ARCHIVUM MATHEMATICUM (BRNO)
Tomus 46 (2010), 1–11

ASYMPTOTIC PROPERTIES OF SOLUTIONS OF NONAUTONOMOUS DIFFERENCE EQUATIONS

Janusz Migda

Abstract. Asymptotic properties of solutions of difference equation of the form

$$\Delta^m x_n = a_n \varphi_n(x_{\sigma(n)}) + b_n$$

are studied. Conditions under which every (every bounded) solution of the equation $\Delta^m y_n = b_n$ is asymptotically equivalent to some solution of the above equation are obtained. Moreover, the conditions under which every polynomial sequence of degree less than m is asymptotically equivalent to some solution of the equation and every solution is asymptotically polynomial are obtained. The consequences of the existence of asymptotically polynomial solution are also studied.

1. Introduction

Let N, $N(0)$, Z, R denote the set of positive integers, the set of nonnegative integers, the set of all integers and the set of real numbers, respectively.
Let $m \in N$. In this paper we consider the difference equation of the form

$$\text{(E)} \qquad \Delta^m x_n = a_n \varphi_n(x_{\sigma(n)}) + b_n$$

$$n \in N\,, \quad a_n, b_n \in R\,, \quad \varphi_n \colon R \to R\,, \quad \sigma \colon N \to Z\,, \quad \lim \sigma(n) = \infty\,.$$

By a solution of (E) we mean a sequence $x\colon N \to R$ satisfying (E) for all large n. Let $n_0 = \min\{n \in N : \sigma(k) \ge 1 \text{ for all } k \ge n\}$. If (E) is satisfied for all $n \ge n_0$ we say that x is a full solution of (E).

Let (X,d), (Y,ρ) be metric spaces, and let Φ be a family of maps $\varphi\colon X \to Y$. Φ is said to be equicontinuous at a point $p \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x,p) < \delta$ then $\rho\big(\varphi(x), \varphi(p)\big) < \varepsilon$ for all $\varphi \in \Phi$. We say that Φ is equicontinuous if it is equicontinuous at every point $p \in X$. If for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho\big(\varphi(x_1), \varphi(x_2)\big) < \varepsilon$ for any pair $x_1, x_2 \in X$ such that $d(x_1,x_2) < \delta$ and all $\varphi \in \Phi$, then Φ is said to be uniformly equicontinuous. Φ is said to be locally bounded if for any point $p \in X$ there exist a neighborhood U of p in X and a constant $M > 0$ such that $|\varphi(t)| \le M$ for all $t \in U$, $\varphi \in \Phi$. If $|\varphi(t)| \le M$ for all $t \in X$, $\varphi \in \Phi$ then we say that Φ is bounded. If $\psi\colon X \to R$, $U \subseteq X$ then $\psi \mid U$ denotes the restriction of ψ. We say that sequences x, y are

2000 *Mathematics Subject Classification*: primary 39A10.
Key words and phrases: difference equation, asymptotic behavior, asymptotically polynomial solution.
Received August 27, 2008. Editor O. Došlý.

asymptotically equivalent if $x_n - y_n = o(1)$. For a given sequence x of real numbers, by $\sum x_n$ we denote the series whose partial sums are: x_1, $x_1 + x_2$, $x_1 + x_2 + x_3$ and so on. We often use the algebraic notation Ax to denote $A(x)$ if A is a linear operator.

Recently, there has been a great interest in the study of asymptotic and oscillatory behavior of solutions of higher order difference equations, see for example [2] [3], [7]–[12]and the references cited therein. The purpose of this paper is to study the asymptotic behavior of solutions of equation (E). Using the Schauder's fixed point theorem and some technical results based on the properties of iterated rest operator we show that if the series $\sum n^{m-1}a_n$ is absolutely convergent and the family $\{\varphi_n\}$ is equicontinuous and locally bounded (uniformly equicontinuous and bounded) then every bounded solution (every solution) of the equation $\Delta^m y_n = b_n$ is asymptotically equivalent to some solution of (E). Moreover, if the series $\sum n^{m-1}b_n$ is also absolutely convergent then every polynomial sequence of degree less than m is asymptotically equivalent to some solution of (E). Similar problem for autonomous difference equation was considered in [4]. We also show that if the series $\sum n^{m-1}a_n$ and $\sum n^{m-1}b_n$ are absolutely convergent and the family $\{\varphi_n\}$ is bounded then every solution of (E) is asymptotically polynomial. On the other hand, under some additional assumptions, we show that if there exists asymptotically polynomial solution of (E) then the series $\sum n^{m-1}a_n$ and $\sum n^{m-1}b_n$ are absolutely convergent.

The results obtained here generalize the results of [1, 6, 7] and some of those contained in [5]. The results obtained here in Theorems 1–4 are analogous to those obtained in [4] for autonomous equations.

2. Preliminary lemmas

In this section we introduce a rest operator and establish some useful properties of their iterations. These results will be used in the proofs of the main theorems in Section 3.

By SQ we denote the space of all sequences $x\colon N \to R$. If $x \in SQ$ then $|x|$ denote the sequence defined by $|x|(n) = |x_n|$ for every $n \in N$. The Banach space of all bounded sequences $x \in SQ$ with the norm $\|x\| = \sup\{|x_n| : n \in N\}$ we denote by BS. For $k \in N$, $\mathrm{Pol}(k)$ denotes the space of all polynomial sequences (with real coefficients) of degree $\leq k$. We identify every sequence $\beta \in \mathrm{Pol}(k)$ with the corresponding polynomial.

