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SEVERAL q-SERIES IDENTITIES FROM THE EULER EXPANSIONS OF $(a;q)_\infty$ AND $\frac{1}{(a;q)_\infty}$

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Abstract. In this paper, we first give several operator identities which extend the results of Chen and Liu, then make use of them to two q-series identities obtained by the Euler expansions of $(a;q)_\infty$ and $\frac{1}{(a;q)_\infty}$. Several q-series identities are obtained involving a q-series identity in Ramanujan's Lost Notebook.

1. Introduction

Throughout this paper, assume that $0<q<1$ and adopt the customary notation in [6] for q-series. Let

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)\,. \tag{1}$$

For any integer n, the q-shifted factorial $(a;q)_n$ is given by

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}\,. \tag{2}$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}\,. \tag{3}$$

We will also use frequently the following equations:

$$(q/a;q)_n = (-a)^{-n} q^{\binom{n+1}{2}} \frac{(q^{-n}a;q)_\infty}{(a;q)_\infty} \tag{4}$$

or

$$(q^{-n}a;q)_\infty = (-a)^n q^{-\binom{n+1}{2}} (q/a;q)_n (a;q)_\infty\,. \tag{5}$$

For convenience, we employ the following notation for multiple q-shifted factorial:

$$(a_1,a_2,\dots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\dots(a_m;q)_n\,,$$

where n is an integer or ∞.

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The q-differential operator D_q and the q-shifted operator are defined by

$$D_q\{f(a)\} = \frac{1}{a}(f(a) - f(aq))$$

and

$$\eta\{f(a)\} = f(aq)\,,$$

respectively. These two operators have been introduced in [9, 10, 11] due to Rogers, who applied them for proving Roger-Ramanujan identities. Built on the D_q and η, the operator appeared in the work of Roman [12] and will be denoted by

$$\theta = \eta^{-1} D_q\,.$$

Two operators introduced in the papers [4] and [5] by Chen and Liu are the exponential operators constructed from D_q and θ:

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q;q)_n}$$

and

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{\binom{n}{2}}}{(q;q)_n}\,.$$

Then the following operator identities were obtained.

Theorem 1.1 (Chen and Liu, [4] and [5]). *Let $T(bD_q)$ and $E(b\theta)$ be respectively defined as above. Then*

$$T(bD_q)\Big\{\frac{1}{(at;q)_\infty}\Big\} = \frac{1}{(at,bt;q)_\infty}\,, \tag{6}$$

$$T(bD_q)\Big\{\frac{1}{(as,at;q)_\infty}\Big\} = \frac{(abst;q)_\infty}{(as,at,bs,bt;q)_\infty}\,, \tag{7}$$

$$E(b\theta)\big\{(at;q)_\infty\big\} = (at,bt;q)_\infty\,, \tag{8}$$

$$E(b\theta)\big\{(as,at;q)_\infty\big\} = \frac{(as,at,bs,bt;q)_\infty}{(abst/q;q)_\infty}\,. \tag{9}$$

By these operator identities, in [4, 5, 7, 8, 13], a lot of q-series identities can be derived. In [14], Theorem 1.1 was extended. Applying generalized operator identities to some terminating basic hypergeometric series and q-integrals, we obtain some summation formulas involving ${}_3\phi_2$ summation and some q-integrals involving ${}_3\phi_2$ summation which extend some famous q-integrals, such as the Ismail-Stanton-Viennot integral, Gasper integral formula.

In this paper, we first give several operator identities which extend the results of Chen and Liu. Then making use of them to two q-series identities obtained by the Euler expansions of $(a;q)_\infty$ and $\frac{1}{(a;q)_\infty}$, several q-series identities are obtained involving a q-series identity in Ramanujan's Lost Notebook.

Throughout this paper, n and k are two nonnegative integers.

2. Several operator identities

First of all, by induction we give a result which is to be used later.

