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HIGHER ORDER LINEAR CONNECTIONS FROM FIRST ORDER ONES

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Abstract. We describe how find all $\mathcal{M}f_m$-natural operators D transforming torsion free classical linear connections ∇ on m-manifolds M into r-th order linear connections $D(\nabla)$ on M.

Introduction

We study the problem how a torsion free classical linear connection ∇ on an m-dimensional manifold M can induce a r-th order linear connection $D(\nabla)\colon TM \to J^rTM$ on M (or equivalently a right invariant connection $D(\nabla)$ in the principal bundle $L^rM = \operatorname{inv} J^r_0(\mathbf{R}^m, M)$). This problem is related to $\mathcal{M}f_m$-natural operators $D\colon Q_\tau \rightsquigarrow Q^r$ in the sense of [3]. We describe how find all operators D in question.

The category of m-dimensional manifolds and their embeddings is denoted by $\mathcal{M}f_m$. All manifolds and maps are assumed to be of class C^∞.

1. Higher order connections on manifolds

Given an m-manifold M we have the principal bundle $L^rM = \operatorname{inv} J^r_0(\mathbf{R}^m, M)$ with the standard group $G^r_m = \operatorname{inv} J^r_0(\mathbf{R}^m, \mathbf{R}^m)_0$ acting on right by the composition of jets. Any $\mathcal{M}f_m$-map $\varphi\colon M \to N$ induces principal bundle map $L^r\varphi\colon L^rM \to L^rN$ by composition of jets. The correspondence $L^r\colon \mathcal{M}f_m \to \mathcal{FM}$ is a natural bundle over m-manifolds, [3].

A principal r-th order connection on M is a G^r_m-invariant section $\Gamma\colon L^rM \to J^1L^rM$ of the first jet prolongation $J^1L^rM \to L^rM$ of $L^rM \to M$, which can be identified with the corresponding (G^r_m-invariant) lifting map $\Gamma\colon TM \times_M L^rM \to TL^rM$, see [3]. A linear r-th order connection on M is a linear section $\lambda\colon TM \to J^rTM$ of the r-jet prolongation $J^rTM \to TM$ of the tangent bundle $TM \to M$.

From the introduction of [2] we have

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Fact 1. A linear r-th order connection $\lambda : TM \to J^rTM$ on M induces a principal r-th order connection $\Gamma^\lambda\colon TM \times_M L^rM \to TL^rM$ on M by $\Gamma^\lambda(v,p) = \mathcal{L}^rV(p)$, $v \in T_xM$, $p \in L^r_xM$, $x \in M$, where $\lambda(v) = j^r_xV \in J^r_xTM$ and $\mathcal{L}^rV$ denotes the flow lifting of V to L^rM. Conversely, any principal connection $\Gamma\colon TM \times_m L^rM \to TL^rM$ on M induces a linear r-th order connection $\lambda^\Gamma\colon TM \to J^rTM$ on M by $\lambda^\Gamma(v) = j^r_xV$, $v \in T_xM$, $x \in M$, where $\mathcal{L}^rV(p) = \Gamma(v,p)$ for some (and then for all) $p \in L^r_xM$. The correspondence $\lambda \to \Gamma^\lambda$ is one to one with the inverse one by $\Gamma \to \lambda^\Gamma$.

Thus a first order linear connection $\lambda\colon TM \to J^1TM$ on M is in fact a classical linear connection on M (which can be also defined by its covariant derivative ∇).

2. Natural operators

The general concept of natural operators is given in [3]. We need only the following partial definition.

Definition 1. A $\mathcal{M}f_m$-natural operator $D\colon Q_\tau \rightsquigarrow Q^r$ is a $\mathcal{M}f_m$-invariant family of regular operators (functions)

$$D\colon Q_\tau(M) \to Q^r(M)$$

for any m-manifold M, where $Q_\tau(M)$ is the set of torsion free classical linear connections on M and $Q^r(M)$ is the set of all r-th order linear connections on M. The invariance means that if $\nabla_1 \in Q_\tau(M_1)$ and $\nabla_2 \in Q_\tau(M_2)$ are φ-related (by a $\mathcal{M}f_m$-map $\varphi\colon M_1 \to M_2$) then $D(\nabla_1)$ and $D(\nabla_2)$ are φ-related, too. The regularity means that D transforms smoothly parametrized families of connections into smoothly parametrized families.