Let $S(0) = \{x \in SQ : \lim x_n = 0\}$,

$$S(1) = \{x \in SQ : \text{the series } \sum x_n \text{ is convergent}\}\,.$$

If $x \in S(1)$, we may define the sequence $r(x)$ by the formula

$$r(x)(n) = \sum_{j=n}^{\infty} x_j\,.$$

Obviously $r(x) \in S(0)$ and the mapping $r\colon S(1) \to S(0)$, which we call the rest operator, is linear. Let $S(2) = \{x \in S(1) : r(x) \in S(1)\}$. Then $S(2)$ is a linear subspace of $S(1)$ and we may define the operator $r^2\colon S(2) \to S(0)$ by

$r^2(x) = r(r(x))$. If $k \in N$ then, by induction, we define the space $S(k+1)$ and the operator $r^{k+1}\colon S(k+1) \to S(0)$ by

$$S(k+1) = \{x \in S(k) \,:\, r^k(x) \in S(1)\}\,, \qquad r^{k+1}(x) = r\big(r^k(x)\big)\,.$$

If $x \in S(k)$, $n \in N$ then the value $r^k(x)(n)$ we denote also by $r^k_n(x)$ or simply $r^k_n x$. Moreover, if $x \in SQ$, $n \in N$ then we define $r^0_n x = x_n$. Note that

$$S(k) \subset S(k-1) \subset \cdots \subset S(1) \subset S(0)$$

are linear subspaces of $S(0)$ and $r^k\colon S(k) \to S(0)$ is a linear operator.

For $n \in N$ we define the numbers $s^0_n = 1$, $s^1_n = s^0_1 + s^0_2 + \cdots + s^0_n = n$. If k, $n \in N$ then, by induction on k, we define numbers

$$s^{k+1}_n = s^k_1 + s^k_2 + \ldots + s^k_n\,.$$

Lemma 1. *If $k \in N$, $|x| \in S(k)$ then $x \in S(k)$ and $|r^k x| \le r^k|x|$.*

Proof. Induction on k. The case $k = 1$ is obvious. Assume the assertion is true for some $k \ge 1$ and $|x| \in S(k+1)$. Then $r^k|x| \in S(1)$. Moreover by inductive assumption, $x \in S(k)$ and $|r^k x| \le r^k|x|$. Hence, by comparison test of convergence of the series, $r^k x \in S(1)$. Therefore, $x \in S(k+1)$. Moreover $|r^{k+1}x| = |r(r^k x)| \le r(|r^k x|) \le r(r^k|x|) = r^{k+1}|x|$. $\square$

Lemma 2. *Assume $x \in SQ$, $k \in N$. Then $|x| \in S(k)$ if and only if the series $\sum s^{k-1}_i x_i$ is absolutely convergent. If $|x| \in S(k)$, $n \in N$ then*

$$r^k_n|x| = s^{k-1}_1|x_n| + s^{k-1}_2|x_{n+1}| + s^{k-1}_3|x_{n+2}| + \ldots\ .$$

Proof. We prove this by induction on k. The case $k = 1$ is obvious. Assume the assertion is true for some $k \ge 1$. If $|x| \in S(k+1)$, $n \in N$ then

$$\begin{aligned}
r^{k+1}_n|x| &= r^k_n|x| + r^k_{n+1}|x| + r^k_{n+2}|x| + r^k_{n+3}|x| + \ldots \\
&= s^{k-1}_1|x_n| + s^{k-1}_2|x_{n+1}| + s^{k-1}_3|x_{n+2}| + s^{k-1}_4|x_{n+3}| + \ldots \\
&\quad + s^{k-1}_1|x_{n+1}| + s^{k-1}_2|x_{n+2}| + s^{k-1}_3|x_{n+3}| + \ldots \\
&\quad + s^{k-1}_1|x_{n+2}| + s^{k-1}_2|x_{n+3}| + \ldots \\
&\quad + s^{k-1}_1|x_{n+3}| + \ldots \\
&= s^{k-1}_1|x_n| + (s^{k-1}_1 + s^{k-1}_2)|x_{n+1}| + (s^{k-1}_1 + s^{k-1}_2 + s^{k-1}_3)|x_{n+2}| + \ldots \\
&= s^k_1|x_n| + s^k_2|x_{n+1}| + s^k_3|x_{n+2}| + s^k_4|x_{n+3}| + \ldots\ .
\end{aligned}$$

Hence

$$r^k_n|x| + r^k_{n+1}|x| + r^k_{n+2}|x| + \ldots = s^k_1|x_n| + s^k_2|x_{n+1}| + s^k_3|x_{n+2}| + \ldots\ . \tag{1}$$

For $n = 1$ we obtain the convergence of the series $\sum_{i=1}^\infty s^k_i x_i$. Conversely, assume the series $\sum_{i=1}^\infty s^k_i x_i$ is absolutely convergent. Since $0 \le s^{k-1}_i \le s^k_i$ for all $i \in N$, the series $\sum_{i=1}^\infty s^{k-1}_i x_i$ is absolutely convergent. Hence, by inductive assumption, $|x| \in S(k)$. Let $n = 1$. By (1), the series $\sum_{i=1}^\infty r^k_i|x|$ is convergent. Therefore $r^k|x| \in S(1)$. Hence $|x| \in S(k+1)$. The proof is complete. $\square$