Lemma 2.1. *We have*

$$\theta^n\{a^k\} = \begin{cases} 0\,, & \text{if} \quad n > k\,, \\ (-1)^n q^n (q^{-k};q)_n a^{k-n}\,, & \text{if} \quad n \le k\,, \end{cases} \tag{10}$$

$$D_q^n\{a^k\} = \begin{cases} 0\,, & \text{if} \quad n > k\,, \\ (q^{k-n+1};q)_n a^{k-n}\,, & \text{if} \quad n \le k\,, \end{cases} \tag{11}$$

$$\theta^n\{a^{-k}\} = (-q)^n (q^k;q)_n a^{-(k+n)}\,, \tag{12}$$

$$D_q^n\{a^{-k}\} = q^{-\binom{n}{2}-kn}(q^k;q)_n a^{-(k+n)}\,. \tag{13}$$

In [13], we obtained the following operator identity.

$$\begin{aligned} &E(d\theta)\big\{a^k(at,as;q)_\infty\big\} \\ &\quad = a^k \frac{(as,at,ds,dt;q)_\infty}{(adst/q)_\infty}\, {}_3\phi_2\left(\begin{matrix} q^{-k}, & q/as, & q/at \\ & 0, & q^2/adst \end{matrix}; q, q\right). \end{aligned} \tag{14}$$

This identity can be rewritten by

$$\begin{aligned} E(d\theta)\big\{a^k(at,as;q)_\infty\big\} &= a^k \frac{(at,as,dt,ds;q)_\infty}{(adst/q;q)_\infty} \\ &\times \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} \frac{(q/at,q/as;q)_j}{(q^2/adst;q)_j} q^{\binom{j+1}{2}-kj}\,. \end{aligned} \tag{15}$$

Next we give three operator identities which are extensions of Theorem 1.1. They specialize to Theorem 1.1 by setting $k = 0$.

Theorem 2.2. *We have*

$$T(dD_q)\Big\{\frac{a^k}{(at,as;q)_\infty}\Big\} = a^k \frac{(adst;q)_\infty}{(at,as,dt,ds;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(at,as;q)_j}{(adst;q)_j}\Big(\frac{d}{a}\Big)^j\,. \tag{16}$$

$$\begin{aligned} &E(d\theta)\big\{a^{-k}(at,as;q)_\infty\big\} \\ &\quad = a^{-k} \frac{(at,as,dtq^k,dsq^k;q)_\infty}{(adstq^{k-1};q)_\infty} \sum_{n=0}^{\infty} \frac{(q^k,adstq^{k-1};q)_n}{(q,dtq^k,dsq^k;q)_n} q^{\binom{n+1}{2}} \Big(-\frac{d}{a}\Big)^n \\ &\quad = a^{-k} \frac{(at,as,dt,ds;q)_\infty}{(adst/q;q)_\infty} \sum_{n=0}^{\infty} \frac{(q^k;q)_n}{(q;q)_n} \frac{(adst/q;q)_{n+k}}{(dt,ds;q)_{n+k}} q^{\binom{n+1}{2}} \Big(-\frac{d}{a}\Big)^n \end{aligned} \tag{17}$$

where $|d/a| < 1$*, and*

$$T(dD_q)\Big\{\frac{a^{-k}}{(at,as;q)_\infty}\Big\}$$
$$= a^{-k}\frac{(adtsq^{-k};q)_\infty}{(at,as,dtq^{-k},dsq^{-k};q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{(q^k,q^{1+k}/adst;q)_n}{(q,q^{1+k}/dt,q^{1+k}/ds;q)_n}q^n$$
$$= \Big(-\frac{1}{d}\Big)^k q^{\binom{k+1}{2}}\frac{(adts;q)_\infty}{(at,as,dt,ds;q)_\infty}\sum_{n=0}^{\infty}\frac{(q^k;q)_n}{(q;q)_n}\frac{(q/adst;q)_{n+k}}{(q/dt,q/ds;q)_{n+k}}(-q)^n\,. \tag{18}$$

Proof. We note that the rule of Leibniz for D_q^n (see [4])

$$D_q^n\{f(a)g(a)\} = \sum_{k=0}^{n} q^{k(k-n)}\begin{bmatrix}n\\k\end{bmatrix}D_q^k\{f(a)\}D_q^{n-k}\{g(aq^k)\}\,. \tag{19}$$

