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ON THE GEOMETRY OF SOME PARA-HYPERCOMPLEX LIE GROUPS

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Abstract. In this paper, firstly we study some left invariant Riemannian metrics on para-hypercomplex 4-dimensional Lie groups. In each Lie group, the Levi-Civita connection and sectional curvature have been given explicitly. We also show these spaces have constant negative scalar curvatures. Then by using left invariant Riemannian metrics introduced in the first part, we construct some left invariant Randers metrics of Berwald type. The explicit formulas for computing flag curvature have been obtained in all cases. Some of these Finsler Lie groups are of non-positive flag curvature.

1. Introduction

Hypercomplex and para-hypercomplex structures are important in differential geometry which have many interesting and effective applications in theoretical physics. For example the background objects of HKT-geometry are hypercomplex manifolds. These spaces appear in $N = 4$ supersymmetric model (see [18, 12]). Also para-hypercomplex structures appear as target manifolds of hypermultiplets in Euclidean theories with rigid $N = 2$ supersymmetry (see [7]).
Like M. L. Barberis who has classified invariant hypercomplex structures on a simply-connected 4-dimensional real Lie group ([4, 5]), N. Blažić and S. Vukmirović have classified 4-dimensional real Lie algebras which admit a para-hypercomplex structure [6]. In the first part (Section 3) of this paper we consider the connected Lie groups corresponding to some of these Lie algebras and assume some left invariant Riemannian metrics on these 4-dimensional Lie groups. Then for the Riemannian manifolds we compute the Levi-Civita connection, sectional curvature and scalar curvature. This shows that all the spaces have constant negative scalar curvature, also some of them have non-positive sectional curvature.
In the second part (Section 4) of the paper we try to construct some invariant Finsler metrics on the Lie groups illustrated in the first part. Finsler manifolds have many applications in physics and biology (see [1, 2]), therefore it can be important to find some Finsler metrics on these manifolds. On the other hand the manifolds assumed in this paper are Lie groups so it is interesting to investigate

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left invariant Finsler metrics. The study of invariant Finsler metrics on Lie groups and homogeneous spaces is one of the attractive fields in Finsler geometry which has been considered in the recent years (see [8, 9, 10, 11, 14, 13, 17]). Invariant structures can extricate us from the complicated local coordinates computations appearing in Finsler geometry. In the previous works we have studied invariant metrics on 4-dimensional hypercomplex Lie groups [15, 16]. In this article we continue our studies on 4-dimensional para-hypercomplex Lie groups and by using Riemannian metrics introduced in the first part, we construct some left invariant Randers metric of Berwald type. One of the fundamental quantities, which is the generalization of sectional curvature in Riemannian geometry, is flag curvature. Computing flag curvature of Finsler manifolds in general state is vary complicated. We obtain the explicit formula for computing the flag curvature of each Finsler manifold constructed in Section 4 and show that some of the spaces admit left invariant Randers metric of non-positive flag curvature.

2. Preliminaries

Let M be a smooth manifold and $\{J_i\}_{i=1,2,3}$ be a family of fiberwise endomorphisms of TM such that

$$J_1^2 = -Id_{TM}\,, \tag{2.1}$$

$$J_2^2 = \mathrm{Id}_{TM}\,, \qquad J_2 \neq \pm\,\mathrm{Id}_{TM} \tag{2.2}$$

$$J_1 J_2 = -J_2 J_1 = J_3\,, \tag{2.3}$$

and

$$N_i = 0\,, \qquad i = 1, 2, 3\,, \tag{2.4}$$

where N_i is the Nijenhuis tensor corresponding to J_i defined as follows:

$$N_1(X, Y) = [J_1X, J_1Y] - J_1([X, J_1Y] + [J_1X, Y]) - [X, Y]\,, \tag{2.5}$$

and

$$N_i(X, Y) = [J_iX, J_iY] - J_i([X, J_iY] + [J_iX, Y]) + [X, Y]\,, \quad i = 2, 3\,, \tag{2.6}$$

for all vector fields X, Y on M. Then the family $\{J_i\}_{i=1,2,3}$ is called a para-hypercomplex structure on M.

In other word a para-hypercomplex structure on a manifold M is a family $\{J_i\}_{i=1,2,3}$ such that J_1 is a complex structure and $J_i, i = 2, 3$, are two non-trivial integrable product structures on M satisfying in the relation 2.3.

