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UNIQUENESS OF MEROMORPHIC FUNCTIONS WHEN TWO NON-LINEAR DIFFERENTIAL POLYNOMIALS SHARE A SMALL FUNCTION

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Abstract. In the paper we deal with the uniqueness of meromorphic functions when two non-linear differential polynomials generated by two meromorphic functions share a small function.

1. Introduction, definitions and results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \{\infty\} \cup \mathbb{C}$ we say that f and g share the value a CM (counting multiplicities) if f, g have the same a-points with the same multiplicity and we say that f, g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities. We denote by $T(r,f)$ the Nevanlinna characteristic function of the meromorphic function f and by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure.

A meromorphic function α is said to be a small function of f if $T(r,\alpha) = S(r,f)$. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. Also we denote by $S(r)$ any quantity satisfying $S(r) = o\{T(r)\}$ as $r \to \infty$, possibly outside a set of finite linear measure.

In the recent past a number of authors worked on the uniqueness problem of meromorphic functions when differential polynomials generated by them share certain values (cf. [1], [2], [3], [4], [6], [9], [10], [11]).

In [6] following question was asked:
What can be said if two non-linear differential polynomials generated by two meromorphic functions share 1 CM*?*

A considerable amount of research has already been done in this direction ([1], [3], [4], [10], [11]). In 2002 Fang-Fang [3] and in 2004 Lin-Yi [11] independently proved the following result.

Theorem A. *Let f and g be two non-constant meromorphic functions and $n\,(\geq 13)$ be an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share* 1 CM*, then $f \equiv g$.*

Also in [3] Fang-Fang proved the following theorem.

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Theorem B. *Let f and g be two non-constant meromorphic functions and $n\,(\geq 28)$ be an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share* 1 IM, *then $f \equiv g$.*

In 2001 an idea of gradation of sharing of values was introduced to measure how close a shared value is to being shared CM or to being shared IM. This notion is known as weighted sharing of values and is defined as follows.

Definition 1.1 ([8, 7])**.** Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m\,(\leq k)$ if and only if it is an a-point of g with multiplicity $m\,(\leq k)$ and z_0 is an a-point of f with multiplicity $m\,(> k)$ if and only if it is an a-point of g with multiplicity $n\,(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ and (a, ∞) respectively.

If $\alpha = \alpha(z)$ is a small function of f and g then f, g share (α, k) means that $f - \alpha$ and $g - \alpha$ share $(0, k)$.

In 2004 Lahiri-Sarkar [10] proved the following theorems.

Theorem C ([10])**.** *Let f and g be two non-constant meromorphic functions such that $2\Theta(\infty; f) + 2\Theta(\infty; g) + \min\{\Theta(\infty; f), \Theta(\infty; g)\} > 4$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share $(1, 2)$ then $f \equiv g$, where $n\,(\geq 7)$ is an integer.*

Theorem D ([10])**.** *Let f and g be two non-constant meromorphic functions such that $2\Theta(\infty; f) + 2\Theta(\infty; g) + \min\{\Theta(\infty; f), \Theta(\infty; g)\} > 4$. If $f^n(f^2-1)f'$ and $g^n(g^2-1)g'$ share $(1, 2)$, then either $f \equiv g$ or $f \equiv -g$, where $n\,(\geq 8)$ is an integer. If n is an even integer then the possibility $f \equiv -g$ does not arise.*

In the paper we investigate uniqueness of meromorphic functions when two non-linear differential polynomials share a small function. We now state the main result of the paper.

Theorem 1.1. *Let f and g be two non-constant meromorphic functions and $\alpha\,(\not\equiv 0, \infty)$ be a small function of f and g. Let n and $k\,(\geq 2)$ be two positive integers such that $f^n(f^k - a)f'$ and $g^n(g^k - a)g'$ share (α, m), where $a\,(\neq 0)$ is a finite complex number. Then $f \equiv g$ or $f \equiv -g$ provided one of the following holds:*

(i) $m \geq 2$ *and* $n > \max\{4, k + 10 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\}$;

(ii) $m = 1$ *and* $n > \max\{4, \frac{3k}{2} + 12 - 3\Theta(\infty; f) - 3\Theta(\infty; g)\}$;

(iii) $m = 0$ *and* $n > \max\{4, 4k + 22 - 5\Theta(\infty; f) - 5\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\}$.