Therefore, we have

$$T(dD_q)\Big\{\frac{a^k}{(at,as;q)_\infty}\Big\} = \sum_{n=0}^{\infty}\frac{d^n}{(q;q)_n}D_q^n\Big\{\frac{a^k}{(at,as;q)_\infty}\Big\}$$
$$= \sum_{n=0}^{\infty}\frac{d^n}{(q;q)_n}\sum_{j=0}^{n}q^{j(j-n)}\begin{bmatrix}n\\j\end{bmatrix}D_q^j\{a^k\}D_q^{n-j}\Big\{\frac{1}{(atq^j,asq^j;q)_\infty}\Big\}$$
$$= \sum_{n=0}^{\infty}\frac{d^n}{(q;q)_n}\sum_{j=0}^{n}q^{j(j-n)}\begin{bmatrix}n\\j\end{bmatrix}\frac{(q;q)_k}{(q;q)_{k-j}}a^{k-j}D_q^{n-j}\Big\{\frac{1}{(atq^j,asq^j;q)_\infty}\Big\}$$
$$= \sum_{j=0}^{\infty}\sum_{n=j}^{\infty}\frac{d^n}{(q;q)_n}q^{j(j-n)}\begin{bmatrix}n\\j\end{bmatrix}\frac{(q;q)_k}{(q;q)_{k-j}}a^{k-j}D_q^{n-j}\Big\{\frac{1}{(atq^j,asq^j;q)_\infty}\Big\}$$
$$= \sum_{j=0}^{\infty}\frac{d^j(q;q)_k}{(q;q)_j(q;q)_{k-j}}a^{k-j}\sum_{n=0}^{\infty}\frac{d^n}{(q;q)_n}q^{-nj}D_q^n\Big\{\frac{1}{(atq^j,asq^j;q)_\infty}\Big\}$$
$$= a^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\Big(\frac{d}{a}\Big)^j\sum_{n=0}^{\infty}\frac{(dq^{-j}D_q)^n}{(q;q)_n}\Big\{\frac{1}{(atq^j,asq^j;q)_\infty}\Big\}$$
$$= a^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\Big(\frac{d}{a}\Big)^j\cdot T(dq^{-j}D_q\Big\{\frac{1}{(atq^j,asq^j;q)_\infty}\Big\}$$
$$= a^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\Big(\frac{d}{a}\Big)^j\frac{(adstq^j;q)_\infty}{(atq^j,asq^j,dt,ds;q)_\infty}$$
$$= a^k\frac{(adst;q)_\infty}{(at,as,dt,ds;q)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(at,as;q)_j}{(adst;q)_j}\Big(\frac{d}{a}\Big)^j\,.$$

We obtain (16). (17) and (18) can be proved similarly. The proof is completed. □

Taking $s = 0$ in Theorem 2.2, we have

Corollary 2.3.

$$T(dD_q)\Big\{\frac{a^k}{(at;q)_\infty}\Big\} = a^k\frac{1}{(at,dt;q)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}(at;q)_j\Big(\frac{d}{a}\Big)^j. \tag{20}$$

$$\begin{aligned}E(d\theta)\{a^{-k}(at;q)_\infty\} &= a^{-k}(at,dtq^k;q)_\infty\\ &\quad\times\sum_{n=0}^{\infty}\frac{(q^k;q)_n}{(q;q)_n}\frac{1}{(dt;q)_{n+k}}q^{\binom{n+1}{2}}\Big(-\frac{d}{a}\Big)^n\end{aligned} \tag{21}$$

where $|d/a| < 1$, *and*

$$\begin{aligned}T(dD_q)\Big\{\frac{a^{-k}}{(at;q)_\infty}\Big\} &= \Big(-\frac{1}{adt}\Big)^k q^{\binom{k+1}{2}}\frac{1}{(at,dt;q)_\infty}\\ &\quad\times\sum_{n=0}^{\infty}\frac{(q^k;q)_n}{(q;q)_n}\frac{1}{(q/dt;q)_{n+k}}\Big(-\frac{q}{at}\Big)^n,\end{aligned} \tag{22}$$

where $|q/at| < 1$.

Further, setting $k = 1$ or $t = 0$ in (21), we get the following results of Chen and Liu.

Corollary 2.4 (Chen and Liu, [5])**.**

$$E(b\theta)\{a^{-1}\} = a^{-1}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}}b^n a^{-n}, \tag{23}$$

$$E(b\theta)\{a^{-1}(-a;q)_\infty\} = a^{-1}(-a;q)_\infty\sum_{m=0}^{\infty}(-b/a)^m q^{\binom{m+1}{2}}(-bq^{m+1};q)_\infty. \tag{24}$$

3. Several q-series identities from expansion of $(a;q)_\infty$

From the Euler expansion of $(a;q)_\infty$ (see [3]):

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}a^n}{(q;q)_n} = (a;q)_\infty,$$

the following identity can be verified.