Suppose that $M = G$ is a Lie group. Then we additionally assume that the para-hypercomplex structure is left invariant, that is:

Definition 2.1. A para-hypercomplex structure $\{J_i\}_{i=1,2,3}$ on a Lie group G is said to be left invariant if for any $a \in G$

$$J_i = Tl_a \circ J_i \circ Tl_{a^{-1}}\,, \quad i = 1, 2, 3\,, \tag{2.7}$$

where Tl_a is the differential function of the left translation l_a.

Also we can consider left invariant metrics on Lie groups.

A Riemannian metric g on a Lie group G is called left invariant if

$$g(a)(Y,Z) = g(e)(T_a l_{a^{-1}} Y, T_a l_{a^{-1}} Z)\,, \qquad \forall\, a \in G\,, \forall\, Y, Z \in T_a G\,, \tag{2.8}$$

where e is the unit element of G.

Let $\mathfrak{g}$ be the Lie algebra of G, then the Levi-Civita connection of the left invariant Riemannian metric g is defined by the following formula:

$$2\langle \nabla_U V, W\rangle = \langle [U,V], W\rangle - \langle [V,W], U\rangle + \langle [W,U], V\rangle\,, \tag{2.9}$$

for any $U, V, W \in \mathfrak{g}$, where $\langle\,,\,\rangle$ is the inner product induced by g on $\mathfrak{g}$.

A Finsler metric on a manifold M is a non-negative function $F\colon TM \longrightarrow \mathbb{R}$ with the following properties:

(1) F is smooth on the slit tangent bundle $TM^0 := TM \setminus \{0\}$,

(2) $F(x, \lambda Y) = \lambda F(x, Y)$ for any $x \in M$, $Y \in T_x M$ and $\lambda > 0$,

(3) the $n \times n$ Hessian matrix $[g_{ij}] = [\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}]$ is positive definite at every point $(x, Y) \in TM^0$.

A special type of Finsler metrics are Randers metrics which have been introduced by G. Randers in 1941 [19]. Randers metrics are constructed on Riemannian metrics and vector fields (1-forms).

Let g and X be a Riemannian metric and a vector field on a manifold M respectively such that $\|X\| = \sqrt{g(X,X)} < 1$. Then a Randers metric F can be defined by g and X as follows:

$$F(x,Y) = \sqrt{g(x)(Y,Y)} + g(x)(X(x), Y)\,, \quad \forall\, x \in M\,,\ Y \in T_x M\,. \tag{2.10}$$

Similar to the Riemannian case, a Finsler metric F on a Lie group G is called left invariant if

$$F(a, Y) = F(e, T_a l_{a^{-1}} Y) \quad \forall\, a \in G\,,\ Y \in T_a G\,. \tag{2.11}$$

We can use left invariant Riemannian metrics and left invariant vector fields for constructing left invariant Randers metrics on Lie groups.

Suppose that G is Lie group, g is a left invariant Riemannian metric and X is a left invariant vector field on G such that $\sqrt{g(X,X)} < 1$. Then we can define a left invariant Riemannian metric on G by using formula (2.10). A special family of Randers metrics (or in general case Finsler metrics) is the family of Berwaldian Randers metrics. A Randers metric of the form (2.10) is of Berwald type if and only if the vector field X is parallel with respect to the Levi-Civita connection of g. In these metrics the Chern connection of the Randers metric F coincide with the Levi-Civita connection of the Riemannian metric g.

One of the important quantities which associates with a Riemannian manifold is sectional curvature which is defined by the following formula:

$$K(U,Y) = K(P) = \frac{g(R(U,Y)Y, U)}{g(Y,Y)\cdot g(U,U) - g^2(Y,U)}\,, \tag{2.12}$$

where P is the 2-plan spanned by U and Y in the tangent space. Sectional curvature defines another curvature named scalar curvature as follows:

$$S(x) = \sum_{j,k=1,j\neq k}^{n} K(e_j, e_k)\,, \tag{2.13}$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal basis for T_xM with respect to the Riemannian metric.