Lemma 3. *Assume $x \in SQ$, $k \in N(0)$. Then $|x| \in S(k+1)$ if and only if the series $\sum n^k x_n$ is absolutely convergent.*

Proof. First we show that $s_n^k \le n^k$ for all $n \in N$. It is obvious if $k = 0$ or $k = 1$. Assume $s_n^k \le n^k$ for some $k \ge 1$ and any $n \in N$. Then $s_n^{k+1} = s_1^k + s_2^k + \ldots + s_n^k \le n^k + n^k + \ldots + n^k = nn^k = n^{k+1}$.
Using the known equality $\sum_{i=1}^n \binom{k+i-1}{k} = \binom{k+n}{k+1}$ it is easy to show that

$$s_n^k = \binom{k+n-1}{k} = \frac{n(n+1)\ldots(n+k-1)}{k!}\,.$$

Hence, $n^k \le n(n+1)\ldots(n+k-1) = k!s_n^k$. Since $s_n^k \le n^k \le k!s_n^k$, absolute convergence of the series $\sum n^k x_n$ is equivalent to the absolute convergence of the series $\sum s_n^k x_n$. The assertion follows now from Lemma 2. □

Lemma 4. *Assume $M > 0$, $k \in N$, $a, b \in SQ$, $|b| \le M$, and the series $\sum n^{k-1}|a_n|$ is convergent. Then $ba, |a| \in S(k)$ and $|r^k(ba)| \le Mr^k|a|$.*

Proof. By Lemma 3, $|a|, |ba| \in S(k)$. Hence, by Lemma 1, $ba \in S(k)$ and $|r^k(ba)| \le r^k|ba|$. By Lemma 2, $r^k|ba| \le r^k(M|a|) = Mr^k|a|$. Hence $|r^k(ba)| \le Mr^k|a|$. □

Lemma 5. *If $k \in N$ and $x \in S(k)$, then $\Delta^k r^k x = (-1)^k x$.*

Proof. We prove this by induction on k. If $k = 1$, then

$$\Delta rx(n) = rx(n+1) - rx(n) = \sum_{k=n+1}^{\infty} x_k - \sum_{k=n}^{\infty} x_k = -x_n\,.$$

Hence $\Delta rx = -x$. Assume the assertion is true for some $k \ge 1$ and let $x \in S(k+1)$. Since $r^k x \in S(1)$, we have $\Delta r(r^k x) = -r^k x$. Hence,

$$\Delta^{k+1} r^{k+1} x = \Delta^k \Delta r r^k x = \Delta^k(-r^k x) = (-1)\Delta^k r^k x = (-1)^{k+1} x\,.$$

□

Lemma 6. *Let x be a sequence convergent to $c \in R$ and let $k \in N$. Then*

$$\Delta^k x \in S(k)\,, \qquad r^k \Delta^k x = (-1)^k (x - c)\,.$$

Proof. Let $k = 1$. Then $\Delta x_1 + \Delta x_2 + \ldots + \Delta x_n = x_{n+1} - x_1$. Hence the series $\sum \Delta x_n$ is convergent i.e., $\Delta x \in S(1)$. Moreover $\Delta x_n + \Delta x_{n+1} + \ldots + \Delta x_p = x_{p+1} - x_n$. Hence $r_n^1 \Delta x = c - x_n$. Therefore the assertion is true for $k = 1$. Assume it is true for some $k \ge 1$. Since the sequence Δx is convergent to zero we have

$$\Delta^{k+1} x = \Delta^k \Delta x \in S(k)\,, \qquad r^k \Delta^{k+1} x = r^k \Delta^k \Delta x = (-1)^k \Delta x\,.$$

Since $\Delta x \in S(1)$, it follows that $(-1)^k \Delta x \in S(1)$. Hence, $r^k \Delta^{k+1} x \in S(1)$. Therefore $\Delta^{k+1} x \in S(k+1)$. Moreover

$$\begin{aligned} r^{k+1}\Delta^{k+1} x &= r r^k \Delta^k \Delta x = r((-1)^k \Delta x) = (-1)^k r \Delta x \\ &= (-1)^k(-1)(x - c) = (-1)^{k+1}(x - c)\,. \end{aligned}$$

The proof is complete. □

Lemma 7. *Let $k \in N$ and let $W = \{x \in SQ : x_n = 0 \text{ for } n \geq k\}$. Then $\Delta^m(W) = W$.*

Proof. It is easy to see that W is a finite dimensional linear subspace of SQ and $\Delta^m(W) \subseteq W$. Since $W \cap \operatorname{Ker} \Delta^m = W \cap \operatorname{Pol}(m-1) = 0$, the linear operator $\Delta^m|W \colon W \to W$ is monomorphic. Hence $\dim(\Delta^m(W)) = \dim W$. Therefore $\Delta^m(W) = W$. $\square$

Remark 1. It is easy to see that if x, $z \in SQ$, $p \in N$, and $z_n = x_n$ for all $n \geq p$ then $\Delta^k z_n = \Delta^k x_n$ for $n \geq p$. Analogously, by easy induction on k, one can show that if $x \in S(k)$, $z \in SQ$, and $z_n = x_n$ for $n \geq p$ then $z \in S(k)$ and $r_n^k z = r_n^k x$ for $n \geq p$.