3. The exponential extension of a classical linear connection

The following construction has been (in equivalent way) presented by I. Kolář [1].

Example 1. Let ∇ be a torsion free classical linear connection on M. We define an r-th order linear connection $\mathrm{Exp}\,(\nabla)\colon TM \to J^rTM$ by

$$\mathrm{Exp}\,(\nabla)(v) = j^r_x\big((\exp^\nabla_x)_*\tilde{v}\big)\,, \tag{1}$$

where $\exp^\nabla_x\colon T_xM \to M$ is the exponent of ∇ in x (defined on some neighborhood of $0 \in T_xM$ onto some neighborhood of x) and where $\tilde{v}$ is the constant vector field on the vector space T_xM corresponding to v ($\tilde{v}(w) = [w+tv]$). The correspondence $\mathrm{Exp}\colon Q_\tau \rightsquigarrow Q^r$ is an $\mathcal{M}f_m$-natural operator.

4. An isomorphism

Example 2. Let ∇ be a torsion free classical linear connection on a manifold M. Define a vector bundle isomorphism

$$\psi_\nabla\colon J^rTM \to \oplus^r_{k=0} S^kT^*M \otimes TM$$

(depending canonically on ∇) as follows. Let $\tau \in J^r_x TM$, $x \in M$. Let φ be a ∇-normal coordinate system on M with center x. We put

$$\psi_\nabla(\tau) = \oplus_{k=0}^r S^k T_0^* \varphi^{-1} \otimes T_0 \varphi^{-1}\big(I(J^r T\varphi(\tau))\big)\,, \tag{2}$$

where $I\colon J^r_0 T\mathbf{R}^m \to \oplus_{k=0}^r S^k T^*_0 \mathbf{R}^m \otimes T_0\mathbf{R}^m$ is the usual identification. If φ_1 is another such ∇-normal coordinate system with center x then $\varphi_1 = A \circ \varphi$ near x for some $A \in GL(m)$. The identification I is $GL(m)$-equivariant. Then standardly we verify that right hand sides of (2) for φ and φ_1 coincide. That is why the definition of $\psi_\nabla(\tau)$ is independent of the choice of φ.

5. The main result

Theorem 1. *Let $D\colon Q_\tau \rightsquigarrow Q^r$ be an $\mathcal{M}f_m$-natural operator transforming torsion free classical linear connections ∇ on m-manifolds M into r-th order linear connections $D(\nabla)\colon TM \to J^r TM$ on M. Then there exist uniquely determined $\mathcal{M}f_m$-natural operators $A_k : Q_\tau \rightsquigarrow T^* \otimes S^k T^* \otimes T$ for $k = 0, \dots, r$ transforming torsion free classical linear connections ∇ on m-manifolds M into tensor fields $A_k(\nabla)$ of type $T^* \otimes S^k T^* \otimes T$ on M such that $A_0 = 0$ and*

$$D(\nabla)(v) = \mathrm{Exp}\,(\nabla)(v) + (\psi_\nabla)^{-1}\big(\langle A_0(\nabla)(x), v\rangle, \dots, \langle A_r(\nabla)(x), v\rangle\big) \tag{3}$$

for any torsion free classical linear connection ∇ on M and any $v \in T_x M$, $x \in M$, where ψ_∇ is the isomorphism from Example 2 and Exp *is the operator from Example 1 and the brackets $\langle \cdot, \cdot \rangle$ denote the obvious contractions $\langle t, v\rangle = t(v, \cdot)$, v in the first position.*

Conversely, given $\mathcal{M}f_m$-natural operators $A_k : Q_\tau \rightsquigarrow T^* \otimes S^k T^* \otimes T$ for $k = 0, \dots, r$ with $A_0 = 0$, the formula (3) *defines an $\mathcal{M}f_m$-natural operator $D\colon Q_\tau \rightsquigarrow Q^r$.*