$$\sum_{n=0}^{\infty}(-aq^{n+1};q)_\infty q^n = -a^{-1} + a^{-1}(-a;q)_\infty. \tag{25}$$

Theorem 3.1. *We have*

$$\sum_{n=0}^{\infty}\frac{(-adt;q)_n}{(-aq,-dq;q)_n}q^n\sum_{j=0}^{k+1}\frac{(q^{-(k+1)},-q^{-n}/a,q/at;q)_j}{(q,-q^{1-n}/adt;q)_j}q^j$$
$$=-\,a^{-1}\frac{(-adt;q)_\infty}{(-aq,-dq;q)_\infty}\sum_{j=0}^{k}\frac{(q^{-k},q/at;q)_j}{(q;q)_j}(dt)^j$$
$$+\,a^{-1}\frac{(1+a)(1+d)}{1+adt/q}\sum_{j=0}^{k}\frac{(q^{-k},-q/a,q/at;q)_j}{(q,-q^2/adt;q)_j}q^j\,. \tag{26}$$

Proof. Multiply both sides in equation (25) by $a^{k+1}(at;q)_\infty$. Then

$$\sum_{n=0}^{\infty}a^{k+1}(-aq^{n+1},at;q)_\infty q^n=-a^k(at;q)_\infty+a^k(-a,at;q)_\infty\,. \tag{27}$$

Apply the operator $E(d\theta)$ to its both sides with respect to the variable a. Then

$$\sum_{n=0}^{\infty}q^nE(d\theta)\{a^{k+1}(-aq^{n+1},at;q)_\infty\}=-E(d\theta)\{a^k(at;q)_\infty\}$$
$$+\,E(d\theta)\{a^k(-a,at;q)_\infty\}\,. \tag{28}$$

Applying Theorem 2.2 and Corollary 2.3, we have

$$E(d\theta)\{a^{k+1}(-aq^{n+1},at;q)_\infty\}=a^{k+1}\frac{(-aq^{n+1},at,-dq^{n+1},dt;q)_\infty}{(-adtq^n;q)_\infty}$$
$$\times\sum_{j=0}^{k+1}\frac{(q^{-(k+1)},-q^{-n}/a,q/at;q)_j}{(q,-q^{1-n}/adt;q)_j}q^j\,,$$

$$E(d\theta)\{a^k(at;q)_\infty\}=a^k(at,dt;q)_\infty\sum_{j=0}^{k}\frac{(q^{-k},q/at;q)_j}{(q;q)_j}(dt)^j\,,$$

$$E(d\theta)\{a^k(-a,at;q)_\infty\}=a^k\frac{(-a,at,-d,dt;q)_\infty}{(-adt/q;q)_\infty}\sum_{j=0}^{k}\frac{(q^{-k},-q/a,q/at;q)_j}{(q,-q^2/adt;q)_j}q^j\,.$$

Substituting these three identities into (27) and then using (4), we obtain (3.1) after simplifying. This completes the proof. □

Taking $k=0$ in Theorem 3.1, we have

Corollary 3.2.

$$a\sum_{n=0}^{\infty}\frac{(-adt;q)_n}{(-aq,-dq;q)_n}q^n+d(1-at/q)\sum_{n=0}^{\infty}\frac{(-adt;q)_n}{(-aq;q)_n(-dq;q)_{n+1}}q^{n+1}$$
$$=-\,\frac{(-adt;q)_\infty}{(-aq,-dq;q)_\infty}+1+a\,. \tag{29}$$

Taking $t=0$ in Corollary 3.2 we have

Corollary 3.3.

$$a\sum_{n=0}^{\infty}\frac{1}{(-aq,-dq;q)_n}q^n+d\sum_{n=0}^{\infty}\frac{1}{(-aq;q)_n(-dq;q)_{n+1}}q^{n+1}$$
$$=-\frac{1}{(-aq,-dq;q)_\infty}+1+a\,. \tag{30}$$

Taking $k=1$ in Theorem 3.1, we have

Corollary 3.4.