Lemma 8 ([6])**.** *If X and Y are metric spaces, X is compact, and Φ is equicontinuous family of maps $\varphi \colon X \to Y$, then Φ is uniformly equicontinuous.*

Lemma 9 ([6])**.** *If X, Y are metric spaces, X is compact, and Φ is a locally bounded family of maps $\varphi \colon X \to Y$, then Φ is bounded.*

3. Main results

Theorem 1. *Assume the series $\sum n^{m-1} a_n$ is absolutely convergent, y is a bounded solution of the equation $\Delta^m y_n = b_n$, and Y is the set of values of the sequence y. If there exists a neighbourhood U of the closure $\overline{Y}$ such that the family $\{\varphi_n \mid U\}$ is locally bounded and equicontinuous, then there exists a solution x of* (E) *such that $x = y + o(1)$.*

Proof. Since the set Y is bounded, the closure $\overline{Y}$ is compact. Hence, there exists an open set V such that $\overline{V}$ is compact and $\overline{Y} \subseteq V \subseteq \overline{V} \subseteq U$. Using Lemma 8 and Lemma 9 one can show that the family $\{\varphi_n \mid V\}$ is bounded and uniformly equicontinuous. Since $\overline{Y}$ is compact, so there exists a number $c > 0$ such that if

$$s \in \overline{Y}, \quad t \in R, \quad |s - t| \leq c$$

then $t \in V$. There exists $M > 0$ such that $|\varphi_n(t)| \leq M$ for all $t \in V$ and all $n \in N$. Let $\rho = r^m|a|$. By definition of r^m, $\rho = o(1)$. Choose $p \in N$ such that $M\rho_n \leq c$ for any $n \geq p$. Let

$$T = \{x \in BS : x_n = 0 \text{ for } n < p \text{ and } |x_n| \leq M\rho_n \text{ for } n \geq p\}.$$

Obviously T is a convex and closed subset of BS. Choose an $\varepsilon > 0$. Then there exists $m \in N$ such that $M\rho_n < \varepsilon$ for any $n \geq m$. For $n = 1, \dots, m$ let G_n denote a finite ε-net for the interval $[-M\rho_n, M\rho_n]$ and let

$$G = \{x \in T : x_n \in G_n \text{ for } n \leq m \text{ and } x_n = 0 \text{ for } n > m\}.$$

Then G is a finite ε-net for T. Hence T is a complete and totally bounded metric space and so, T is compact. Hence T is a convex and compact subset of the Banach space BS. Let

$$S = \{x \in SQ : x_n = y_n \text{ for } n < p \text{ and } |x_n - y_n| \leq M\rho_n \text{ for } n \geq p\}$$

and let $F \colon T \to S$ be a map given by $F(x)(n) = x_n + y_n$. The formula $d(x, z) = \sup_{n \in N} |x_n - z_n|$ defines a metric on S such that F is an isometry of T onto S. By

Schauder's fixed point theorem, every continuous map $B\colon T \to T$ has a fixed point. Since the space S is homeomorphic to T, every continuous map $A\colon S \to S$ has a fixed point too.
Let $x \in S$. Then $|x_i - y_i| \le c$ for any $i \in N$. Hence $x_i \in V$ for all $i \in N$. Therefore $|\varphi_n(x_i)| \le M$ for all $n \in N$ and all $i \in N$. Hence, if $x \in S$ then $|\varphi_n(x_{\sigma(n)})| \le M$ for all $n \ge n_0$. For $x \in SQ$ let $\bar{x}$ be defined by

$$\bar{x}_n = \begin{cases} 0 & \text{for} \quad n < n_0 \\ a_n\varphi_n(x_{\sigma(n)}) & \text{for} \quad n \ge n_0\,. \end{cases}$$

If $x \in S$ then $|\bar{x}| \le M|a|$. Hence, by Lemma 3, $|\bar{x}| \in S(m)$ for all $x \in S$. By Lemma 1, $\bar{x} \in S(m)$ for all $x \in S$. Since $\lim \sigma(n) = \infty$, there exists $p_1 \ge p$ such that $\sigma(n) \ge p$ for all $n \ge p_1$. For $x \in S$ we define the sequence $A(x)$ by

$$A(x)(n) = \begin{cases} y_n & \text{for} \quad n < p_1 \\ y_n + (-1)^m r_n^m \bar{x} & \text{for} \quad n \ge p_1\,. \end{cases}$$

Then $|A(x) - y| \le |r^m\bar{x}| \le r^m|\bar{x}|$. Hence, by Lemma 2,

$$|A(x) - y| \le r^m(M|a|) = M\rho\,.$$

Therefore $A(x) \in S$ for all $x \in S$.