Also the possibility $f \equiv -g$ does not arise if n and k are both even or both odd or if n is even and k is odd.

For standard definitions and notations of the value distribution theory we refer the reader to [5].

2. Lemmas

In this section we present some lemmas which will be needed to prove the theorem.

Lemma 2.1 ([12, 13])**.** *Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ $(\neq 0)$ are constants. Then*

$$T\big(r, P(f)\big) = nT(r, f) + S(r, f)\,.$$

Lemma 2.2 ([14])**.** *Let f be a non-constant meromorphic function. Then*

$$N(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty; f) + N(r, 0; f) + S(r, f)\,.$$

Lemma 2.3 ([8])**.** *Let f and g be two non-constant meromorphic functions sharing $(1, 2)$. Then one of the following cases holds:*

(i) $T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r)$,

(ii) $f \equiv g$,

(iii) $fg \equiv 1$.

Lemma 2.4 ([1])**.** *Let f and g be two non-constant meromorphic functions sharing $(1, m)$ and*

$$\frac{f''}{f'} - \frac{2f'}{f-1} \not\equiv \frac{g''}{g'} - \frac{2g'}{g-1}.$$

Now the following hold:

(i) *if $m = 1$ then $T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + \frac{1}{2}\overline{N}(r, 0; f) + \frac{1}{2}\overline{N}(r, \infty; f) + S(r, f) + S(r, g)$;*

(ii) *if $m = 0$ then $T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + 2\overline{N}(r, 0; f) + \overline{N}(r, 0; g) + 2\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f) + S(r, g)$.*

Lemma 2.5 ([15])**.** *If*

$$\frac{f''}{f'} - \frac{2f'}{f-1} \equiv \frac{g''}{g'} - \frac{2g'}{g-1}$$

and

$$\limsup_{r\to\infty, r\notin E} \frac{\overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)}{T(r)} < 1$$

then $f \equiv g$ or $fg \equiv 1$, where E is a set of finite linear measure and not necessarily the same at each of its occurrence.

Lemma 2.6. *Let f and g be two non-constant meromorphic functions and α $(\not\equiv 0, \infty)$ be a small function of f and g. Let $n\,(\geq 4)$ and $k\,(\geq 2)$ be positive integers. Then for any non-zero constant a,*

$$f^n(f^k - a)f'g^n(g^k - a)g' \not\equiv \alpha^2\,.$$

Proof. We suppose that

$$f^n(f^k-a)f'g^n(g^k-a)g'\equiv\alpha^2\,. \tag{2.1}$$

Let z_0 ($\alpha(z_0)\neq 0,\infty$) be a zero of f with multiplicity p. Then z_0 is a pole of g with multiplicity q, say. From (2.1) we get

$$np+p-1=nq+kq+q+1$$

and so

$$kq+2=(n+1)(p-q)\,. \tag{2.2}$$

From (2.2) we get $q\geq\frac{n-1}{k}$ and again from (2.2) we obtain

$$p\geq\frac{1}{n+1}\Big[\frac{(n+k+1)(n-1)}{k}+2\Big]=\frac{n+k-1}{k}\,.$$

Let z_1 $\big(\alpha(z_1)\neq 0,\infty\big)$ be a zero of f^k-a with multiplicity p. Then z_1 is a pole of g with multiplicity q, say. So from (2.1) we get

$$\begin{aligned}2p-1&=(n+k+1)q+1\\&\geq n+k+2\end{aligned}$$

i.e.,

$$p\geq\frac{n+k+3}{2}\,.$$

Since a pole of f (which is not a pole of α) is either a zero of $g^n(g^k-a)$ or a zero of g', we have

$$\begin{aligned}\overline{N}(r,\infty;f)&\leq\overline{N}(r,0;g)+\overline{N}(r,a;g^k)+\overline{N}_0(r,0;g')+S(r,f)+S(r,g)\\&\leq\frac{k}{n+k-1}N(r,0;g)+\frac{2}{n+k+3}N(r,a;g^k)+\overline{N}_0(r,0;g')\\&\quad+S(r,f)+S(r,g)\\&\leq\Big(\frac{k}{n+k-1}+\frac{2k}{n+k+3}\Big)T(r,g)+\overline{N}_0(r,0;g')+S(r,f)+S(r,g)\,,\end{aligned}$$

where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $g(g^k-a)$.