The concept of sectional curvature is developed to Finsler manifolds as follows:

$$K(P, Y) = \frac{g_Y(R(U, Y)Y, U)}{g_Y(Y, Y).g_Y(U, U) - g_Y^2(Y, U)}\,, \tag{2.14}$$

where $g_Y(U, V) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}(F^2(Y + sU + tV))|_{s=t=0}$, $P = span\{U, Y\}$, $R(U, Y)Y = \nabla_U\nabla_Y Y - \nabla_Y\nabla_U Y - \nabla_{[U,Y]}Y$ and ∇ is the Chern connection induced by F (see [3] and [20]). This generalization of sectional curvature to Finsler manifolds is named flag curvature.

3. Left invariant Riamnnian metrics on 4-dimensional para-hypercomlex Lie groups

N. Blažić and S. Vukmirović have studied 4-dimensional Lie algebras with para-hypercomplex structures. They classified these spaces in [6]. In this section by using some of these Lie algebras, we construct some Riemannian Lie groups with some curvature properties. In the next section by using these Riemannian metrics, we give some Randers spaces of non-positive flag curvature.

In each case let G_i be the connected 4-dimensional Lie group corresponding to the considered Lie algebra $\mathfrak{g}_i$ and $<,>$ is an inner product on $\mathfrak{g}_i$ such that $\{X, Y, Z, W\}$ is an orthonormal basis for $\mathfrak{g}_i$. Also we use g for the left invariant Riemannian metric on any G_i induced by $\langle\,,\rangle$.

In any case we suppose that $U = aX+bY+cZ+dW$ and $V = \tilde{a}X+\tilde{b}Y+\tilde{c}Z+\tilde{d}W$ are any two independent vectors in $\mathfrak{g}_i$.

Case 1. Let $\mathfrak{g}_1$ be the Lie algebra spanned by the basis $\{X, Y, Z, W\}$ with the following Lie algebra structure:

$$[X, Y] = Y\,, \qquad [X, W] = W\,. \tag{3.1}$$

(In any case when we do not write a commutator between the elements of the basis, the commutator is zero.)

Now consider the Riemannian manifold (G_1, g). By using formula (2.9) for the Levi-Civita connection of g we have:

$$\begin{array}{llll} \nabla_X X = 0\,, & \nabla_X Y = 0\,, & \nabla_X Z = 0\,, & \nabla_X W = 0\,, \\ \nabla_Y X = -Y\,, & \nabla_Y Y = X\,, & \nabla_Y Z = 0\,, & \nabla_Y W = 0\,, \\ \nabla_Z X = 0\,, & \nabla_Z Y = 0\,, & \nabla_Z Z = 0\,, & \nabla_Z W = 0\,, \\ \nabla_W X = -W\,, & \nabla_W Y = 0\,, & \nabla_W Z = 0\,, & \nabla_W W = X\,. \end{array} \tag{3.2}$$

A direct computation for the curvature tensor shows that:

$$\begin{aligned} R(X,Y)Y &= R(X,W)W = -X\,, \\ R(X,Y)X &= -R(Y,W)W = Y\,, \\ R(X,W)X &= R(Y,W)Y = W\,, \end{aligned} \tag{3.3}$$

and in other cases $R = 0$. Therefore for U and V we have:

$$\begin{aligned} R(V,U)U = &- \{(a\tilde{b} - b\tilde{a})(aY - bX) + (a\tilde{d} - d\tilde{a})(aW - dX) \\ &+ (b\tilde{d} - d\tilde{b})(bW - dY)\}\,. \end{aligned} \tag{3.4}$$

The last equation shows that the sectional curvature is obtained with the formula:

$$K(U,V) = -\{(a\tilde{b} - b\tilde{a})^2 + (a\tilde{d} - d\tilde{a})^2 + (b\tilde{d} - d\tilde{b})^2\} \le 0\,. \tag{3.5}$$

Also for the scalar curvature S_1 we have:

$$S_1(a) = -6 \quad \forall\, a \in G_1\,. \tag{3.6}$$

Case 2. The Lie algebra of case 2 is of the following form:

$$[X,Y] = Z\,. \tag{3.7}$$

Therefore for the Levi-Civita connection of the Riemannian manifold (G_2, g) we have:

$$\begin{array}{llll} \nabla_X X = 0\,, & \nabla_X Y = \frac{1}{2}Z\,, & \nabla_X Z = -\frac{1}{2}Y\,, & \nabla_X W = 0\,, \\ \nabla_Y X = -\frac{1}{2}Z\,, & \nabla_Y Y = 0\,, & \nabla_Y Z = \frac{1}{2}X\,, & \nabla_Y W = 0\,, \\ \nabla_Z X = -\frac{1}{2}Y\,, & \nabla_Z Y = \frac{1}{2}X\,, & \nabla_Z Z = 0\,, & \nabla_Z W = 0\,, \\ \nabla_W X = 0\,, & \nabla_W Y = 0\,, & \nabla_W Z = 0\,, & \nabla_W W = 0\,. \end{array} \tag{3.8}$$