$$a^2q\sum_{n=0}^{\infty}\frac{(-adt;q)_n}{(-aq,-dq;q)_n}q^n-ad(1+q)(at-q)\sum_{n=0}^{\infty}\frac{(-adt;q)_n}{(-aq;q)_n(-dq;q)_{n+1}}q^n$$
$$+d^2(at-q)(at-q^2)\sum_{n=0}^{\infty}\frac{(-adt;q)_n}{(-aq;q)_n(-dq;q)_{n+2}}q^n$$
$$=\frac{(1+a)(aq+dq+adq^2-adt)}{1+dq}-(aq+dq-adt)\frac{(-adt;q)_\infty}{(-aq,-dq;q)_\infty}\,. \tag{31}$$

Taking $t=0$ in Corollary 3.4, we have

Corollary 3.5.

$$a^2\sum_{n=0}^{\infty}\frac{1}{(-aq,-dq;q)_n}q^n+ad(1+q)\sum_{n=0}^{\infty}\frac{1}{(-aq;q)_n(-dq;q)_{n+1}}q^n$$
$$+d^2q^2\sum_{n=0}^{\infty}\frac{1}{(-aq;q)_n(-dq;q)_{n+2}}q^n$$
$$=\frac{(1+a)(a+d+adq)}{1+dq}-(a+d)\frac{1}{(-aq,-dq;q)_\infty}\,. \tag{32}$$

Theorem 3.6. *We have*

$$(1-dtq^k)\sum_{n=0}^{\infty}\frac{(-adtq^k;q)_n}{(-aq,-dq^{k+1};q)_n}q^n\sum_{i=0}^{\infty}\frac{(q^k,-adtq^{n+k};q)_i}{(q,-dq^{n+k+1},dtq^k;q)_i}q^{\binom{i+1}{2}}\Big(-\frac{d}{a}\Big)^i$$
$$=-a^{-1}\frac{(-adtq^k;q)_\infty}{(-aq,-dq^{k+1};q)_\infty}\sum_{i=0}^{\infty}\frac{(q^{k+1};q)_i}{(q,dtq^{k+1};q)_i}q^{\binom{i+1}{2}}\Big(-\frac{d}{a}\Big)^i$$
$$+(1+a^{-1})\sum_{i=0}^{\infty}\frac{(q^{k+1},-adtq^k;q)_i}{(q,-dq^{k+1},dtq^{k+1};q)_i}q^{\binom{i+1}{2}}\Big(-\frac{d}{a}\Big)^i, \tag{33}$$

where $|d/a|<1$.

Proof. (25) can be rewritten by

$$\sum_{n=0}^{\infty}q^na^{-k}(-aq^{n+1},at;q)_\infty=-a^{-(k+1)}(at;q)_\infty+a^{-(k+1)}(-a,at;q)_\infty \tag{34}$$

Apply the operator $E(d\theta)$ to its both sides with respect to the variable a. Then

$$\sum_{n=0}^{\infty} q^n E(d\theta)\{a^{-k}(-aq^{n+1}, at; q)_\infty\} = -E(d\theta)\{a^{-(k+1)}(at; q)_\infty\} + E(d\theta)\{a^{-(k+1)}(-a, at; q)_\infty\}.$$

By Theorem 2.2 and Corollary 2.3, we have

$$E(d\theta)\{a^{-k}(-aq^{n+1}, at; q)_\infty\} = a^{-k}\frac{(-aq^{n+1}, at, -dq^{n+k+1}, dtq^k; q)_\infty}{(-adtq^{n+k}; q)_\infty} \times \sum_{i=0}^{\infty} \frac{(q^k, -adtq^{n+k}; q)_i}{(q, -dq^{n+k+1}, dtq^k; q)_i} q^{\binom{i+1}{2}} \left(-\frac{d}{a}\right)^i,$$

$$E(d\theta)\{a^{-(k+1)}(at; q)_\infty\} = a^{-(k+1)}(at, dtq^{k+1}; q)_\infty \times \sum_{i=0}^{\infty} \frac{(q^{k+1}; q)_i}{(q, dtq^{k+1}; q)_i} q^{\binom{i+1}{2}} \left(-\frac{d}{a}\right)^i,$$

$$E(d\theta)\{a^{-(k+1)}(-a, at; q)_\infty\} = a^{-(k+1)}\frac{(-a, at, -dq^{k+1}, dtq^{k+1}; q)_\infty}{(-adtq^k; q)_\infty} \times \sum_{i=0}^{k} \frac{(q^{k+1}, -adtq^k; q)_i}{(q, -dq^{k+1}, dtq^{k+1}; q)_i} q^{\binom{i+1}{2}} \Big(-\frac{d}{a}\Big)^i.$$

