ARCHIVUM MATHEMATICUM (BRNO)
Tomus 44 (2008), 37–40

NOTES ON COUNTABLE EXTENSIONS OF $p^{\omega+n}$-PROJECTIVES

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Abstract. We prove that if G is an Abelian p-group of length not exceeding ω and H is its $p^{\omega+n}$-projective subgroup for $n \in \mathbb{N} \cup \{0\}$ such that G/H is countable, then G is also $p^{\omega+n}$-projective. This enlarges results of ours in (Arch. Math. (Brno), 2005, 2006 and 2007) as well as a classical result due to Wallace (J. Algebra, 1971).

Unless we do not specify some else, by the term "*group*" we mean "*an Abelian p-group*", written additively as is the custom when dealing with such groups, for some arbitrary but a fixed prime p. All unexplained exclusively, but however used, notions and notations are standard and follow essentially those from [7]. For instance, a group is called *separable* if it does not contain elements of infinite height. As usual, for any group A, A_r denotes the reduced part of A.

A recurring theme is the relationship between the properties of a given group and its countable extension (see, e.g., [1]). The study in that aspect starts incidentally by Wallace [12] in order to establish a complete set of invariants for a concrete class of mixed Abelian groups. Specifically, his remarkable achievement states as follows.

Theorem (Wallace, 1971)**.** *Let G be a reduced group with a totally projective subgroup H so that G/H is countable. Then G is totally projective.*

Since any reduced group is summable precisely when its socle is a free valuated vector space, as application of ([8], Lemma 7) one can derive the following.

Theorem (Fuchs, 1977)**.** *Let G be a reduced group with a summable isotype subgroup H so that G/H is countable. Then G is summable.*

Without knowing then the cited attainment of Fuchs, we have proved in [1] an analogous assertion for summable groups of countable length via the usage of a more direct group-theoretical approach. In [5] was also showed via the construction of a concrete example that when the summable subgroup H is not isotype in G, G may not be summable.

Likewise, in [5] (see [1] too) it was obtained the following affirmation.

2000 *Mathematics Subject Classification:* Primary: 20K10; Secondary: 20K25.
Key words and phrases: abelian groups, countable factor-groups, $p^{\omega+n}$-projective groups.
Received May 28, 2007. Editor J. Trlifaj.

Theorem (Danchev and Keef, 2005 and 2008)**.** *Let G be a group with a subgroup H so that G/H is countable. If*

(a) *H is σ-summable, then G is σ-summable provided that it is of limit length and H is isotype in G (when H is not isotype in G, G may not be σ-summable);*

(b) *H is a Σ-group, then G is a Σ-group;*

(c) *H is a Q-group, then G is a Q-group provided that it is separable;*

(d) *H is weakly ω_1-separable, then G is weakly ω_1-separable provided that it is separable.*

In [1] and [3] we have shown the following statement as well.

Theorem (Danchev, 2005 and 2006)**.** *Let G be a group with a $p^{\omega+n}$-projective subgroup H so that G/H is countable and $n \in \mathbb{N} \cup \{0\}$. If*

(e) *H is pure and nice in G, then G is $p^{\omega+n}$-projective;*

(f) *H is pure in the separable G, then G is $p^{\omega+n}$-projective.*

Note that in [4] we have established such type results for ω-elongations of totally projective groups by $p^{\omega+n}$-projective groups or summable groups by $p^{\omega+n}$-projective groups, respectively.

The purpose of the present brief work is to discuss some questions as those alluded to above concerning when a given separable group is $p^{\omega+n}$-projective provided that it has a modulo countable proper $p^{\omega+n}$-projective subgroup, but by removing the pureness of the subgroup in the whole group.

We are now in a position to proceed by proving the next extension of point (f) (see [5] as well).

Theorem. *Suppose that G is a group of length at most ω which contains a subgroup H such that G/H is countable. Then G is $p^{\omega+n}$-projective if and only if H is $p^{\omega+n}$--projective, whenever $n \in \mathbb{N} \cup \{0\}$.*