Hence for the curvature tensor of this Levi-Civita connection we have:

$$\begin{aligned} -\frac{4}{3}R(X,Y)Y &= 4R(X,Z)Z = X\,, \\ \frac{4}{3}R(X,Y)X &= 4R(Y,Z)Z = Y\,, \\ -4R(X,Z)X &= -4R(Y,Z)Y = Z\,, \end{aligned} \tag{3.9}$$

and so:

$$\begin{aligned} R(V,U)U = &\frac{3}{4}(a\tilde{b} - b\tilde{a})(bX - aY) + \frac{1}{4}(a\tilde{c} - c\tilde{a})(aZ - cX) \\ &+ \frac{1}{4}(b\tilde{c} - c\tilde{b})(bZ - cY)\,. \end{aligned} \tag{3.10}$$

This equation shows that the sectional curvature can be obtained with the formula:

$$K(U,V) = \frac{1}{4}((a\tilde{c} - c\tilde{a})^2 + (b\tilde{c} - c\tilde{b})^2) - \frac{3}{4}(a\tilde{b} - b\tilde{a})^2\,. \tag{3.11}$$

Also using formula (2.13) for computing scalar curvature shows that:

$$S_2(a) = -\frac{1}{2} \quad \forall\, a \in G_2\,. \tag{3.12}$$

Case 3. The Lie algebra structure of $\mathfrak{g}_3$ is of the following form:

$$[X, Y] = X\,. \tag{3.13}$$

Therefore for (G_3, g) we have:

$$\begin{array}{llll} \nabla_X X = -Y\,, & \nabla_X Y = X\,, & \nabla_X Z = 0\,, & \nabla_X W = 0\,, \\ \nabla_Y X = 0\,, & \nabla_Y Y = 0\,, & \nabla_Y Z = 0\,, & \nabla_Y W = 0\,, \\ \nabla_Z X = 0\,, & \nabla_Z Y = 0\,, & \nabla_Z Z = 0\,, & \nabla_Z W = 0\,, \\ \nabla_W X = 0\,, & \nabla_W Y = 0\,, & \nabla_W Z = 0\,, & \nabla_W W = 0\,, \end{array} \tag{3.14}$$

and

$$R(X, Y)X = Y\,, \qquad R(X, Y)Y = -X\,. \tag{3.15}$$

Hence for U and V we have:

$$R(V, U)U = (a\tilde{b} - b\tilde{a})(bX - aY)\,. \tag{3.16}$$

So, like to case (1) we have another Riemannian Lie group of non-positive sectional curvature with formula:

$$K(U, V) = -(a\tilde{b} - b\tilde{a})^2 \leq 0\,. \tag{3.17}$$

The scalar curvature S_3 of this manifold is of the form:

$$S_3(a) = -2 \quad \forall\, a \in G_3\,. \tag{3.18}$$

Case 4. In the Lie algebra structure of case (4) there are two real parameters α and β. This Lie algebra has the following structure:

$$[X, Z] = X\,, \quad [X, W] = Y\,, \quad [Y, Z] = Y\,, \quad [Y, W] = \alpha X + \beta Y\,, \quad \alpha, \beta \in \mathbb{R}\,. \tag{3.19}$$

Existence of these parameters makes the computing a little complicated. In this case we have:

