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OSCILLATORY AND NONOSCILLATORY SOLUTIONS FOR FIRST ORDER IMPULSIVE DYNAMIC INCLUSIONS ON TIME SCALES

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Abstract. In this paper we discuss the existence of oscillatory and nonoscillatory solutions for first order impulsive dynamic inclusions on time scales. We shall rely of the nonlinear alternative of Leray-Schauder type combined with lower and upper solutions method.

1. Introduction

This paper is concerned with the existence of oscillatory and nonoscillatory solutions of first order impulsive dynamic inclusions on time scales. More precisely, we consider the following problem,

$$y^{\Delta}(t) \in F\big(t, y(t)\big)\,,\ t \in J_T := [t_0, \infty) \cap \mathbb{T}\,, \quad t \neq t_k\,, \quad k = 1, \dots, m, \dots\,, \tag{1}$$

$$y(t_k^+) = I_k\big(y(t_k^-)\big)\,, \quad k = 1, \dots, m, \dots\,, \tag{2}$$

$$y(t_0) = y_0\,, \tag{3}$$

where $\mathbb{T}$ is time scale which is assumed to be unbounded from above, $F : J_T \times \mathbb{R} \to CK(\mathbb{R})$, is a multivalued map, $CK(\mathbb{R})$ denotes the set of nonempty, closed, and convex subsets of $\mathbb{R}$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $y_0 \in \mathbb{R}$, $t_k \in \mathbb{T}$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} < \dots \uparrow \infty$, $y(t_k^+) = \lim\limits_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim\limits_{h\to 0^+} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$ and $\lim\limits_{k\to\infty} y(t_k) = \infty$ in the sense of the time scales. Impulsive differential equations have become important in recent years in mathematical models of real processes and they rise in phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There have been significant developments in impulse theory also in recent years, especially in the area of impulsive differential equations and first

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order impulsive inclusions, with fixed moments; see the monographs of Bainov and Simeonov [6], Benchohra et al [10], Lakshmikantham et al [28], Samoilenko and Perestyuk [31] and the references therein. In recent years dynamic equations on times scales have received much attention. We refer the reader to the books by Bohner and Peterson [14, 15], Lakshmikantham et al [29] and the references therein. The time scales calculus has tremendous potential for applications in mathematical models of real processes, for example, in physics, chemical technology, population dynamics, biotechnology and economics, neural networks, social sciences, see Aulbach and Hilger [5], Bohner and Peterson [14, 15], Lakshmikantham et al [29], and the references therein. The existence of solutions of boundary value problem on a measure chain (i.e. time scale) was recently studied by Bohner and Tisdell [17], Henderson [24] and Henderson and Tisdell [26]. The question of existence of solutions to some classes of impulsive dynamic equations and impulsive dynamic inclusions on time scales was treated very recently by Henderson [25] and Benchohra et al in [1, 11, 12]. Recently (see [9]) we have initiated the study of oscillation and nonoscillatory of solutions to impulsive dynamic equations on time scales. The aim of this paper is to continue this study for impulsive dynamic inclusions on time scales. For oscillation and nonoscillation of impulsive differential equations, see for instance the monograph of Bainov and Simeonov [7] and the papers of Graef and Karsai [21, 22]. The purpose of this paper is to give some sufficient conditions for existence of oscillatory and nonoscillatory solutions of the first order dynamic impulsive problem (1)–(3) on time scales. There has been, in fact, a good deal of research already devoted to oscillation questions for dynamic equations on time scales; see, for example [2, 4, 13, 16, 19, 20, 30]. For the purposes of this paper, we shall rely on the nonlinear alternative of Leray-Schauder type combined with a lower and upper solutions method. Our results can be considered as contributions to this emerging field.

2. Preliminaries

We will briefly recall some basic definitions and facts from times scales calculus that we will use in the sequel.

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. The notations $[a, b], [a, b)$, and so on, will denote time scales intervals

$$[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\},$$

where $a, b \in \mathbb{T}$ with $a < \rho(b)$.