Let $\varepsilon > 0$. Since the family $\{\varphi_n|V\}$ is uniformly continuous, there exists $\delta > 0$ such that if $t,\, s \in V$ and $|t - s| < \delta$ then $|\varphi_n(t) - \varphi_n(s)| < \varepsilon$ for any $n \in N$. Let $x,\, z \in S$, $\|x - z\| < \delta$. Then $x_i, z_i \in V$ and $|x_i - z_i| < \delta$ for all $i \in N$. Hence $|\varphi_n(x_i) - \varphi_n(z_i)| < \varepsilon$ for all $n \in N$ and all $i \in N$. Hence $|\bar{x} - \bar{z}| \le \varepsilon|a|$. Therefore

$$\begin{aligned} \|A(x) - A(z)\| &= \sup_{n \ge p_1} |r_n^m\bar{x} - r_n^m\bar{z}| \\ &= \sup_{n \ge p_1} |r_n^m(\bar{x} - \bar{z})| \le r_{p_1}^m|\bar{x} - \bar{z}| \le \varepsilon r_{p_1}^m|a| = \varepsilon\rho_{p_1}\,. \end{aligned}$$

Hence, the mapping $A\colon S \to S$ is continuous. Therefore there exists $x \in S$ such that $A(x) = x$. Then $x_n = y_n + (-1)^m r_n^m\bar{x}$ for any $n \ge p_1$. Hence, using Lemma 5 we obtain

$$\Delta^m x_n = \Delta^m y_n + \Delta^m\big((-1)^m r_n^m\bar{x}\big) = b_n + \bar{x}_n = a_n\varphi_n(x_{\sigma(n)}) + b_n$$

for $n \ge p_1$. Moreover, since $r^m\bar{x} = o(1)$, we have $x = y + o(1)$. The proof is complete. □

Corollary 1. *If the series $\sum n^{m-1}a_n$ is absolutely convergent, and the family $\{\varphi_n\}$ is equicontinuous and locally bounded, then for any bounded solution y of the equation $\Delta^m y_n = b_n$ there exists a solution x of (E) such that $x = y + o(1)$.*

Proof. Take $U = R$ in Theorem 1. □

Corollary 2. *If the series $\sum n^{m-1}a_n$ is absolutely convergent, and the family $\{\varphi_n\}$ is equicontinuous and bounded, then for any bounded full solution y of the equation $\Delta^m y = b$ there exists a full solution x of* (E) *such that $x = y + o(1)$.*

Proof. Choose $M > 0$ such that $|\varphi_n(t)| \le M$ for all $n \in N$, $t \in R$. In the proof of Theorem 1 we can choose V and c such that $c > M\rho_1$. Then we can take $p = 1$, $p_1 = n_0$. □

The next Corollary generalizes Theorem 1 of [7].

Corollary 3. *If the series $\sum n^{m-1}a_n$, $\sum n^{m-1}b_n$ are absolutely convergent, and the family $\{\varphi_n\}$ is equicontinuous and locally bounded, then for any $c \in R$ there exists a solution x of* (E) *such that* $\lim x_n = c$. *If moreover, the family $\{\varphi_n\}$ is bounded, then for every $c \in R$ there exists a full solution x of* (E) *such that* $\lim x_n = c$.

Proof. Let $c \in R$, $u = (-1)^m r^m b$, $y = c + u$. Then $u = o(1)$ and $\Delta^m y = \Delta^m c + \Delta^m u = \Delta^m u = b$. Hence y is a bounded solution of the equation $\Delta^m y = b$. By Corollary 1 there exists a solution x of (E) such that $x = y + o(1)$. Hence $x = c + u + o(1) = c + o(1)$. This means that $\lim x_n = c$. The second assertion follows from Corollary 2. □

Remark 2. Assume $f, g\colon R \to R$ are continuous functions and (α_n), (β_n) are bounded sequences of real numbers. Then the family $\{\varphi_n\}$ defined by $\varphi_n = \alpha_n f + \beta_n g$ is equicontinuous and locally bounded. Moreover if f ang g are uniformly continuous then the family $\{\varphi_n\}$ is uniformly equicontinuous.

Remark 3. Assume $f_0, f_1, \dots, f_{p-1}\colon R \to R$ are continuous functions. Then the family $\{\varphi_n\}$ defined by $\varphi_{jp+k} = f_k$ for $k = 0, 1, \dots, p-1$, $j = 0, 1, \dots$ is equicontinuous and locally bounded.

Example 1. Assume c_0, c_1, $c_2 \in R$. Let $b_0 = 9(c_0 - c_2)$, $b_1 = 9(c_1 - c_0)$, $b_2 = 9(c_0 - c_2)$ and let (c_n), (b_n) be defined by

$$c_0, c_1, c_2, c_0, c_1, c_2, \dots \qquad b_0, b_1, b_2, b_0, b_1, b_2, \dots$$

respectively. It is easy to see that for every $\alpha \in R$ the sequence $(\alpha + c_n)$ is a bounded solution of the equation $\Delta^5 y_n = b_n$. Hence, by Remark 2 and by Corrollary 1, for every $\alpha \in R$ there exists a solution x of the equation

$$\Delta^5 x_n = \frac{1}{n^6}\Big(\Big(1 + \frac{1}{n}\Big)e^{x_n} + \Big(\sin\frac{n\pi}{3}\Big)x_n^2\Big) + b_n$$

such that $\lim x_{3n} = \alpha + c_0$, $\lim x_{3n+1} = \alpha + c_1$, $\lim x_{3n+2} = \alpha + c_2$.