Proof. We must define $\mathcal{M}f_m$-natural operators $A_k\colon Q_\tau \rightsquigarrow T^* \otimes S^k T^* \otimes T$ by $\big(\langle A_k(\nabla)(x), v\rangle\big)_{k=0}^r = \psi_\nabla\big(D(\nabla)(v) - \mathrm{Exp}\,(\nabla)(v)\big)$, $v \in T_x M$, $x \in M$. Clearly $A_0 = 0$ and we have (3). □

Remark 1. Theorem 1 together with the result of Section 33.4 in [3] gives a complete description of all $\mathcal{M}f_m$-natural operators $D\colon Q_\tau \rightsquigarrow Q^r$. In fact, each covariant derivative of the curvature $R(\nabla) \in C^\infty(TM \otimes T^*M \otimes \wedge^2 T^*M)$ of a classical linear connection ∇ is an ($\mathcal{M}f_m$-)natural tensor. Further every tensor multiplication of two natural tensors and every contraction on one covariant and one contravariant entry of a natural tensor give new natural tensor. Finally, we can tensor any natural tensor with a connection independent natural tensor, we can permute any number of entries in the tensor prodduct and we can repeat of these steps and take linear combinations. In this way we can obtain any natural tensor of type (p, q) (in particular of type $(1, k+1)$). Then each natural tensor of type $T^* \otimes S^k T^* \otimes T$ (i.e. $\mathcal{M}f_m$-natural operator $Q_\tau \rightsquigarrow T^* \otimes S^k T^* \otimes T$) can be obtained from a natural tensor of type $(1, k+1)$ by using the respective symmetrization.

6. Natural operators $Q_\tau \rightsquigarrow Q^r_\tau$

By [4], an r-th order linear connection $\lambda \in Q^r(M)$ on M is called torsion-free if its torsion tensor $\tau^\lambda\colon \wedge^2 TM \to J^{r-1}TM$, $\tau^\lambda(u,v) = \{\lambda(u), \lambda(v)\}$, $u, v \in T_xM$, $x \in M$, where $\{j^r_x X, j^r_x Y\} := j^{r-1}_x([X,Y])$, $j^r_x X, j^r_x Y \in J^r_x TM$, vanishes. An equivalent notion of torsion free r-th order linear connections is presented in [1]. The construction of [1] clarify that the exponential prolongation (equivalently defined in Example 1) is a torsion-free connection on L^rM. By Proposition 5 in [1], the difference of torsion-free connections on L^rM over the same connection on $L^{r-1}M$ is an arbitrary section of the tensor bundle $S^{r+1}T^*M \otimes TM$. Now we easily observe that if $D\colon Q_\tau \rightsquigarrow Q^r_\tau$ is an $\mathcal{M}f_m$-natural operator sending torsion free classical linear connections $\nabla \in Q_\tau(M)$ into torsion-free r-th order linear connections $D(\nabla) \in Q^r_\tau(M)$ then the defined in the proof of Theorem 1 operators A_k have values in tensor fields of type $S^{k+1}T^* \otimes T$, i.e. $A_k\colon Q_\tau \rightsquigarrow S^{k+1}T^* \otimes T$. Thus we have

Theorem 2. *Let $D\colon Q_\tau \rightsquigarrow Q^r_\tau$ be an $\mathcal{M}f_m$-natural operator transforming torsion free classical linear connections ∇ on m-manifolds M into torsion free r-th order linear connections $D(\nabla)$ on M. Then there exist uniquely determined $\mathcal{M}f_m$-natural operators $A_k\colon Q_\tau \rightsquigarrow S^{k+1}T^* \otimes T$ for $k = 0, \dots, r$ transforming torsion free classical linear connections ∇ on m-manifolds M into tensor fields $A_k(\nabla)$ of type $S^{k+1}T^* \otimes T$ on M such that $A_0 = 0$ and we have (3) for any torsion free classical linear connection ∇ on M and any $v \in T_xM$, $x \in M$.*

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