Definition 2.1. Let X be a Banach space. The function $f\colon \mathbb{T} \to X$ will be called rd-continuous provided it is continuous at each right-dense point and has a left-sided limit at each point, we write $f \in C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, X)$. Let $t \in \mathbb{T}^k$, the Δ derivative of f at t, denoted $f^\Delta(t)$, be the number(provided it exists) if for all $\varepsilon > 0$ there exists a neighborhood U of t such that

$$\big|f\big(\sigma(t)\big) - f(s) - f^\Delta(t)[\sigma(t) - s]\big| \le \varepsilon\big|\sigma(t) - s\big|$$

for all $s \in U$, at fix t. Let F be a function and it is called antiderivative of $f\colon \mathbb{T} \to X$ provided

$$F^\Delta(t) = f(t) \quad \text{for each} \quad t \in \mathbb{T}^k .$$

A function $p\colon \mathbb{T} \to \mathbb{R}$ is called regressive if

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T} ,$$

where $\mu(t) = \sigma(t) - t$ which called the graininess function. The set of all rd-continuous function f that satisfy $1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}$ will be denoted by $\mathcal{R}^+$. The generalized exponential function e_p is defined as the unique solution $y(t) = e_p(t, a)$ of the initial value problem $y^\Delta = p(t)y$, $y(a) = 1$, where p is a regressive function. An explicit formula for $e_p(t, a)$ is given by

$$e_p(t,s) = \exp\Big\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\Big\} \quad \text{with} \quad \xi_h(z) = \begin{cases} \dfrac{\log(1+hz)}{h} & \text{if } h \neq 0 \\ z & \text{if } h = 0 \end{cases}$$

For more details, see [14]. Clearly, $e_p(t, s)$ never vanishes. $C([a, b], \mathbb{R})$ is the Banach space of all continuous functions from $[a, b]$ into $\mathbb{R}$ where $[a, b] \subset \mathbb{T}$ with the norm

$$\|y\|_\infty = \sup\big\{|y(t)| : t \in [a, b]\big\} .$$

Remark 2.1. (i) If f is continuous, then f rd-continuous.
(ii) If f is delta differentiable at t then f is continuous at t.

For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(t)\Delta t = f(b) - f(a) .$$

Note that in the case $\mathbb{T} = \mathbb{R}$ we have

$$\sigma(t) = t , \quad \mu(t) \equiv 0 , \quad f^\Delta(t) = f'(t)$$

and in the case $\mathbb{T} = Z$ we have

$$\sigma(t) = t + 1 , \quad \mu(t) \equiv 1 , \quad f^\Delta(t) = f(t + 1) - f(t) .$$

Another important time scale is $\mathbb{T} = \{q^k\colon k \in \mathbb{N}\}$ with $q > 1$, for which

$$\sigma(t) = qt , \quad \mu(t) = (q - 1)t , \quad f^\Delta(t) = \frac{f(qt) - f(t)}{(q - 1)t} ,$$

and this time scale gives rise to so-called q-difference equations.

The condition

$$y \le \overline{y} \quad \text{if and only if} \quad y(t) \le \overline{y}(t) \quad \text{for all} \quad t \in [a, b] ,$$

defines a partial ordering in $C\big([a,b],\mathbb{R}\big)$. If $\alpha,\beta\in C\big([a,b],\mathbb{R}\big)$ and $\alpha\le\beta$, we denote

$$[\alpha,\beta]=\big\{y\in C\big([a,b],\mathbb{R}\big):\alpha(t)\le y(t)\le\beta(t)\big\}\,.$$

Let $(X,|\cdot|)$ be a Banach space. A multivalued map $G\colon X\longrightarrow 2^X$ has convex (closed) values if $G(x)$ is convex (closed) for all $x\in X$. G is bounded on bounded sets if $G(B)$ is bounded in X for each bounded set B of X (i.e. $\sup\limits_{x\in B}\big\{\sup\{|y|\colon y\in G(x)\}\big\}<\infty$).

A map $G\colon X\to CK(X)$ is called upper semicontinuous provided $\{u_k\}_{k\in\mathbb{N}}$, $\{v_k\}_{k\in\mathbb{N}}\subset X$ with $u_k\to u$, $v_k\to v$ $(k\to\infty)$ and $v_k\in G(u_k)$ for all $k\in\mathbb{N}$ always implies $v\in G(u)$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B\subseteq X$. G has a fixed point if there is $x\in X$ such that $x\in G(x)$. For more details on multivalued maps see the books of Deimling [18] and Hu and Papageorgiou [27].

Lemma 2.1 (Nonlinear Alternative [23]). *Let X be a Banach spaces with $C\subset X$ convex. Assume U is a open subset of C with $0\in U$ and $G\colon\overline{U}\to\mathcal{P}(C)$ is a compact multivalued map, u.s.c. with convex closed values. Then either,*
(i) *G has a fixed point in $\overline{U}$; or*
(ii) *there is a point $u\in\partial U$ and $\lambda\in(0,1)$ with $u\in\lambda G(u)$.*

3. Main result

We will assume for the remainder of the paper that, for each $k=1,\dots,$ the points of impulse t_k are right dense. In order to define the solution of (1)–(3) we shall consider the following space

$$PC=\big\{y\colon[t_0,\infty)\to\mathbb{R}\colon y_k\in C(J_k,\mathbb{R})\,,\quad k=0,\dots\,,\ \text{and there exist}$$
$$y(t_k^-)\ \text{and}\ y(t_k^+),\quad k=1,\dots,\ \text{with}\ \ y(t_k^-)=y(t_k^+)\big\}\,,$$

which is a Banach space with the norm

$$\|y\|_{PC}=\max\big\{\|y_k\|_k,\ k=0,\dots,\big\}\,,$$

where y_k is the restriction of y to $J_k=[t_k,t_{k+1}]$, and $\|y\|_k=\sup\limits_{t\in J_k}|y(t)|$, $k=0,\dots$.