Proof. The necessity is immediate because $p^{\omega+n}$-projectives are closed with respect to subgroups (see, for example, [10]). As for the sufficiency, according to the classical Nunke's criterion for $p^{\omega+n}$-projectivity (see [10]), there exists $P \leq H[p^n]$ with H/P a direct sum of cyclic groups. But observing that $(G/P)_r/((G/P)_r \cap H/P) \cong ((G/P)_r + H/P)/H/P \subseteq G/P/H/P \cong G/H$ is countable with $(G/P)_r \cap H/P \subseteq H/P$ a direct sum of cyclic groups, we appeal to Wallace's theorem, quoted above, to infer that $(G/P)_r$ is totally projective. Hence G/P is simply presented. Referring now to [11] (see [7], v. II, too), we deduce that $G/P/(G/P)^1 = G/P/P_G^-/P \cong G/P_G^-$ is a direct sum of cycles, where $P_G^- = \cap_{i<\omega}(P + p^iG)$ is the closure of P in G. It is a straightforward argument that $p^nP_G^- \subseteq p^\omega G$. Since G is separable, that is $p^\omega G = 0$, we derive that $p^nP_G^- = 0$, so employing once again the Nunke's criterion we are finished. □

The condition on separability may be avoided if the following strategy is realizable: Since P is bounded, one can write $P = \cup_{m<\omega} P_m$, where $P_m \subseteq P_{m+1} \leq P$ with $p^kP_m = 0$ for each $m < \omega$ and some $k \in \mathbb{N}$. It is readily seen that

$P_G^- = \cup_{m<\omega} K_m$, where $K_m = \cap_{i<\omega}(P_m + p^i G)$. The crucial moment is whether we may choose a nice subgroup N of G such that $N \subseteq P_m$ and such that $P_m \cap p^m G \subseteq N$ for each integer $m \geq 1$; thus P/N is strongly bounded in G/N in terms of [2]. Consequently, complying with the modular law from [7], we calculate that $K_m \cap p^m G = \cap_{m\leq i<\omega}(P_m + p^i G) \cap p^m G = \cap_{m\leq i<\omega}(P_m \cap p^m G + p^i G) \subseteq \cap_{m\leq i<\omega}(N + p^i G) = N + \cap_{m\leq i<\omega} p^i G = N + p^\omega G \leq P_G^-[p^n]$. Furthermore, we elementarily observe that $P_G^-/(N + p^\omega G) = \cup_{m<\omega}[K_m/(N + p^\omega G)]$ where, for each $m < \omega$, we compute with the aid of the modular law in [7] and the foregoing calculations that $(K_m/(N+p^\omega G)) \cap p^m(G/(N+p^\omega G)) = [K_m \cap (p^m G+N)]/(N+p^\omega G) = (N + K_m \cap p^m G)/(N + p^\omega G) \subseteq (N + p^\omega G)/(N + p^\omega G) = \{0\}$. Besides, by what we have already shown above, $G/(N + p^\omega G)/P_G^-/(N + p^\omega G) \cong G/P_G^-$ is a direct sum of cyclic groups. Knowing this, we apply the Dieudonné's criterion from [6] (see also [2]) to deduce that $G/(N + p^\omega G)$ is, in fact, a direct sum of cycles. Hence and from Nunke's criterion in [10], we conclude that G is $p^{\omega+n}$-projective, as asserted. This completes our conclusions in all generality.

Remark. Actually, $G/P = (G/P)_r$ since $p^{\omega+n}(G/P) = 0$ by seeing that $(G/P)^1 = \cap_{i<\omega}(p^i G + P)/P \subseteq G[p^n]/P$ with $P \leq G[p^n]$ and $G^1 = 0$. However, our approach in the proof gives a more general strategy even for inseparable groups. Nevertheless, this general case is still in question.

A group A is said to be C-decomposable if $A = B \oplus K$, where B is a direct sum of cycles with $\operatorname{fin} r(B) = \operatorname{fin} r(A)$.

We also pose the following conjecture.

Conjecture. Suppose G is a group whose subgroup H is C-decomposable and G/H is countable. Then G is C-decomposable.

In closing, we notice that Hill jointly with Megibben have found in ([9], Proposition 1) that if G is a reduced group which possesses a torsion-complete subgroup H such that $G/H \cong \mathbf{Z}(p^\infty)$, then G is torsion-complete.

So, we are ready to state the following.

Problem. Suppose G is a group with a subgroup H which belongs to the class $\mathcal{K}$ of Abelian p-groups. If $(G/H)[p]$ is finite, then whether or not G also belongs to $\mathcal{K}$?

Investigate with a priority when $\mathcal{K}$ coincides with the class of thick groups, torsion-complete groups, semi-complete groups, quasi-complete groups or pure-complete groups, respectively.

It is worthwhile noticing that according to the main result, stated above, the results from [4] can be improved by dropping off some unnecessary additional limitations.

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