Substituting these three identities into (34) and then using (4), we obtain the theorem. □

Taking $k = 0$ in Theorem 3.6 we have

Corollary 3.7.

$$(1-dt)\sum_{n=0}^{\infty} \frac{(-adt; q)_n}{(-aq, -dq; q)_n} q^n = -a^{-1}\frac{(-adt; q)_\infty}{(-aq, -dq; q)_\infty} \sum_{i=0}^{\infty} \frac{1}{(dtq; q)_i} q^{\binom{i+1}{2}} \Big(-\frac{d}{a}\Big)^i + (1+a^{-1})\sum_{i=0}^{\infty} \frac{(-adt; q)_i}{(-dq, dtq; q)_i} q^{\binom{i+1}{2}} \Big(-\frac{d}{a}\Big)^i, \tag{35}$$

where $|d/a| < 1$.

Taking $t = 0$ in Corollary 3.7 we obtain the following Ramanujan's identity.

Corollary 3.8.

$$\sum_{n=0}^{\infty} \frac{1}{(-aq, -dq; q)_n} q^n = -a^{-1}\frac{1}{(-aq, -dq; q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{\binom{n+1}{2}} (d/a)^n + (1+a^{-1})\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}} (d/a)^n}{(-dq; q)_n}, \tag{36}$$

where $|d/a| < 1$.

Note. This identity is a formula in Ramanujan's Lost Notebook, and its proofs are given by Andrews [1, 2]. In [5], Chen and Liu gave also a simple proof by using the method of the operator identity.

4. Several q-series identities from expansion of $\frac{1}{(a;q)_\infty}$

From the Euler expansion of $\frac{1}{(a;q)_\infty}$ (see [3]):

$$\sum_{n=0}^{\infty} \frac{1}{(q;q)_n} a^n = \frac{1}{(a;q)_\infty},$$

the following identity can be verified.

$$\sum_{n=0}^{\infty} \frac{1}{(aq^n;q)_\infty} q^n = a^{-1} \frac{1}{(a;q)_\infty} - a^{-1}. \tag{37}$$

From this identity, by the operator $T(dD_q$ we can obtain the following results.

Theorem 4.1. *We have*

$$\sum_{n=0}^{\infty} q^n (d;q)_n \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} (at;q)_j \frac{(a;q)_{n+j}}{(adt;q)_{n+j}} \Big(\frac{d}{a}\Big)^j = a^{-1} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(a,at;q)_j}{(adt;q)_j} \Big(\frac{d}{a}\Big)^j$$
$$- a^{-1} \frac{(a,d;q)_\infty}{(adt;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (at;q)_j \left(\frac{d}{a}\right)^j. \tag{38}$$

Proof. By (37), we have

$$\sum_{n=0}^{\infty} \frac{a^{k+1}}{(aq^n, at;q)_\infty} q^n = \frac{a^k}{(a,at;q)_\infty} - \frac{a^k}{(at;q)_\infty}. \tag{39}$$

Apply the operator $T(dD_q)$ to its both sides with respect to the variable a. Then

$$\sum_{n=0}^{\infty} q^n \cdot T(dD_q)\Big\{\frac{a^{k+1}}{(aq^n, at;q)_\infty}\Big\} = T(dD_q)\Big\{\frac{a^k}{(a,at;q)_\infty}\Big\}$$
$$- T(dD_q)\Big\{\frac{a^k}{(at;q)_\infty}\Big\}. \tag{40}$$

By Theorem 2.2 and Corollary 2.3, we have

$$T(dD_q)\Big\{\frac{a^{k+1}}{(aq^n, at;q)_\infty}\Big\} = a^{k+1} \frac{(adtq^n;q)_\infty}{(aq^n, at, dq^n, dt;q)_\infty}$$
$$\times \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} \frac{(aq^n, at;q)_j}{(adtq^n;q)_j} \Big(\frac{d}{a}\Big)^j, \tag{41}$$

$$T(dD_q)\Big\{\frac{a^k}{(a,at;q)_\infty}\Big\} = a^k \frac{(adt;q)_\infty}{(a,at,d,dt;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(a,at;q)_j}{(adt;q)_j} \Big(\frac{d}{a}\Big)^j \tag{42}$$

and

$$T(dD_q)\Big\{\frac{a^k}{(at;q)_\infty}\Big\} = a^k \frac{1}{(at,dt;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (at;q)_j \Big(\frac{d}{a}\Big)^j . \tag{43}$$

