U\,.$$

Since w is nonzero central element, we have $U \subset Z$. Thus N is commutative ring by [2, Lemma 1.5].

If $U \cap Z = \{0\}$, in which case (2.3) shows that $D(u) \neq 0$ for all $u \in U\backslash\{0\}$. For each such u, $D(u^2) = \tau(u)\,D(u) + D(u)\,\sigma(u) = D(u)\big(\tau(u) + \sigma(u)\big) \in Z$. Since $D(u)$ is noncentral element of N, we get

$$\tau(u) + \sigma(u) \in Z\,, \quad \text{for all} \quad u \in U\,.$$

Suppose that $\tau(u) + \sigma(u) = 0$, for all $u \in U\backslash\{0\}$. By the hypothesis, $D(u^3) \in Z$, we get

$$\begin{aligned} D(u^3) &= \tau(u\,)D(u^2) + D(u)\,\sigma(u^2) \\ &= \tau(u)\,\tau(u)\,D(u) + \tau(u)\,D(u)\,\sigma(u) + D(u)\,\sigma(u^2) \in Z\,. \end{aligned}$$

Using $\tau(u) = -\sigma(u)$ and $D(u) \in Z$, we get

$$\tau(u)\,\tau(u)\,D(u) - \tau(u)\,\tau(u)\,D(u) + D(u)\,\sigma(u^2) \in Z$$

which implies $u^2 \in Z$. Hence we get $u^2 \in U \cap Z = \{0\}$, and so $u^2 = 0$.

Now, for any $x \in N$, $D(xu) = \tau(x)\,D(u) + D(x)\,\sigma(u) \in Z$. Hence,

$$\sigma(u)\,\big(\tau(x)\,D(u) + D(x)\,\sigma(u)\big) = \big(\tau(x)\,D(u) + D(x)\,\sigma(u)\big)\,\sigma(u)\,.$$

Using Lemma 1 and $u^2 = 0$, we have

$$\sigma(u)\,\big(\tau(x)\,D(u) + D(x)\,\sigma(u)\big) = \tau(x)\,D(u)\,\sigma(u)\,.$$

Left-multiplying this relation by $\sigma(u)$ and using $u^2 = 0$, we obtain

$$\sigma(u)\,\tau(x)\,D(u)\sigma(u) = 0\,, \quad \text{for all} \quad u \in U,\ x \in N\,.$$

The primenessly of the near-ring N, we obtain that $D(u)\sigma(u) = 0$. Since $0 \neq D(u) \in Z$, we have $\sigma(u) = 0$, a contradiction. Thus, we must have $z = \sigma(u_0) + \tau(u_0) \in Z\backslash\{0\}$ for an $u_0 \in U$. Since $D(zu) = \tau(z)\,D(u) + D(z)\,\sigma(u) \in Z$, for any $u \in U$, we have

$$\big(\tau(z)\,D(u) + D(z)\,\sigma(u)\big)\,\sigma(y) = \sigma(y)\big(\tau(z)\,D(u) + D(z)\,\sigma(u)\big)\,, \quad \text{for all} \quad y \in N$$

and so,

$$\tau(z)\,D(u)\,\sigma(y) + D(z)\,\sigma(u)\,\sigma(y) = \sigma(y)\,\tau(z)\,D(u) + \sigma(y)\,D(z)\,\sigma(u)\,.$$

Using $z, D(u) \in Z$, we get

$$D(u)\,\tau(z)\,\sigma(y) + D(z)\,\sigma(u)\,\sigma(y) = D(u)\,\tau(z)\,\sigma(y) + \sigma(y)\,\sigma(u)\,D(z)\,.$$

That is,

$$D(z)\,\sigma\big([u,y]\big) = 0\,, \quad \text{for all} \quad u \in U\,,\ y \in N\,.$$

Since $D(z)$ is a nonzero central element of N, we have $U \subset Z$. Thus, we conclude that N is commutative ring from [2, Lemma 1.5]. □