(3.20)

$$\begin{array}{llll} \nabla_X X = -Z\,, & \nabla_X Y = \frac{-(1+\alpha)}{2} W\,, & \nabla_X Z = X\,, & \nabla_X W = \frac{1+\alpha}{2} Y\,, \\ \nabla_Y X = \frac{-(1+\alpha)}{2} W\,, & \nabla_Y Y = -(Z + \beta W)\,, & \nabla_Y Z = Y\,, & \nabla_Y W = \frac{(1+\alpha)}{2} X + \beta Y\,, \\ \nabla_Z X = 0\,, & \nabla_Z Y = 0\,, & \nabla_Z Z = 0\,, & \nabla_Z W = 0\,, \\ \nabla_W X = \frac{\alpha - 1}{2} Y\,, & \nabla_W Y = \frac{1 - \alpha}{2} X\,, & \nabla_W Z = 0\,, & \nabla_W W = 0\,, \end{array}$$

and

$$
\begin{aligned}
&R(X,Y)X = \frac{4-(1+\alpha)^2}{4}Y\,, && R(X,Y)Y = \frac{(1+\alpha)^2-4}{4}X\,,\\
&R(X,Y)Z = 0\,, && R(X,Y)W = 0\,,\\
&R(X,Z)X = Z\,, && R(X,Z)Y = \frac{1+\alpha}{2}W\,,\\
&R(X,Z)Z = -X\,, && R(X,Z)W = -\frac{1+\alpha}{2}Y\,,\\
&R(X,W)X = \Big(\frac{1-\alpha^2}{4}+\frac{1+\alpha}{2}\Big)W\,, && R(X,W)Y = \frac{1+\alpha}{2}Z+\beta W\,,\\
&R(X,W)Z = -\frac{1+\alpha}{2}Y\,, && R(X,W)W = \frac{(\alpha+1)(\alpha-3)}{4}X-\beta Y\,,\\
&R(Y,Z)X = \frac{1+\alpha}{2}W\,, && R(Y,Z)Y = Z+\beta W\,,\\
&R(Y,Z)Z = -Y\,, && R(Y,Z)W = -\frac{1+\alpha}{2}X-\beta Y\,,\\
&R(Y,W)X = \frac{1+\alpha}{2}Z+\beta W\,, && R(Y,W)Y = \frac{3\alpha^2+4\beta^2+2\alpha-1}{4}W+\beta Z\,,\\
&R(Y,W)Z = -\frac{1+\alpha}{2}X-\beta Y\,, && R(Y,W)W = \frac{-3\alpha^2-4\beta^2-2\alpha+1}{4}Y-\beta X\,.
\end{aligned}
\tag{3.21}
$$

Therefore for U and V we have

$$
\begin{aligned}
R(V,U)U = &-\Big\{(a\tilde{b}-b\tilde{a})\Big(b\frac{(1+\alpha)^2-4}{4}X+a\frac{4-(1+\alpha)^2}{4}Y\Big)\\
&+(a\tilde{c}-c\tilde{a})\Big(aZ+b\frac{1+\alpha}{2}W-cX-d\frac{1+\alpha}{2}Y\Big)+(a\tilde{d}-d\tilde{a})\\
&\times\Big(a\frac{-\alpha^2+2\alpha+3}{4}W+b\frac{1+\alpha}{2}Z+b\beta W-c\frac{1+\alpha}{2}Y\\
&+d\frac{(1+\alpha)(\alpha-3)}{4}X-d\beta Y\Big)\\
&+(b\tilde{c}-c\tilde{b})\Big(a\frac{1+\alpha}{2}W+bZ+b\beta W-cY-d\frac{1+\alpha}{2}X-d\beta Y\Big)\\
&+(b\tilde{d}-d\tilde{b})\Big(a\frac{1+\alpha}{2}Z+a\beta W\\
&+b\beta Z+b\frac{3\alpha^2+4\beta^2+2\alpha-1}{4}W-c\frac{1+\alpha}{2}X\\
&-c\beta Y-d\frac{3\alpha^2+4\beta^2+2\alpha-1}{4}Y-d\beta X\Big)\Big\}\,.
\end{aligned}
\tag{3.22}
$$

The last equation shows that the sectional curvature can be obtained with the following formula:

$$
\begin{aligned}
K(U,V) = &-\Big\{(a\tilde{b}-b\tilde{a})\Big(\tilde{a}b\frac{(1+\alpha)^2-4}{4}+\tilde{b}a\frac{4-(1+\alpha)^2}{4}\Big)\\
&+(a\tilde{c}-c\tilde{a})\Big(a\tilde{c}+b\tilde{d}\frac{1+\alpha}{2}-c\tilde{a}-d\tilde{b}\frac{1+\alpha}{2}\Big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+(a\tilde{d}-d\tilde{a})\Big(a\tilde{d}\frac{-\alpha^2+2\alpha+3}{4}+b\tilde{c}\frac{1+\alpha}{2}+b\tilde{d}\beta-c\tilde{b}\frac{1+\alpha}{2}\\
&+d\tilde{a}\frac{(1+\alpha)(\alpha-3)}{4}-d\tilde{b}\beta\Big)\\
&+(b\tilde{c}-c\tilde{b})\Big(a\tilde{d}\frac{1+\alpha}{2}+b\tilde{c}+b\tilde{d}\beta-c\tilde{b}-d\tilde{a}\frac{1+\alpha}{2}-d\tilde{b}\beta\Big)\\
&+(b\tilde{d}-d\tilde{b})\Big(a\tilde{c}\frac{1+\alpha}{2}+a\tilde{d}\beta+b\tilde{c}\beta\\
&+b\tilde{d}\frac{3\alpha^2+4\beta^2+2\alpha-1}{4}-c\tilde{a}\frac{1+\alpha}{2}-c\tilde{b}\beta\\
&-d\tilde{b}\frac{3\alpha^2+4\beta^2+2\alpha-1}{4}-d\tilde{a}\beta\Big)\Big\}\,.
\end{aligned}
\tag{3.23}
$$

Also for the scalar curvature S_4 we have:

$$S_4(a)=-\frac{(1+\alpha)^2}{2}-2\beta^2-6<0\quad\forall\ a\in G_4\,. \tag{3.24}$$

Case 5. The Lie algebra structure of $\mathfrak{g}_5$ is:

$$[X,Z]=X\,,\qquad [Y,W]=Y\,. \tag{3.25}$$

The Levi-Civita connection of (G_5,g) is as follows:

$$
\begin{array}{llll}
\nabla_X X=-Z\,, & \nabla_X Y=0\,, & \nabla_X Z=X\,, & \nabla_X W=0\,,\\
\nabla_Y X=0\,, & \nabla_Y Y=-W\,, & \nabla_Y Z=0\,, & \nabla_Y W=Y\,,\\
\nabla_Z X=0\,, & \nabla_Z Y=0\,, & \nabla_Z Z=0\,, & \nabla_Z W=0,\\
\nabla_W X=0\,, & \nabla_W Y=0\,, & \nabla_W Z=0\,, & \nabla_W W=0\,.
\end{array}
\tag{3.26}
$$

A simple computation for the curvature tensor shows that:

$$
\begin{array}{ll}
R(X,Z)X=Z\,, & R(X,Z)Z=-X\,,\\
R(Y,W)Y=W\,, & R(Y,W)W=-Y\,,
\end{array}
\tag{3.27}
$$

and

$$R(V,U)U=-\big\{(a\tilde{c}-c\tilde{a})(-cX+aZ)+(b\tilde{d}-d\tilde{b})(-dY+bW)\big\}\,. \tag{3.28}$$

The above equation shows that the formula for computing sectional curvature is of the form:

$$K(U,V)=-\big\{(a\tilde{c}-c\tilde{a})^2+(b\tilde{d}-d\tilde{b})^2\big\}\leq 0\,. \tag{3.29}$$

Therefore (G_5,g) is of non-positive sectional curvature and constant negative scalar curvature S_5:

$$S_5(a)=-4\quad\forall a\in G_5\,. \tag{3.30}$$

Case 6. The last Lie algebra is $\mathfrak{g}_6$ with the following Lie algebra structure:

$$[X,Y]=W\,,\quad [X,W]=-Y\,,\quad [Y,W]=-X\,. \tag{3.31}$$

By using formula 2.9 for the Levi-Civita connection of g we have:

$$
\begin{aligned}
&\nabla_X X = 0\,, && \nabla_X Y = \frac{3}{2}W\,, && \nabla_X Z = 0\,, && \nabla_X W = -\frac{3}{2}Y\,,\\
&\nabla_Y X = \frac{1}{2}W\,, && \nabla_Y Y = 0\,, && \nabla_Y Z = 0\,, && \nabla_Y W = -\frac{1}{2}X\,,\\
&\nabla_Z X = 0\,, && \nabla_Z Y = 0\,, && \nabla_Z Z = 0\,, && \nabla_Z W = 0\,,\\
&\nabla_W X = -\frac{1}{2}Y\,, && \nabla_W Y = \frac{1}{2}X\,, && \nabla_W Z = 0\,, && \nabla_W W = 0\,.
\end{aligned}
\tag{3.32}
$$