Let $f^k-a=(f-a_1)(f-a_2)\ldots(f-a_k)$. Then by the second fundamental theorem we get

$$\begin{aligned}kT(r,f)&\leq\overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\sum_{j=1}^{k}\overline{N}(r,a_j;f)-\overline{N}_0(r,0;f')+S(r,f)\\&=\overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\overline{N}(r,a;f^k)-\overline{N}_0(r,0;f')+S(r,f)\\&\leq\Big(\frac{k}{n+k-1}+\frac{2k}{n+k+3}\Big)T(r,g)+\overline{N}_0(r,0;g')+\frac{k}{n+k-1}N(r,0;f)\end{aligned}$$

$$+\frac{2}{n+k+3}N(r,a;f^k)-\overline{N}_0(r,0;f')+S(r,f)+S(r,g)$$
$$\le\Big(\frac{k}{n+k-1}+\frac{2k}{n+k+3}\Big)\{T(r,f)+T(r,g)\}+\overline{N}_0(r,0;g')$$
$$-\overline{N}_0(r,0;f')+S(r,f)+S(r,g)\,. \tag{2.3}$$

Similarly we get

$$kT(r,g)\le\Big(\frac{k}{n+k-1}+\frac{2k}{n+k+3}\Big)\{T(r,f)+T(r,g)\}+\overline{N}_0(r,0;f')$$
$$-\overline{N}_0(r,0;g')+S(r,f)+S(r,g)\,. \tag{2.4}$$

Adding (2.3) and (2.4) we obtain

$$\Big(1-\frac{2}{n+k-1}-\frac{4}{n+k+3}\Big)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g)\,,$$

which is a contradiction. This proves the lemma. □

Lemma 2.7. *Let f and g be two non-constant meromorphic functions and $F=f^{n+1}\Big(\frac{f^k}{n+k+1}-\frac{a}{n+1}\Big)$ and $G=g^{n+1}\Big(\frac{g^k}{n+k+1}-\frac{a}{n+1}\Big)$, where a is a non-zero constant. Further let $F_0=\frac{F'}{\alpha}$ and $G_0=\frac{G'}{\alpha}$, where α ($\not\equiv 0,\infty$) is a small function of f and g. Then $S(r,F_0)$ and $S(r,G_0)$ are replaceable by $S(r,f)$ and $S(r,g)$ respectively.*

Proof. By Lemma 2.1 we get

$$\begin{aligned}T(r,F_0)&\le T(r,F')+S(r,f)\\&\le 2T(r,F)+S(r,f)\\&=2(n+k+1)T(r,f)+S(r,f)\end{aligned}$$

and similarly

$$T(r,G_0)\le 2(n+k+1)T(r,g)+S(r,g)\,.$$

This proves the lemma. □

Lemma 2.8. *Let F, G, F_0 and G_0 be defined as in Lemma 2.7. Then*

(i) $T(r,F)\le T(r,F_0)+N(r,0;f)+N\big(r,\frac{n+k+1}{n+1}a;f^k\big)-N(r,a;f^k)$ $-N(r,0;f')+S(r,f)$,

(ii) $T(r,G)\le T(r,G_0)+N(r,0;g)+N\big(r,\frac{n+k+1}{n+1}a;g^k\big)-N(r,a;g^k)$ $-N(r,0;g')+S(r,g)$.