Theorem 2. *If the series $\sum n^{m-1}a_n$ is absolutely convergent, and the family $\{\varphi_n\}$ is bounded and uniformly equicontinuous, then for every full solution y of the equation $\Delta^m y_n = b_n$ there exists a full solution x of (E) such that $x = y + o(1)$.*

Proof. Let y be a full solution of the equation $\Delta^m y_n = b_n$. Choose $M > 0$ such that $|\varphi_n(t)| \le M$ for every $n \in N$ and every $t \in R$. Let $\rho = r^m|a|$,

$$T = \{x \in BS : |x| \le M\rho\}, \qquad S = \{x \in SQ : |x - y| \le M\rho\}.$$

Let $x \in S$, $u \in SQ$, $u_n = a_n\varphi_n(x_{\sigma(n)})$ for $n \ge n_0$, $A(x) = y + (-1)^m r^m u$. As in the proof of Theorem 1 one can show that there exists $x \in S$ such that $A(x) = x$. Then $x = y + (-1)^m r^m u = x + o(1)$ and $\Delta^m x = \Delta^m y + u = b + u$. Hence $\Delta^m x_n = b_n + a_n\varphi_n(x_{\sigma(n)})$ for $n \ge n_0$. Therefore x is a full solution of (E). □

The next Corollary generalizes Theorem 2 of [7].

Corollary 4. *If the family $\{\varphi_n\}$ is bounded and uniformly equicontinuous, and the series $\sum n^{m-1}a_n$, $\sum n^{m-1}b_n$ are absolutely convergent, then for every polynomial $\beta \in \mathrm{Pol}(m-1)$ there exists a full solution x of* (E) *such that $x = \beta + o(1)$.*

Proof. Let $\beta \in \mathrm{Pol}(m-1)$, $u = (-1)^m r^m b$, $y = \beta + u$. Then $u = o(1)$ and $\Delta^m y = \Delta^m \beta + \Delta^m u = \Delta^m u = b$. Hence y is a full solution of the equation $\Delta^m y = b$. By Theorem 2 there exists a full solution x of (E) such that $x = y + o(1)$. Hence $x = \beta + u + o(1) = \beta + o(1)$. □

Theorem 3. *Assume $\lambda \in R$, and the family $\{\varphi_n|[\lambda,\infty)\}$ is bounded and uniformly equicontinuous. If the series $\sum n^{m-1}a_n$, $\sum n^{m-1}b_n$ are absolutely convergent, then for every polynomial $\beta \in \mathrm{Pol}(m-1)$ such that $\lim \beta(n) = \infty$ there exists a solution x of (E) such that $x = \beta + o(1)$.*

Proof. Assume $\beta \in \mathrm{Pol}(m-1)$, $\lim \beta(n) = \infty$. Choose $M \geq 1$ such that $|\varphi_n(t)| \leq M$ for all $t \in [\lambda, \infty)$ and all $n \in N$. Let $\rho = r^m(|a| + |b|)$. Then $\rho = o(1)$. Choose $p \in N$ such that $\beta(n) \geq \lambda + M\rho_1$ for $n \geq p$. Let

$$T = \{x \in BS \,:\, x_n = 0 \quad \text{for} \quad n < p \quad \text{and} \quad |x_n| \leq M\rho_n \quad \text{for} \quad n \geq p\}\,.$$

$$S = \{x \in SQ \,:\, x_n = \beta(n) \quad \text{for} \quad n < p \quad \text{and} \quad |x_n - \beta(n)| \leq M\rho_n \quad \text{for} \quad n \geq p\}\,.$$

If $x \in S$, $n \geq p$ then $x_n \geq \beta(n) - M\rho_n \geq \lambda + M\rho_1 - M\rho_n \geq \lambda$. Choose $p_1 \geq p$ such that $\sigma(n) \geq p$ for any $n \geq p_1$. Then $x_{\sigma(n)} \geq \lambda$ for all $n \geq p_1$. Hence $|\varphi_n(x_{\sigma(n)})| \leq M$ for $n \geq p_1$. For $x \in SQ$ we define $\bar{x}$ by

$$\bar{x}_n = \begin{cases} 0 & \text{for} \quad n < n_0 \\ a_n\varphi_n(x_{\sigma(n)}) + b_n & \text{for} \quad n \geq n_0\,. \end{cases}$$

If $x \in S$ then $|\bar{x}| \leq M|a| + |b| \leq M(|a| + |b|)$. Hence, by Lemma 3, $|\bar{x}| \in S(m)$ for all $x \in S$. By Lemma 1, $\bar{x} \in S(m)$ for all $x \in S$. For $x \in S$ let

$$A(x)(n) = \begin{cases} \beta(n) & \text{for} \quad n < p_1 \\ \beta(n) + (-1)^m r_n^m \bar{x} & \text{for} \quad n \geq p_1\,. \end{cases}$$

Then $|A(x) - \beta| \leq |r^m \bar{x}| \leq r^m|\bar{x}|$. Hence, by Lemma 2,

$$|A(x) - \beta| \leq r^m(M(|a| + |b|)) = M\rho\,.$$

As in the proof of Theorem 1 one can show that there exists $x \in S$ such that $A(x) = x$. Then $x_n = \beta(n) + (-1)^m r_n^m \bar{x}$ for $n \geq p_1$. Hence $x = \beta + o(1)$ and

$$\Delta^m x_n = \Delta^m \beta(n) + \Delta^m((-1)^m r_n^m \bar{x}) = 0 + \bar{x}_n = a_n\varphi_n(x_{\sigma(n)}) + b_n$$

for $n \geq p_1$. The proof is complete. □

The proof of the following theorem is analogous to that of Theorem 3 and hence it is omitted.