Let us start by defining what we mean by a solution of problem (1)–(3).

Definition 3.1. A function $y\in PC\cap C^1((t_k,t_{k+1}),\mathbb{R})$, $k=0,\dots$, is said to be a solution of (1)–(3) if y satisfies the inclusion $y^{\Delta}(t)\in F(t,y(t))$ on $J-\{t_1,\dots,\}$ and the conditions $y(t_0)=y_0$, $y(t_k^+)=I_k\big(y(t_k^-)\big)$, $k=1,\dots$.

For any $y\in PC$ we define the set

$$S_{F,y}=\{v\in C(J_T,\mathbb{R}):v(t)\in F\big(t,y(t)\big)\ \text{ for all }\ t\in J_T\}\,.$$

The following concept of lower and upper solutions for (1)–(3) was introduced by Benchohra and Boucherif [8] for initial initial value problems for impulsive differential inclusions of first order. These will the basic tools in the approach that follows.

Definition 3.2. A function $\alpha \in PC \cap C^1\big((t_k, t_{k+1}), \mathbb{R}\big)$, $k = 0, \ldots$ is said to be a lower solution of (1)–(3) if there exists $v_1 \in C(J_T, \mathbb{R})$ such that $v_1(t) \in F\big(t, \alpha(t)\big)$ on J_T, $\alpha^\Delta(t) \leq v_1(t)$ on J_T, $\alpha(t_k^+) \leq I_k\big(\alpha(t_k^-)\big)$, $k = 1, \ldots$, and $\alpha(t_0) \leq y_0$. Similarly a function $\beta \in PC \cap C^1\big((t_k, t_{k+1}), \mathbb{R}\big)$, $k = 0, \ldots$ is said to be an upper solution of (1)–(3) if there exists $v_2 \in C(J_T, \mathbb{R})$ such that $v_2(t) \in F\big(t, \beta(t)\big)$ on J_T, $\beta^\Delta(t) \geq v_2(t)$ on J_T, $\beta(t_k^+) \geq I_k\big(\beta(t_k^-)\big)$, $k = 1, \ldots$ and $\beta(t_0) \geq y_0$.

For the study of this problem we first list the following hypotheses:

(H1) $F\colon J_{\mathbb{T}} \times \mathbb{R} \to CK(\mathbb{R})$ is such that $F(t, \cdot)$ is upper semicontinuous for all $t \in J_{\mathbb{T}}$ and $S_{F,y} \neq \emptyset$ for all $y \in C(J_{\mathbb{T}}, \mathbb{R})$.

(H2) For all $r > 0$ there exists a nonnegative function $h_r \in C(J_{\mathbb{T}}, \mathbb{R}^+)$ with
$$\big|F(t, y)\big| \leq h_r(t) \quad \text{for all} \quad t \in J_{\mathbb{T}} \quad \text{and all} \quad |y| \leq r\,;$$

(H3) there exist α and $\beta \in PC\big((t_k, t_{k+1}), \mathbb{R}\big)$, $k = 0, \ldots$, lower and upper solutions for the problem (1)–(3) such that $\alpha \leq \beta$;

(H4)
$$\alpha(t_k^+) \leq \min_{y \in [\alpha(t_k^-), \beta(t_k^-)]} I_k(y) \leq \max_{y \in [\alpha(t_k^-), \beta(t_k^-)]} I_k(y) \leq \beta(t_k^+)\,, \quad k = 1, \ldots.$$

Theorem 3.1. *Assume that hypotheses* (H1)–(H4) *hold. Then the problem* (1)–(3) *has at least one solution* y *such that*
$$\alpha(t) \leq y(t) \leq \beta(t) \quad \textit{for all} \quad t \in J_{\mathbb{T}}\,.$$

Proof. The proof will be given in several steps.