Proof. We prove (i) only as the proof of (ii) is similar. By Nevanlinna's first fundamental theorem and lemma 2.1 we get

$$\begin{aligned} T(r,F) &= T\Big(r,\frac{1}{F}\Big)+O(1) \\ &= N(r,0;F)+m\Big(r,\frac{1}{F}\Big)+O(1) \\ &\le N(r,0;F)+m\Big(r,\frac{F_0}{F}\Big)+m(r,0;F_0)+O(1) \\ &= N(r,0;F)+T(r,F_0)-N(r,0;F_0)+S(r,F) \\ &= T(r,F_0)+N(r,0;f)+N\Big(r,\frac{n+k+1}{n+1}a;f^k\Big) \\ &\quad -N(r,a;f^k)-N(r,0;f')+S(r,f)\,. \end{aligned}$$

This proves the lemma. □

Following lemma can be proved in the line of Lemma 2.10 [10].

Lemma 2.9. *Let F and G be defined as in Lemma 2.7, where k and $n\,(\ge 3+k)$ are positive integers. Then $F'\equiv G'$ implies $F\equiv G$.*

Lemma 2.10. *Let F and G be defined as in Lemma 2.7 and $F\equiv G$. If $k\ge 2$ and $n+k\ge 5$ then either $f\equiv g$ or $f\equiv -g$. Also if n and k are both even or both odd or if n is even and k is odd then the possibility $f\equiv -g$ does not arise.*

Proof. Clearly if n and k are both even or both odd or if n is even and k is odd, then $f\equiv -g$ contradicts $F\equiv G$.

Let neither $f\equiv g$ nor $f\equiv -g$. We put $h=\frac{g}{f}$. Then $h\not\equiv 1$ and $h\not\equiv -1$. Also $F\equiv G$ implies

$$f^k = a\frac{n+k+1}{n+1}\,\frac{h^{n+1}-1}{h^{n+k+1}-1}\,.$$

Since f is non-constant, we see that h is not a constant. Again since f^k has no simple pole, $h-\alpha_m$ has no simple zero, where $\alpha_m=\exp\big(\frac{2m\pi i}{n+k+1}\big)$ and $m=1,2,\dots,n+k$. Hence $\Theta(\alpha_m;h)\ge\frac{1}{2}$ for $m=1,2,\dots,n+k$, which is impossible. Therefore either $f\equiv g$ or $f\equiv -g$. This proves the lemma. □

3. Proof of the Theorem

Proof of Theorem 1.1. Let F, G, F_0 and G_0 be defined as in Lemma 2.7. We consider the following three cases of the theorem separately.

Case (i). Since F_0 and G_0 share $(1,2)$, one of the possibilities of Lemma 2.3 holds. We suppose that

$$\begin{aligned} T_0(r) \le\ & N_2(r,0;F_0)+N_2(r,0;G_0)+N_2(r,\infty;F_0)+N_2(r,\infty;G_0) \\ & +S(r,F_0)+S(r,G_0)\,, \end{aligned} \tag{3.1}$$

where $T_0(r)=\max\big\{T(r,F_0),T(r,G_0)\big\}$. We now choose a number ϵ such that

$$0<2\epsilon<n-k-10+2\Theta(\infty;f)+2\Theta(\infty;g)+\min\big\{\Theta(\infty;f),\Theta(\infty;g)\big\}\,.$$

Now by Lemma 2.2, Lemma 2.7 and Lemma 2.8 we get from (3.1)