Theorem 4. *Assume there exists $\lambda \in R$ such that the family $\{\varphi_n|(-\infty,\lambda]\}$ is bounded and uniformly equicontinuous. If the series $\sum n^{m-1}a_n$, $\sum n^{m-1}b_n$ are absolutely convergent, then for every polynomial $\beta \in \mathrm{Pol}(m-1)$ such that $\lim \beta(n) = -\infty$ there exists a solution x of (E) such that $x = \beta + o(1)$.*

The following corollary is an immediate consequence of Theorems 3 and 4.

Corollary 5. *Assume the series* $\sum n^{m-1}a_n$, $\sum n^{m-1}b_n$ *are absolutely convergent, and there exists* $\lambda \in R$ *such that the family* $\{\varphi_n|(-\infty,-\lambda]\cup[\lambda,\infty)\}$ *is bounded and uniformly equicontinuous. Then for every nonconstant polynomial* $\beta \in \mathrm{Pol}(m-1)$ *there exists a solution* x *of* (E) *such that* $x = \beta + o(1)$.

Theorem 5. *Assume the series* $\sum n^{m-1}a_n$, $\sum n^{m-1}b_n$ *are absolutely convergent and the family* $\{\varphi_n\}$ *is bounded. Then for every solution* x *of* (E) *there exist a polynomial* $\beta \in \mathrm{Pol}(m-1)$ *such that* $x = \beta + o(1)$. *If, moreover,* $\{\varphi_n\}$ *is uniformly equicontinuous, then for every polynomial* $\beta \in \mathrm{Pol}(m-1)$ *there exists a full solution* x *of* (E) *such that* $x = \beta + o(1)$.

Proof. Let x be a solution of (E). For $n \geq n_0$ let $u_n = a_n\varphi_n(x_{\sigma(n)})$. The series $\sum n^{m-1}a_n$ is absolutely convergent and there exists a constant M such that $|\varphi_n(x_{\sigma(n)})| \leq M$ for $n \geq n_0$. Hence the series $\sum n^{m-1}u_n$ is absolutely convergent. Therefore the series $\sum n^{m-1}(u_n + b_n)$ is also absolutely convergent. Let $z = (-1)^m r^m(u+b)$. Then, by definition of r^m, $z = o(1)$ and, by Lemma 5, $\Delta^m z = u + b$. Hence there exists $k \in N$ such that $\Delta^m z_n = \Delta^m x_n$ for all $n \geq k$. Let $v = \Delta^m x - \Delta^m z$. Then $v_n = 0$ for $n \geq k$. By Lemma 7, there exists a sequence $w \in SQ$ such that $\Delta^m w = v$ and $w_n = 0$ for $n \geq k$. Then $0 = \Delta^m x - \Delta^m z - v = \Delta^m x - \Delta^m z - \Delta^m w = \Delta^m(x - z - w)$. Hence $x - z - w = \beta$ for some $\beta \in \mathrm{Pol}(m-1)$. Then $x = \beta + z + w = \beta + o(1)$. The second assertion follows from Corollary 4. □

The next theorem generalizes Theorem 1 of [1].

Theorem 6. *Assume* U *is a neighborhood of some* $c \in R$, $\varepsilon \geq 0$, *the sequences* (a_n), (b_n) *are nonoscillatory and one of the following conditions holds*

(a) $a_nb_n \geq 0$, *for large* n, $\varphi_n(t) \geq \varepsilon$ *for large* n *and* $t \in U$,

(b) $a_nb_n \leq 0$, *for large* n, $\varphi_n(t) \leq -\varepsilon$ *for large* n *and* $t \in U$.

If there exists a solution x *of* (E) *such that* $\lim x_n = c$, *then the series* $\sum n^{m-1}b_n$ *is absolutely convergent. If moreover* $\varepsilon > 0$, *then the series* $\sum n^{m-1}a_n$ *is absolutely convergent.*

Proof. Assume the condition (a) is satisfied and $a_n \geq 0$ for all large n. The proof in other cases is similar and will be omitted. Let x be a solution of (E) such that $\lim x_n = c$. Then $x = c + z$ for some $z = o(1)$. Let $u = \Delta^m x$. Then, by Lemma 6, $u = \Delta^m(c+z) = \Delta^m z \in S(m)$. There exists $p \in N$ such that

$$u_n = a_n\varphi_n(x_{\sigma(n)}) + b_n\,, \qquad \varphi_n(x_{\sigma(n)}) \geq \varepsilon\,, \qquad a_n \geq 0\,, \qquad b_n \geq 0$$

for $n \geq p$. Let $h = |u| - u$. Then $h_n = 0$ for $n \geq p$. Hence, by Lemma 3, $h \in S(m)$. Therefore $|u| = u + h \in S(m)$. Hence by Lemma 3 the series $\sum n^{m-1}|u_n|$ is convergent. Since $|b_n| \leq |u_n|$ for large n, the series $\sum n^{m-1}|b_n|$ is convergent too. Now assume $\varepsilon > 0$. Then $a_n\varphi_n(x_{\sigma(n)}) \leq u_n$ for $n \geq p$. Hence $0 \leq a_n \leq \varepsilon^{-1}u_n$ for $n \geq p$. Therefore the series $\sum n^{m-1}|a_n|$ is convergent. □