Step 1: Consider the following problem:
$$y^\Delta(t) \in F\big(t, y(t)\big)\,, \quad t \in J_1 := [t_0, t_1]\,, \tag{4}$$
$$y(t_0) = y_0\,. \tag{5}$$

Transform the problem (4)–(5) into a fixed point problem. Consider the following modified problem
$$y^\Delta(t) \in F\big(t, (\tau y)(t)\big)\,, \quad t \in J_1\,, \tag{6}$$
$$y(t_0) = y_0\,, \tag{7}$$
where $\tau\colon C(J_1, \mathbb{R}) \longrightarrow C(J_1, \mathbb{R})$ be the truncation operator defined by
$$(\tau y)(t) = \begin{cases} \alpha(t), & y(t) < \alpha(t)\,, \\ y(t), & \alpha(t) \leq y(t) \leq \beta(t)\,, \\ \beta(t)(t), & y(t) > \beta(t)\,. \end{cases}$$

A solution to (6)–(7) is a fixed point of the operator $N\colon C\big([t_0, t_1], \mathbb{R}\big) \longrightarrow CK\big(C\big([t_0, t_1], \mathbb{R}\big)\big)$ defined by
$$N(y) = \Big\{h \in C\big([t_0, t_1], \mathbb{R}\big)\colon h(t) = y_0 + \int_{t_0}^t g(s)\Delta s\Big\}\,,$$

where $g \in \tilde{S}^1_{F,\tau y}$ and

$$\tilde{S}^1_{F,\tau y} = \big\{ g \in S^1_{F,\tau y} \colon g(t) \geq v_1(t) \ \text{ a.e. on } \ A_1 \ \text{ and } \ g(t) \leq v_2(t) \ \text{ a.e. on } \ A_2 \big\}\,,$$

$$S^1_{F,\tau y} = \big\{ g \in C^1(J_1,\mathbb{R}) \colon g(t) \in F\big(t,(\tau y)(t)\big) \ \text{ for a.e. } \ t \in J_1 \big\}\,,$$

$$A_1 = \big\{ t \in J_1 \colon y(t) < \alpha(t) \leq \beta(t) \big\}\,, \qquad A_2 = \big\{ t \in J_1 \colon \alpha(t) \leq \beta(t) < y(t) \big\}\,.$$

Remark 3.1. (i) For each $y \in C\big([t_0,t_1],\mathbb{R}\big)$, the set $\tilde{S}^1_{F,\tau y}$ is nonempty. In fact, (H1) implies there exists $g_3 \in S^1_{F,\tau y}$, so we set

$$g = v_1\chi_{A_1} + v_2\chi_{A_2} + v_3\chi_{A_3}\,,$$

where

$$A_3 = \big\{ t \in J_1 \colon \alpha(t) \leq y(t) \leq \beta(t) \big\}\,.$$

Then, by decomposability, $g \in \tilde{S}^1_{F,\tau y}$.

(ii) By the definition of τ it is clear that there exists a nonnegative function $h \in C(J_1,\mathbb{R}^+)$ with

$$\big|F\big(t,\tau(y)(t)\big)\big| \leq h(t) \quad \text{for each} \quad t \in J_1 \quad \text{and each} \quad y \in \mathbb{R}\,.$$

We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type [23]. The proof will be given in a couple of claims.

Claim 1: A priori bounds on solutions.

Let $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. Then there exists $g \in \tilde{S}^1_{F,\tau y}$ such that for some $\lambda \in (0,1)$ we have, for each $t \in J_1$

$$y(t) = \lambda y_0 + \lambda\Big[\int_{t_0}^{t} g(s)\Delta s\Big]\,.$$

This implies by (H2) and Remark 3.1 (ii) that for each $t \in J_1$ we have

$$\big|y(t)\big| \leq |y_0| + \int_{t_0}^{t} \big|g(s)\big|\Delta s \leq |y_0| + \|h\|_{L^1} := M\,.$$

Set

$$U = \big\{ y \in C\big([t_0,t_1],\mathbb{R}\big) \colon \|y\|_\infty < M+1 \big\}\,.$$

From the choice of U there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. We first show that $N \colon \overline{U} \to CK\big(C\big([t_0,t_1],\mathbb{R}\big)\big)$ is compact.

Claim 2: $N(y)$ is convex for each $y \in C([t_0,t_1],\mathbb{R})$.