Substituting these three identities into (40) and then using (4), we obtain the theorem. □

Taking $k = 0$ in Theorem 4.1 we have

Corollary 4.2.

$$a\sum_{n=0}^{\infty} \frac{(a,d;q)_n}{(adt;q)_n} q^n + d(1-at)\sum_{n=0}^{\infty} (d;q)_n \frac{(a;q)_{n+1}}{(adt;q)_{n+1}} q^n = 1 - \frac{(a,d;q)_\infty}{(adt;q)_\infty} . \tag{44}$$

Specially, taking $t = 0$ we have

Corollary 4.3.

$$a\sum_{n=0}^{\infty} (a,d;q)_n q^n + d\sum_{n=0}^{\infty} (d;q)_n (a;q)_{n+1} q^n = 1 - (a,d;q)_\infty . \tag{45}$$

Taking $k = 1$ in Theorem 4.1 we have

Corollary 4.4.

$$\begin{aligned} a^2 \sum_{n=0}^{\infty} \frac{(a,d;q)_n}{(adt;q)_n} q^n &+ ad(1+q)(1-at) \sum_{n=0}^{\infty} \frac{(a;q)_{n+1}(d;q)_n}{(adt;q)_{n+1}} q^n \\ &+ d^2(1-at)(1-atq) \sum_{n=0}^{\infty} \frac{(a;q)_{n+2}(d;q)_n}{(adt;q)_{n+2}} q^n \\ &= a + d\frac{(1-a)(1-at)}{1-adt} - \frac{(a,d;q)_\infty}{(adt;q)_\infty}(a + d - adt) . \end{aligned} \tag{46}$$

Specially, taking $t = 0$ we have

Corollary 4.5.

$$\begin{aligned} a^2 \sum_{n=0}^{\infty} (a,d;q)_n q^n &+ ad(1+q) \sum_{n=0}^{\infty} (a;q)_{n+1}(d;q)_n q^n + d^2 \sum_{n=0}^{\infty} (a;q)_{n+2}(d;q)_n q^n \\ &= (a+d)(1-(a,d;q)_\infty) - ad . \end{aligned} \tag{47}$$

Theorem 4.6. *We have*

$$\begin{aligned} \sum_{n=0}^{\infty} q^n \frac{(a,d;q)_n}{(adt;q)_n} \sum_{j=0}^{\infty} \frac{(q^k;q)_j}{(q;q)_j} &\frac{(q^{1-n}/adt;q)_{k+j}}{(q/dt, q^{1-n}/d;q)_{k+j}} (-q)^j \\ &= -\frac{q^{k+1}}{d} \sum_{j=0}^{\infty} \frac{(q^{k+1};q)_j}{(q;q)_j} \frac{(q/adt;q)_{k+1+j}}{(q/d, q/dt;q)_{k+1+j}} (-q)^j \\ &\quad + \frac{1}{d}\frac{(a,d;q)_\infty}{(adt;q)_\infty} \sum_{j=0}^{\infty} \frac{(q^{k+1};q)_j}{(q;q)_j} \frac{(-1)^j}{(q/dt;q)_{k+1+j}} (q/at)^{k+1+j} , \end{aligned} \tag{48}$$

where $|q/at| < 1$.

Proof. We rewrite (37) into the following form:

$$\sum_{n=0}^{\infty} \frac{a^{-k}}{(aq^n, at; q)_\infty} q^n = \frac{a^{-(k+1)}}{(a, at; q)_\infty} - \frac{a^{-(k+1)}}{(at; q)_\infty}. \tag{49}$$

Apply the operator $T(dD_q)$ to its both sides with respect to the variable a. Then

$$\sum_{n=0}^{\infty} q^n \cdot T(dD_q)\Big\{\frac{a^{-k}}{(aq^n, at; q)_\infty}\Big\} = T(dD_q)\Big\{\frac{a^{-(k+1)}}{(a, at; q)_\infty}\Big\} - T(dD_q)\Big\{\frac{a^{-(k+1)}}{(at; q)_\infty}\Big\}. \tag{50}$$