Theorem 3. *Let N be a 3-prime near-ring, D be a (σ,τ)-derivation of N such that $\sigma D = D\sigma, \tau D = D\tau$ and U a nonzero semigroup ideal of N. If $D^2(U) = 0$ then $D = 0$.*

Proof. For arbitrary $u, v \in U$, we have $0 = D^2(uv) = D\big(D(uv)\big) = D\big(\tau(u)\,D(v) + D(u)\,\sigma(v)\big) = \tau^2(u)\,D^2(v) + D\big(\tau(u)\big)\,\sigma\big(D(v)\big) + \tau\big(D(u)\big)\,D\big(\sigma(v)\big) + D^2(u)\,\sigma^2(v)$. Using the hypothesis and $\sigma D = D\sigma, \tau D = D\tau$, we get

$$2D\big(\tau(u)\big)\,D\big(\sigma(v)\big) = 0\,, \quad \text{for all} \quad u, v \in U$$

and so,

$$2\sigma^{-1}\big(\tau(D(u))\big)\,D(U) = 0\,, \quad \text{for all} \quad u \in U\,.$$

From Lemma 5(ii), we obtain that

$$2D(u) = 0\,, \quad \text{for all} \quad u \in U\,.$$

Now for any $y \in N$ and $u \in U$, $0 = D^2(yu) = D\big(\tau(y)\,D(u) + D(y)\,\sigma(u)\big) = \tau^2(y)\,D^2(v) + 2\tau\big(D(y)\big)\,\sigma\big(D(u)\big) + D^2(y)\,\sigma^2(u)$. Hence,

$$D^2(y)\,\sigma^2(u) = 0\,, \quad \text{for all} \quad y \in N,\ u \in U\,.$$

Using σ is an automorphism on N and [2, Lemma 1.4 (i)], we can take $D^2(N) = 0$. That is $D = 0$ by [3, Lemma 4]. □

Theorem 4. *Let N be a 3-prime near-ring, D be a (σ,τ)-derivation of N such that $\sigma D = D\sigma, \tau D = D\tau$ and U a nonzero semigroup ideal of N. If $a \in N$ and $\big[D(U), a\big]_{\sigma,\tau} = 0$ then $D(a) = 0$ or $a \in Z$.*

Proof. For $u \in U$, we get $D(au)\,\sigma(a) = \tau(a)\,D(au)\,\tau(a)$. Expanding this equation, one can obtain,

$$\tau(a)\,D(u)\,\sigma(a) + D(a)\,\sigma(u)\,\sigma(a) = \tau(a)\,\tau(a)\,D(u) + \tau(a)\,D(a)\,\sigma(u)\,.$$

By the hypothesis, we get $D(u)\,\sigma(a) = \tau(a)\,D(u)$. Hence,

$$\tau(a)\,\tau(a)\,D(u) + D(a)\,\sigma(u)\,\sigma(a) = \tau(a)\,\tau(a)\,D(u) + \tau(a)\,D(a)\,\sigma(u)\,.$$

That is,

$$D(a)\,\sigma(u)\,\sigma(a) = \tau(a)\,D(a)\,\sigma(u)\,, \quad \text{for all} \quad u \in U\,. \tag{2.4}$$

Replacing $ux, x \in N$ by u in (2.4) and using (2.4), we have

$$D(a)\,\sigma(u)\,\sigma(x)\,\sigma(a) = \tau(a)\,D(a)\,\sigma(u)\,\sigma(x) = D(a)\,\sigma(u)\,\sigma(a)\,\sigma(x)$$

and so,

$$D(a)\,\sigma(u)\,\sigma\big([x,a]\big) = 0\,, \quad \text{for all} \quad x \in N,\ u \in U\,.$$

Since σ is an automorphism of N, this relation can be written,

$$\sigma^{-1}\big(D(a)\,U[x,a]\big) = 0\,, \quad \text{for all} \quad x \in N\,.$$

Thus we conclude that $a \in Z$ or $D(a) = 0$ by [2, Lemma 1.4 (i)]. $\square$

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