Indeed, if h_1, h_2 belong to $N(y)$, then there exist $g_1, g_2 \in \tilde{S}^1_{F,\tau y}$ such that for each $t \in J_1$ we have

$$h_i(t) = y_0 + \int_{t_0}^{t} g_i(s)\Delta s\,, \quad i = 1,2\,.$$

Let $0 \le d \le 1$. Then for each $t \in J_1$ we have

$$\big(dh_1 + (1-d)h_2\big)(t) = y_0 + \int_{t_0}^{t} \big[dg_1(s) + (1-d)g_2(s)\big]\Delta s\,.$$

Since $\tilde{S}^1_{F_1,\tau y}$ is convex (because $F\big(t,(\tau y)(t)\big)$ has convex values) then

$$dh_1 + (1-d)h_2 \in N(y)\,.$$

Claim 3: *N maps bounded sets into sets in $C\big([t_0,t_1],\mathbb{R}\big)$.*

Indeed, it is enough to show that for each $q > 0$ there exists a positive constant ℓ such that for each $y \in B_q = \big\{y \in C\big([t_0,t_1],\mathbb{R}\big) : |y| \le q\big\}$ one has $|N(y)| \le \ell$.

Let $y \in B_q$ and $h \in N(y)$ then there exists $g \in \tilde{S}^1_{F,\tau y}$ such that for each $t \in J_1$ we have

$$h(t) = y_0 + \int_{t_0}^{t} g(s)\Delta s\,.$$

By (H2) we have for each $t \in J_1$

$$\big|h(t)\big| \le |y_0| + \int_{t_0}^{t} \big|g(s)\big|\Delta s \le |y_0| + \|h_q\|_{L^1} := \ell\,.$$

Claim 4: *N maps bounded set into equicontinuous sets of $C\big([t_0,t_1],\mathbb{R}\big)$.*

Let $u_1, u_2 \in J_1$, $u_1 < u_2$ and B_q be a bounded set of $C\big([t_0,t_1],\mathbb{R}\big)$ as in Claim 3. Let $y \in B_q$ and $h \in N(y)$ then for each $t \in J_1$ we have

$$\big|h(u_2) - h(u_1)\big| = \Big|\int_{t_0}^{u_2} g(s)\Delta s - \int_{t_0}^{u_1} g(s)\Delta s\Big| \le \int_{u_1}^{u_2} \big|g(s)\big|\Delta s \le \int_{u_1}^{u_2} h_q(s)\Delta s\,.$$

As $u_2 \to u_1$ the right-hand side of the above inequality tends to zero. Claims 2 to 4 together with the Arzela-Ascoli theorem imply that $N\colon \overline{U} \to CK\big(C\big([t_0,t_1],\mathbb{R}\big)\big)$ is a compact multivalued map.

Claim 5: *N is upper semicontinuous maps.*

We define a linear and continuous operator $\Gamma\colon C\big([t_0,t_1],\mathbb{R}\big) \to C\big([t_0,t_1],\mathbb{R}\big)$ by

$$(\Gamma h)(t) = \int_{t_0}^{t} h(s)\Delta s\,, \quad t \in J_1\,.$$

Let $\{u_k\}_{k\in\mathbb{N}}, \{w_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $u_k \to u_0$, $w_k \to w_0$ $(k \to \infty)$ and $w_k \in N(u_k)$ for all $k \in \mathbb{N}$. Thus there exists $v_k \in \tilde{S}^1_{F,\tau u_k}$ with $w_k = \Gamma v_k$. Since $u_k \in \overline{U}$ for all $K \in \mathbb{N}$, (H1) and (H2) imply that there exists a compact set (see [3]) $\Omega \subset C\big([t_0,t_1],\mathbb{R}\big)$ with $\{v_k\}_{k\in\mathbb{N}} \subset \Omega$. Therefore there exists a convergent subsequence $\{v_{k_\nu}\}_{\nu\in\mathbb{N}}$ of $\{v_k\}_{k\in\mathbb{N}}$, say $v_{k_\nu} \to v_0$ as $\nu \to \infty$. Now $v_{k_\nu} \to v_0$ and $u_{k_\nu} \to u_0$ as $\nu \to \infty$ and $v_{k_\nu}(t) \in F\big(t, \tau u_{k_\nu}(t)\big)$ for all $t \in J_1$. Thus, since $F(t,\cdot)$ is upper semicontinuous for all $t \in J_1$, we may conclude $v_0(t) \in F\big(t,\tau u_0(t)\big)$ for all $t \in J_1$ and therefore $v_0 \in \tilde{S}^1_{F,\tau u_0}$. Since $v_{k_\nu} \to v_0$ as $\nu \to \infty$ and $\Gamma\colon C\big([t_0,t_1],\mathbb{R}\big) \to C\big([t_0,t_1],\mathbb{R}\big)$ is continuous, we say that $w_{k_\nu} = \Gamma v_{k_\nu} \to \Gamma v_0$ as $\nu \to \infty$, and hence

$$w_0 = \Gamma v_0 \in N(u_0)\,.$$

Therefore $N\colon \overline{U} \to CK\big(C([t_0,t_1],\mathbb{R})\big)$ is upper semicontinuous. As it is also compact, we deduce from the nonlinear alternative of Leray Schauder type [23] that N has a fixed point y in U is a solution of the problem (6)–(7).