A computation for the curvature tensor shows that:

$$
\begin{aligned}
4R(X,Y)Y &= 4R(X,W)W = X\,,\\
-4R(X,Y)X &= -\frac{4}{7}R(Y,W)W = Y\,,\\
-4R(X,W)X &= \frac{4}{7}R(Y,W)Y = W\,.
\end{aligned}
\tag{3.33}
$$

So we have

$$
\begin{aligned}
R(V,U)U = &-\frac{1}{4}\big\{(a\tilde{b}-b\tilde{a})(bX-aY)+(a\tilde{d}-d\tilde{a})(dX-aW)\\
&+7(b\tilde{d}-d\tilde{b})(-dY+bW)\big\}\,,
\end{aligned}
\tag{3.34}
$$

and

$$
K(U,V) = \frac{1}{4}\big\{(a\tilde{b}-b\tilde{a})^2+(a\tilde{d}-d\tilde{a})^2-7(b\tilde{d}-d\tilde{b})^2\big\}\,. \tag{3.35}
$$

The scalar curvature of (G_6, g) is

$$
S_6(a) = -\frac{7}{2} \quad \forall\, a \in G_6\,. \tag{3.36}
$$

4. Left invariant Randers metrics on 4-dimensional para-hypercomplex Lie groups

In this section we try to use the cases (1) to (6) to construct invariant Randers metrics of Berwald type. Also in possible cases we give the explicit formula for computing flag curvature of these metrics.

Case 1. A direct computation shows that a left invariant vector field Q is parallel with respect to the Levi-Civita connection if and only if $Q = qZ$. Now let $0 < \|Q\| < 1$ because we would like to construct invariant Randers metric. This shows that $0 < |q| < 1$. So the Randers metric defined by g and Q on G_1 is of Berwald type. For $g_{\tilde{Y}}(U,V)$ we have (see [10])

$$
\begin{aligned}
g_{\tilde{Y}}(U,V) = &g(U,V) + g(X,U)\cdot g(X,V)\\
&-\frac{g(X,\tilde{Y})\cdot g(\tilde{Y},V)\cdot g(\tilde{Y},U)}{g(\tilde{Y},\tilde{Y})^{\frac{3}{2}}} + \frac{1}{\sqrt{g(\tilde{Y},\tilde{Y})}}\\
&\times\big\{g(X,U)\cdot g(\tilde{Y},V) + g(X,\tilde{Y})\cdot g(U,V) + g(X,V)\cdot g(\tilde{Y},U)\big\}\,.
\end{aligned}
\tag{4.1}
$$

Equation (4.1) shows that

$$g_U(R(V,U)U,V) = -(1+qc)\big\{(a\tilde{b}-b\tilde{a})^2 + (a\tilde{d}-d\tilde{a})^2 + (b\tilde{d}-d\tilde{b})^2\big\} \tag{4.2}$$

$$g_U(U,U) = (1+qc)^2 \tag{4.3}$$

$$g_U(V,V) = 1 + (q\tilde{c})^2 + qc \tag{4.4}$$

$$g_U(U,V) = q\tilde{c}(1+qc)\,. \tag{4.5}$$

Now let $P = span\{U,V\}$, therefore for the flag curvature we have

$$K(P,U) = \frac{-\{(a\tilde{b}-b\tilde{a})^2 + (a\tilde{d}-d\tilde{a})^2 + (b\tilde{d}-d\tilde{b})^2\}}{(1+qc)^2} \leq 0\,. \tag{4.6}$$

Therefore in the case 1, (G_1, F) is of non-positive flag curvature.

Case 2. Let $\nabla Q = 0$ for a left invariant vector field Q. A simple computation shows that $Q = qW$. Also the assumption $0 < \|Q\| < 1$ forces us to let $0 < |q| < 1$. Therefore we have a left invariant Randers metric of Berwald type which is constructed by g and Q like formula (2.10). Also we have:

$$g_U\big(R(V,U)U,V\big) = \Big\{\frac{1}{4}\big((a\tilde{c}-c\tilde{a})^2 + (b\tilde{c}-c\tilde{b})^2\big) - \frac{3}{4}(a\tilde{b}-b\tilde{a})^2\Big\}(1+qd) \tag{4.7}$$

$$g_U(U,U) = (1+qd)^2 \tag{4.8}$$

$$g_U(V,V) = 1 + (q\tilde{d})^2 + qd \tag{4.9}$$

$$g_U(U,V) = q\tilde{d}(1+qd)\,. \tag{4.10}$$

Therefore the flag curvature formula of this Randers metric is of the form:

$$K(P,U) = \frac{(a\tilde{c}-c\tilde{a})^2 + (b\tilde{c}-c\tilde{b})^2 - 3(a\tilde{b}-b\tilde{a})^2}{4(1+qd)^2}\,. \tag{4.11}$$