$$
\begin{aligned}
T(r,F) &\le T(r,F_0) + N(r,0;f) + N\Big(r,\frac{n+k+1}{n+1}a;f^k\Big) - N(r,a;f^k)\\
&\quad - N(r,0;f') + S(r,f)\\
&\le N_2(r,0;F_0) + N_2(r,0;G_0) + N_2(r,\infty;F_0) + N_2(r,\infty;G_0) + N(r,0;f)\\
&\quad + N\Big(r,\frac{n+k+1}{n+1}a;f^k\Big) - N(r,a;f^k) - N(r,0;f') + S(r,f) + S(r,g)\\
&\le 2\overline{N}(r,0;f) + N(r,a;f^k) + N(r,0;f') + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,0;g)\\
&\quad + N(r,a;g^k) + N(r,0;g') + 2\overline{N}(r,\infty;g) + N(r,0;f)\\
&\quad + N\Big(r,\frac{n+k+1}{n+1}a;f^k\Big)\\
&\quad - N(r,a;f^k) - N(r,0;f') + S(r,f) + S(r,g)\\
&= 2\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + N(r,0;f) + N\Big(r,\frac{n+k+1}{n+1}a;f^k\Big)\\
&\quad + 2\overline{N}(r,0;g) + N(r,a;g^k) + N(r,0;g') + 2\overline{N}(r,\infty;g) + S(r,f) + S(r,g)\\
&\le \big\{5+k-2\Theta(\infty,f)+\epsilon\big\}T(r,f) + \big\{6+k-3\Theta(\infty,g)+\epsilon\big\}T(r,g)\\
&\quad + S(r,f) + S(r,g)\,.
\end{aligned}
$$

So by Lemma 2.1 we obtain

$$
\begin{aligned}
(n+k+1)T(r,f) \le \big\{&11+2k-2\Theta(\infty,f)-3\Theta(\infty,g)+2\epsilon\big\}\\
&\times T(r)+S(r)\,.
\end{aligned}
\tag{3.2}
$$

Similarly we get

$$
\begin{aligned}
(n+k+1)T(r,g) \le \big\{&11+2k-3\Theta(\infty,f)-2\Theta(\infty,g)+2\epsilon\big\}\\
&\times T(r)+S(r)\,.
\end{aligned}
\tag{3.3}
$$

From 3.2 and 3.3 we see that

$$
\big[n-k-10+2\Theta(\infty;f)+2\Theta(\infty;g)+\min\big\{\Theta(\infty;f),\Theta(\infty;g)\big\}-2\epsilon\big]\,T(r)\le S(r)\,,
$$

which is a contradiction. Hence 3.1 does not hold. So by Lemma 2.3 either $F_0G_0 \equiv 1$ or $F_0 \equiv G_0$. Since by Lemma 2.6 $F_0G_0 \not\equiv 1$, we get $F_0 \equiv G_0$. Now the result follows from Lemma 2.9 and Lemma 2.10.

Case (ii). We put

$$
H = \Big(\frac{F_0''}{F_0'} - \frac{2F_0'}{F_0-1}\Big) - \Big(\frac{G_0''}{G_0'} - \frac{2G_0'}{G_0-1}\Big)\,.
$$

Also we choose a number ϵ such that

$$
0 < 2\epsilon < n - \frac{3k}{2} - 12 + 3\Theta(\infty;f) + 3\Theta(\infty;g)\,.
$$

We suppose that $H \not\equiv 0$. Since F_0 and G_0 share $(1,1)$, by Lemma 2.2, Lemma 2.4(i), Lemma 2.7 and Lemma 2.8 we get

$$\begin{aligned}
T(r,F) &\le T(r,F_0) + N(r,0;f) + N(r,\frac{n+k+1}{n+1}a;f^k) - N(r,a;f^k) \\
&\quad - N(r,0;f') + S(r,f) \\
&\le N_2(r,0;F_0) + N_2(r,0;G_0) + N_2(r,\infty;F_0) + N_2(r,\infty;G_0) \\
&\quad + \frac{1}{2}\overline{N}(r,0;F_0) + \frac{1}{2}\overline{N}(r,\infty;F_0) + N(r,0;f) + N\Big(r,\frac{n+k+1}{n+1}a;f^k\Big) \\
&\quad - N(r,a;f^k) - N(r,0;f') + S(r,f) + S(r,g) \\
&\le 2\overline{N}(r,0;f) + N(r,a;f^k) + N(r,0;f') + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,0;g) + \\
&\quad N(r,a;g^k) + N(r,0;g') + 2\overline{N}(r,\infty;g) + \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,a;f^k) \\
&\quad + \frac{1}{2}\overline{N}(r,0;f') + \frac{1}{2}\overline{N}(r,\infty;f) + N(r,0;f) + N\Big(r,\frac{n+k+1}{n+1}a;f^k\Big) \\
&\quad - N(r,a;f^k) - N(r,0;f') + S(r,f) + S(r,g) \\
&\le \Big\{\frac{3k}{2} + 7 - 3\Theta(\infty,f) + \epsilon\Big\}T(r,f) + \{6 + k - 3\Theta(\infty,g) + \epsilon\}T(r,g) \\
&\quad + S(r,f) + S(r,g) \\
&\le \Big\{13 + \frac{5k}{2} - 3\Theta(\infty,f) - 3\Theta(\infty,g) + 2\epsilon\Big\}T(r) + S(r)\,.
\end{aligned}$$