Claim 6: The solution y of (6)–(7) *satisfies*

$$\alpha(t) \le y(t) \le \beta(t) \quad \text{for all} \quad t \in J_1 .$$

Let y be a solution to (6)–(7). We prove that

$$\alpha(t) \le y(t) \quad \text{for all} \quad t \in J_1 .$$

Suppose not. Then there exist $e_1, e_2 \in J_1$, $e_1 < e_2$ such that $\alpha(e_1) = y(e_1)$ and

$$y(t) < \alpha(t) \quad \text{for all} \quad t \in [e_1, e_2] . \tag{8}$$

In view of the definition of τ one has

$$y(t) - y(e_1) \in \int_{e_1}^{t} F\big(s, \alpha(s)\big)\Delta s .$$

Thus there exists $g(s) \in F\big(s,\alpha(s)\big)$ for all $s \in (e_1, e_2)$ with $g(s) \ge v_1(s)$ for all $s \in (e_1, e_2)$.

$$y(t) = y(e_1) + \int_{e_1}^{t} g(s)\Delta s . \tag{9}$$

Using (8)–(9) and the fact that α is a lower solution to (4)–(5) we get for $t \in (e_1, e_2]$

$$\begin{aligned}
0 &< \alpha(t) - y(t) \\
&\le \alpha(e_1) + \int_{e_1}^{t} v_1(s)\Delta s - y(t) \\
&= \alpha(e_1) + \int_{e_1}^{t} v_1(s)\Delta s - \Big(y(e_1) + \int_{e_1}^{t} g(s)\Delta s\Big) \\
&= \int_{e_1}^{t} \big(v_1(s) - g(s)\big)\Delta s \\
&\le 0
\end{aligned}$$

which is a contradiction. Analogously, we can prove that

$$y(t) \le \beta(t) \quad \text{for all} \quad t \in [t_0, t_1] .$$

This shows that the problem (6)–(7) has a solution in the interval $[\alpha, \beta]$ which is solution of (4)–(5). Denote this solution by y_0.

Step 2: Consider the following problem:

$$y^{\Delta}(t) \in F\big(t, y(t)\big) , \quad t \in J_2 := [t_1, t_2] , \tag{10}$$

$$y(t_1^{+}) = I_1\big(y_0(t_1^{-})\big) . \tag{11}$$

Transform the problem (10)–(11) into a fixed point problem. Consider the following modified problem

$$y^{\Delta}(t) \in F_1\big(t, \tau(t)\big), \quad t \in J_2, \tag{12}$$

$$y(t_1^+) = I_1\big(y_0(t_1^-)\big). \tag{13}$$

A solution to (12)–(13) is a fixed point of the operator $N_1 \colon C\big([t_1, t_2], \mathbb{R}\big) \to CK\big(C\big([t_1, t_2], \mathbb{R}\big)\big)$ defined by

$$N_1(y) = \Big\{h \in C([t_0, t_1], \mathbb{R}) \colon h(t) = \int_{t_1}^{t} g(s)\Delta s + I_1\big(y_0(t_1^-)\big)\Big\},$$

where $g \in \tilde{S}^1_{F,\tau y}$. Since $y_0(t_1) \in \big[\alpha(t_1^-), \beta(t_1^-)\big]$, then (H4) implies that

$$\alpha(t_1^+) \le I_1(y_0(t_1^-)) \le \beta(t_1^+),$$

that is

$$\alpha(t_1^+) \le y(t_1^+) \le \beta(t_1^+).$$

Claim 1: A priori bounds on solutions.

Let $y \in \lambda N_1(y)$ for some $\lambda \in (0, 1)$. Then there exists $g \in \tilde{S}^1_{F,\tau y}$ such that for some $\lambda \in (0, 1)$ we have, for each $t \in J_2$

$$y(t) = \lambda\Big[\int_{t_1}^{t} g(s)\Delta s + I_1\big(y_0(t_1^-)\big)\Big].$$

This implies by (H2) that for each $t \in J_2$ we have

$$\big|y(t)\big| \le \int_{t_1}^{t} \big|g(s)\big|\Delta s + \big|I_1\big(y_0(t_1^-)\big)\big| \le \|h\|_{L^1} + \max\big(|\alpha(t_1^+)|, |\beta(t_1^+)|\big) := M_1.$$

Set

$$U_1 = \big\{y \in C\big([t_1, t_2], \mathbb{R}\big) \colon \|y_1\|_\infty < M_1 + 1\big\}.$$

From the choice of U_1 there is no $y \in \partial U_1$ such that $y \in \lambda N_1(y)$ for some $\lambda \in (0, 1)$. Using the same reasoning as that used for problem (6)–(7) we can conclude that $N_1 \colon \overline{U}_1 \to CK\big(C\big([t_1, t_2], \mathbb{R}\big)\big)$ is upper semicontinuous and compact. As a consequence of the nonlinear alternative of Leray Schauder type [23] we deduce that N_1 has a fixed point y_1 in U_1 is a solution of the problem (12)–(13).