By Theorem 2.2 and Corollary 2.3, we have

$$T(dD_q)\Big\{\frac{a^{-k}}{(aq^n, at; q)_\infty}\Big\} = (-d)^{-k} q^{\binom{k+1}{2}} \frac{(adtq^n; q)_\infty}{(aq^n, at, dq^n, dt; q)_\infty} \times \sum_{j=0}^{\infty} \frac{(q^k; q)_j}{(q; q)_j} \frac{(q^{1-n}/adt; q)_{k+j}}{(q/dt, q^{1-n}/d; q)_{k+j}} (-q)^j, \tag{51}$$

$$T(dD_q)\Big\{\frac{a^{-(k+1)}}{(a, at; q)_\infty}\Big\} = (-d)^{-(k+1)} q^{\binom{k+2}{2}} \frac{(adt; q)_\infty}{(a, at, d, dt; q)_\infty} \times \sum_{j=0}^{\infty} \frac{(q^{k+1}; q)_j}{(q; q)_j} \frac{(q/adt; q)_{k+j+1}}{(q/d, q/dt; q)_{k+j+1}} (-q)^j \tag{52}$$

and

$$T(dD_q)\Big\{\frac{a^{-(k+1)}}{(at; q)_\infty}\Big\} = \Big(-\frac{1}{adt}\Big)^{-(k+1)} q^{\binom{k+2}{2}} \frac{1}{(at, dt; q)_\infty} \times \sum_{j=0}^{\infty} \frac{(q^{k+1}; q)_j}{(q; q)_j} \frac{1}{(q/dt; q)_{k+j+1}} \Big(-\frac{q}{at}\Big)^j. \tag{53}$$

Substituting these three identities into (50) and then using (4), we obtain the theorem. □

Taking $k = 0$ in Theorem 4.6 we have

Corollary 4.7.

$$\sum_{n=0}^{\infty} q^n \frac{(a, d; q)_n}{(adt; q)_n} = -\frac{q}{d} \sum_{j=0}^{\infty} \frac{(q/adt; q)_{j+1}}{(q/d, q/dt; q)_{j+1}} (-q)^j + \frac{1}{d} \frac{(a, d; q)_\infty}{(adt; q)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j}{(q/dt; q)_{j+1}} \Big(\frac{q}{at}\Big)^{j+1}, \tag{54}$$

where $|q/at| < 1$.

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References

[1] Andrews, G. E., *Ramanujan's "Lost" Notebook I, Partial θ-function*, Adv. Math. **41** (1981), 137–172.

[2] Andrews, G. E., *Ramanujan: Essays and Surveys*, ch. An introduction to Ramanujan's "Lost" Notebook, pp. 165–184, 2001.

[3] Andrews, G. E., Askey, R., Roy, R., *Special Function*, Cambridge University Press, Cambridge, 1999.

[4] Chen, W. Y. C., Liu, Z.-G., *Parameter augmentation for basic hypergeometric series, II*, J. Combin. Theory Ser. A **80** (1997), 175–195.

[5] Chen, W. Y. C., Liu, Z.-G., *Mathematical essays in honor of Gian-Carlo Rota*, ch. Parameter augmentation for basic hypergeometric series, I, pp. 111–129, Birkhäuser, Basel, 1998.

[6] Gasper, G. G., Rahman, M., *Basic Hypergeometric Series*, Encyclopedia of Mathematics and Its Applications, Second edition, vol. 96, Cambridge University Press, 2004.

[7] Liu, Z.-G., *A new proof of the Nassrallah-Rahman integral*, Acta Math. Sinica **41** (2) (1998), 405–410.

[8] Liu, Z.-G., *Some operator identities and q-series transformation formulas*, Discrete Math. **265** (2003), 119–139.

[9] Rogers, L. J., *On the expansion of some infinite products*, Proc. London Math. Soc. **24** (1893), 337–352.

[10] Rogers, L. J., *Second Memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **25** (1894), 318–343.

[11] Rogers, L. J., *Third Memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **26** (1896), 15–32.

[12] Roman, S., *More on the umbral calculus, with emphasis on the q-umbral calculus*, J. Math. Anal. Appl. **107** (1985), 222–254.

[13] Zhang, Z. Z., Liu, M. X., *Applications of operator identities to the multiple q-binomial theorem and q-Gauss summation theorem*, Discrete Math. **306** (2006), 1424–1437.

[14] Zhang, Z. Z., Wang, J., *Two operator identities and their applications to terminating basic hypergeometric series and q-integrals*, J. Math. Anal. Appl. **312** (2) (2005), 653–665.

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