Case 3. A computation for finding left invariant parallel vector fields with respect to the Levi-Civita connection shows that the only vector fields which are of the form $Q = q_1Z + q_2W$ are parallel. We consider the vector fields $Q = q_1Z + q_2W$ such that $0 < q_1^2 + q_2^2 < 1$, because we need the condition $0 < \|Q\| < 1$. In this case we have:

$$g_U\big(R(V,U)U,V\big) = -(a\tilde{b}-b\tilde{a})^2(1+cq_1+dq_2) \tag{4.12}$$

$$g_U(U,U) = (1+cq_1+dq_2)^2 \tag{4.13}$$

$$g_U(V,V) = 1 + (\tilde{c}q_1+\tilde{d}q_2)^2 + (cq_1+dq_2) \tag{4.14}$$

$$g_U(U,V) = (\tilde{c}q_1+\tilde{d}q_2)(1+cq_1+dq_2)\,. \tag{4.15}$$

Now for $P = span\{U,V\}$, the flag curvature is as follows:

$$K(P,U) = \frac{-(a\tilde{b}-b\tilde{a})^2}{(1+cq_1+dq_2)^2} \leq 0\,. \tag{4.16}$$

So in this case (G_3, F) is of non-positive flag curvature.

Case 4. A simple computation shows that (G_4, g) admits a parallel left invariant vector field Q if and only if $\alpha = -1$ and $\beta = 0$. In this situation $Q = qW$ is the only family of left invariant vector fields which are parallel with respect to the Levi-Civita connection of (G_4, g). Let $0 < |q| < 1$ because of the condition $0 < \|Q\| < 1$. So we have:

$$g_U(R(V,U)U,V) = -(1+dq)\big\{(a\tilde{b} - b\tilde{a})^2 + (a\tilde{c} - c\tilde{a})^2 + (b\tilde{c} - c\tilde{b})^2\big\} \tag{4.17}$$

$$g_U(U,U) = (1+dq)^2 \tag{4.18}$$

$$g_U(V,V) = 1 + dq + (\tilde{d}q)^2 \tag{4.19}$$

$$g_U(U,V) = \tilde{d}q(1+dq)\,. \tag{4.20}$$

Hence for the flag curvature we have:

$$K(P,U) = \frac{-\{(a\tilde{b} - b\tilde{a})^2 + (a\tilde{c} - c\tilde{a})^2 + (b\tilde{c} - c\tilde{b})^2\}}{(1+dq)^2} \leq 0\,, \tag{4.21}$$

This case is exactly the case 2 of [16].

Case 5. In this case a simple computation shows that the Levi-Civita connection does not admit a left invariant vector parallel field. Therefore we can not construct Randers metrics as above cases.

Case 6. The Riemannian Lie group (G_6, g) admits parallel left invariant vector fields of the form $Q = qZ$. Let $0 < |q| < 1$ because we need $0 < \|Q\| < 1$. Therefore we have:

$$g_U\big(R(V,U)U,V\big) = \frac{1+qc}{4}\big\{(a\tilde{b} - b\tilde{a})^2 + (a\tilde{d} - d\tilde{a})^2 - 7(b\tilde{d} - d\tilde{b})^2\big\} \tag{4.22}$$

$$g_U(U,U) = (1+qc)^2 \tag{4.23}$$

$$g_U(V,V) = 1 + (q\tilde{c})^2 + qc \tag{4.24}$$

$$g_U(U,V) = q\tilde{c}(1+qc)\,, \tag{4.25}$$

and so

$$K(P,U) = \frac{(a\tilde{b} - b\tilde{a})^2 + (a\tilde{d} - d\tilde{a})^2 - 7(b\tilde{d} - d\tilde{b})^2}{4(1+qc)^2}\,. \tag{4.26}$$

The last equation shows that the flag curvature of (G_6, F) has not constant sign.

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