Claim 2: We show that this solution satisfies

$$\alpha(t) \le y(t) \le \beta(t) \quad \text{for all} \quad t \in J_2.$$

Let y be a solution to (12)–(13). We first show that

$$\alpha(t) \le y(t) \quad \text{for all} \quad t \in J_2.$$

Assume this false, then since $y(t_1^+) \ge \alpha(t_1^+)$, there exist $e_3, e_4 \in J_2$ with $e_3 < e_4$ such that $\alpha(e_3) = y(e_3)$ and

$$y(t) > \alpha(t) \quad \text{for all} \quad t \in (e_3, e_4]. \tag{14}$$

Thus there exists $g(s) \in F(s, \alpha(s))$ on J_2 with $g(s) \geq v_1(s)$ on (e_3, e_4), and

$$y(t) = y(e_3) + \int_{e_3}^{t} g(s)\Delta s. \tag{15}$$

Using (14)–(15) and the fact that α is a lower solution to (10)–(11) we get for $t \in (e_3, e_4]$

$$\begin{aligned} 0 &< \alpha(t) - y(t) \\ &\leq \alpha(e_3) + \int_{e_3}^{t} v_1(s)\Delta s - y(t) \\ &= \alpha(e_3) + \int_{e_3}^{t} v_1(s)\Delta s - \Big(y(e_3) + \int_{e_3}^{t} g(s)\Delta s\Big) \\ &= \int_{e_3}^{t} \big(v_1(s) - g(s)\big)\Delta s \\ &\leq 0 \end{aligned}$$

which is a contradiction. Analogously, we can prove that

$$y(t) \leq \beta(t) \quad \text{for all} \quad t \in [t_1, t_2] \,.$$

This shows that the problem (12)–(13) has a solution in the interval $[\alpha, \beta]$ which is solution of (10)–(11). Denote this solution by y_1.

Step 3: We continue this process and into account that $y_m := y|_{[t_{m-1}, t_m]}$ is a solution the problem

$$y^{\Delta}(t) \in F\big(t, y(t)\big)\,, \quad t \in J_m := [t_{m-1}, t_m]\,, \tag{16}$$

$$y(t_m^+) = I_m\big(y_{m-1}(t_{m-1}^-)\big)\,. \tag{17}$$

Consider the following modified problem

$$y^{\Delta}(t) \in F_1\big(t, y(t)\big)\,, \quad t \in J_m\,, \tag{18}$$

$$y(t_m^+) = I_m\big(y_{m-1}(t_{m-1}^-)\big)\,. \tag{19}$$

A solution to (18)–(19) is a fixed point of the operator $N_m \colon C\big([t_{m-1}, t_m], \mathbb{R}\big) \to p\big(C\big([t_{m-1}, t_m], \mathbb{R}\big)\big)$ defined by

$$N_m(y)(t) = \Big\{h \in C\big([t_{m-1}, t_m], \mathbb{R}\big) \colon h(t) = \int_{t_m}^{t} g(s)\Delta s + I_m\big(y_{m-1}(t_{m-1}^-)\big)\Big\}\,,$$

where $g \in \tilde{S}^1_{F,\tau y}$. Using the same reasoning as that used for problem (4)–(5) and (10)–(11) we can conclude to the existence of at least one solution y to (16)–(17). Denote this solution by y_{m-1}. The solution y of the problem (1)–(3) is then defined by

$$y(t) = \begin{cases} y_1(t), & t \in [t_0, t_1], \\ y_2(t), & t \in (t_2, t_1], \\ \vdots & \\ y_m(t), & t \in (t_m, t_{m+1}], \\ \vdots & \end{cases}$$

The following theorem gives sufficient conditions to ensure the nonoscillatory of the solutions of problem (1)–(3).