So by Lemma 2.1 we get

$$(n+k+1)T(r,f) \le \Big\{13 + \frac{5k}{2} - 3\Theta(\infty,f) - 3\Theta(\infty,g) + 2\epsilon\Big\}T(r) + S(r)\,.$$

Similarly we get

$$(n+k+1)T(r,g) \le \Big\{13 + \frac{5k}{2} - 3\Theta(\infty,f) - 3\Theta(\infty,g) + 2\epsilon\Big\}T(r) + S(r)\,.$$

Combining the above two inequalities we obtain

$$\Big\{n - \frac{3k}{2} - 12 + 3\Theta(\infty;f) + 3\Theta(\infty;g) - 2\epsilon\Big\}T(r) \le S(r)\,,$$

which is a contradiction. Hence $H \equiv 0$. Now by Lemma 2.1 we get

$$\begin{aligned}
(n+k)T(r,f) &= T(r,f^n(f^k-a)) + S(r,f) \\
&\le T(r,F') + T(r,f') + S(r,f) \\
&\le T(r,F_0) + 2T(r,f) + S(r,f)
\end{aligned}$$

and so

$$T(r,F_0) \ge (n+k-2)T(r,f) + S(r,f)\,.$$

Similarly we get

$$T(r,G_0) \ge (n+k-2)T(r,g) + S(r,g)\,.$$

Also we see by Lemma 2.2 that

$$\begin{aligned}
&\overline{N}(r,0;\,F_0)+\overline{N}(r,\infty;F_0)+\overline{N}(r,0;G_0)+\overline{N}(r,\infty;G_0)\\
&\quad\le \overline{N}(r,0;f)+\overline{N}(r,a;f^k)+\overline{N}(r,0;f')+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)\\
&\qquad+\overline{N}(r,a;g^k)+\overline{N}(r,0;g')+\overline{N}(r,\infty;g)+S(r,f)+S(r,g)\\
&\quad\le (k+2)T(r,f)+2\overline{N}(r,\infty;f)+(k+2)T(r,g)+2\overline{N}(r,\infty;g)\\
&\qquad+S(r,f)+S(r,g)\\
&\quad\le \big\{k+4-2\Theta(\infty;f)+\epsilon\big\}T(r,f)+\big\{k+4-2\Theta(\infty;g)+\epsilon\big\}T(r,g)\\
&\qquad+S(r,f)+S(r,g)\\
&\quad\le \frac{2k+8-2\Theta(\infty;f)-2\Theta(\infty;g)+2\epsilon}{n+k-2}T_0(r)+S(r)\,,
\end{aligned}$$

where $S_0(r)=o\big\{T_0(r)\big\}$ as $r\to\infty$ possibly outside a set of finite linear measure and $\epsilon\,(>0)$ is sufficiently small.

In view of the hypothesis we get from above

$$\limsup_{r\to\infty,r\notin E}\frac{\overline{N}(r,0;F_0)+\overline{N}(r,\infty;F_0)+\overline{N}(r,0;G_0)+\overline{N}(r,\infty;G_0)}{T_0(r)}<1\,.$$

So by Lemma 2.5 we obtain either $F_0G_0\equiv 1$ or $F_0\equiv G_0$. Hence the result follows from Lemma 2.6, Lemma 2.9 and Lemma 2.10.

Case (iii). Using Lemma 2.4(ii) this case can be proved as case II. This proves the theorem. □

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