Theorem 3.2. *Let α and β be lower and upper solutions respectively of* (1)–(3) *and assume that*

(H5) *α is eventually positive nondecreasing or β is eventually negative nonincreasing*

Then every solution y of (1)–(3) *such that $y \in [\alpha, \beta]$ is nonoscillatory.*

Proof. Assume α be eventually positive. Thus there exist $T_\alpha > t_0$ such that

$$\alpha(t) > 0 \quad \text{for all} \quad t > T_\alpha .$$

Hence $y(t) > 0$ for all $t > T_\alpha$, and $t \neq t_k, k = 1, \dots$ For some $k \in N$ and $t > t_\alpha$ we have $y(t_k^+) = I_k\big(y(t_k)\big)$. From (H4) we get $y(t_k^+) > \alpha(t_k^+)$. Since for each $h > 0, \alpha(t_k + h) \geq \alpha(t_k) > 0$, then $I_k\big(y(t_k)\big) > 0$ for all $t_k > T_\alpha, k = 1, \dots$ which means that y is nonoscillatory. Analogously, if β eventually negative, then there exists $T_\beta > t_0$ such that

$$y(t) < 0 \quad \text{for all} \quad t > T_\beta ,$$

which means that y is nonoscillatory. □

The following theorem discusses the oscillatory of the solutions of problem (1)–(3).

Theorem 3.3. *Let α and β be lower and upper solutions respectively of* (1)–(3) *and assume that the sequences $\alpha(t_k)$ and $\beta(t_k), k = 1, \dots$ are oscillatory then every solution y of* (1)–(3) *such that $y \in [\alpha, \beta]$ is oscillatory.*

Proof. Suppose on the contrary that y is nonoscillatory solution of (1)–(3). Then there exists $T_y > 0$ such that $y(t) > 0$ for all $t > T_y$ or $y(t) < 0$ for all $t > T_y$. In the case $y(t) > 0$ for all $t > T_y$ we have $\beta(t_k) > 0$ for all $t_k > T_y, k = 1, \dots$ which is a contradiction since $\beta(t_k)$ is an oscillatory upper solution. Analogously in the case $y(t) < 0$ for all $t > T_y$ we have $\alpha(t_k) < 0$ for all $t_k > T_y, k = 1, \dots$ which is also a contradiction since $\alpha(t_k)$ is an oscillatory lower solution. □

4. Example

As an application of our results, we consider the following impulsive dynamic inclusion

$$y^{\Delta}(t) \in [f_1(t, y(t)), f_2(t, y(t))] , \quad t \in J_T := [t_0, \infty) \cap \mathbb{T} , \qquad (20)$$
$$t \neq t_k, \quad k = 1, \dots, m, \dots ,$$

$$y(t_k^+) = I_k\big(y(t_k^-)\big)\,, \quad k = 1, \dots, m, \dots\,, \tag{21}$$

where $t \to f_1(t,y)$ is lower semicontinuous and $t \to f_2(t,y)$ is upper semicontinuous for each $y \in \mathbb{R}$. From [18] the multivalued $F(t,y) := \big[f_1(t,y), f_2(t,y)\big]$ is upper semicontinuous with respect to its second variable and with closed, convex values. Assume that

$$\int_0^t f_1(s,y)\Delta s \le I_k\Big(\int_0^t f_1(s,y)\Delta s\Big)\,, \quad k \in \mathbb{N}\,,$$

$$\int_0^t f_2(s,y)\Delta s \ge I_k\Big(\int_0^t f_2(s,y)\Delta s\Big)\,, \quad k \in \mathbb{N}\,.$$

Consider the functions

$$\alpha(t) := \int_0^t f_1(s,y)\Delta s\,,$$

and

$$\beta(t) := \int_0^t f_2(s,y)\Delta s\,.$$

Clearly, α and β are lower abd upper solutions for the problem (20)–(21), respectively; that is,

$$\alpha^{\Delta}(t) \le f_1\big(t, \alpha(t)\big)\,, \quad t \in J_T,\ t \ne t_k,\ k = 1, \dots, m, \dots\,,$$

$$\beta^{\Delta}(t) \ge f_2\big(t, \beta(t)\big)\,, \quad t \in J_T,\ t \ne t_k,\ k = 1, \dots, m, \dots\,.$$

Since all conditions of Theorem 3.1 are satisfied, then problem (20)–(21) has at leat one solution y satisfying

$$\alpha(t) \le y(t) \le \beta(t) \quad \text{for each} \quad t \in J_{\mathbb{T}}\,.$$

If $f_1(t,y) > 0$ for each $y \in \mathbb{R}$, then α is positive and nondecreasing, thus the solution y is nonoscillatory. If $f_2(t,y) < 0$ for each $y \in \mathbb{R}$, then β is negative and nonincreasing, thus the solution y is nonoscillatory. If the sequences $\alpha(t_k)$, $\beta(t_k)$ are both oscillatory, then y is oscillatory.

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