Theorem 7. *Assume* $\lambda \in R$, $\varepsilon \geq 0$, $U = (\lambda, \infty)$, $(U = (-\lambda, -\infty))$ *the sequences* (a_n), (b_n) *are nonoscillatory and one of the following conditions holds*

(a) $\quad a_n b_n \ge 0$ *for large* n, $\varphi_n(t) \ge \varepsilon$ *for large* n *and* $t \in U$,

(b) $\quad a_n b_n \le 0$ *for large* n, $\varphi_n(t) \le -\varepsilon$ *for large* n *and* $t \in U$.

If there exists a solution x *of* (E) *and* $\beta \in \mathrm{Pol}(m-1)$ *such that*

$$x = \beta + o(1), \qquad \lim x_n = \infty \qquad (\lim x_n = -\infty),$$

then the series $\sum n^{m-1} b_n$ *is absolutely convergent. If moreover* $\varepsilon > 0$ *then the series* $\sum n^{m-1} a_n$ *is absolutely convergent.*

Proof. Assume the condition (a) is satisfied, $a_n \ge 0$ for all large n and $U = (\lambda, \infty)$. The proof in other cases is similar. Let x be a solution of (E) such that $x = \beta + z$ for some $\beta \in \mathrm{Pol}(m-1)$ and $z = o(1)$. Let $u = \Delta^m x$. Then $u = \Delta^m(\beta + z) = \Delta^m \beta + \Delta^m z = \Delta^m z \in S(m)$. The rest of the proof is the same as the second part of the proof of Theorem 6. $\square$

Remark 4. It is easy to see that if x is a convergent sequence such that $\Delta^m x_n \ge 0$ for $n \ge p$ then $(-1)^{m+k} \Delta^k x_n \ge 0$ for $k \in \{1, 2, \ldots, m\}$ and $n \ge p$.

Example 2. Assume $k \in N$, $m = 2k$, $a_n > 0$ for $n \in N$, the series $\sum n^{m-1} a_n$ is convergent, $c_0 \in R$,

$$\varphi(t) = \begin{cases} 1 & \text{for} \quad t \le c_0 \\ 0 & \text{for} \quad t > c_0 \, . \end{cases}$$

By Corollary 5 for every nonconstant polynomial $\beta \in \mathrm{Pol}(m-1)$ there exists a solution x of the equation

$$\text{(E1)} \qquad \Delta^m x_n = a_n \varphi(x_n)$$

such that $x = \beta + o(1)$. By Theorem 1 for every constant $c \ne c_0$ there exists a solution x of (E1) such that $\lim x_n = c$. We will show the equation (E1) has no solutions convergent to c_0. Assume x is a solution of (E1), $\lim x_n = c_0$. Then $\Delta^m x_n = a_n \varphi(x_n) \ge 0$ for large n. Hence, by Remark 4, there exists $p \in N$ such that $\Delta x_n \le 0$ for all $n \ge p$. Then $x_n \ge c_0$ for $n \ge p$. If $x_q = c_0$ for some $q \ge p$ then $x_n = c_0$ for all $n \ge q$. Hence $\Delta^m x_n = 0$ for $n \ge q$. On the other hand $\Delta^m x_n = a_n \varphi(x_n) = a_n > 0$ for $n \ge q$. This is impossible. Hence $x_n > c_0$ for all $n \ge p$. Therefore $\Delta^m x_n = a_n \varphi(x_n) = 0$ for $n \ge p$. Hence x is for large n a polynomial sequence convergent to c_0. It follows, $x_n = c_0$ for large n. As above it is impossible.

Example 3. Assume $a_n > 0$ for $n \in N$, and the series $\sum n^3 a_n$ is convergent. By Corollary 1, for every $c \in R$ there exists a solution x of the equation

$$\text{(E2)} \qquad \Delta^4 x_n = a_n x_n^2$$

such that $\lim x_n = c$. Let $A = 8 \inf\{a_n^{-1} : n \in N\}$. We will show that if $c \ge A$ then (E2) has no full solutions convergent to c. Assume x is a full solution of (E2) such that $\lim x_n = c \ge A$. Then $\Delta^4 x_n = a_n x_n^2$ for all $n \in N$. Hence $\Delta^4 x \ge 0$. By Remark 4, $\Delta x \le 0$. Therefore $x \ge c > 0$. There exists $p \in N$ such that $A = 8 a_p^{-1}$.

Then $ca_p \geq 8$ and

$$x_{p+4} - 4x_{p+3} + 6x_{p+2} - 4x_{p+1} + x_p = \Delta^4 x_p = a_p x_p^2 \,,$$

$$x_{p+4} = 4x_{p+3} - 6x_{p+2} + 4x_{p+1} - x_p + a_p x_p^2 > a_p x_p^2 - 6x_{p+2} - x_p \,.$$

Since $\Delta x \leq 0$, we have $x_{p+2} \leq x_p$. Hence $-6x_{p+2} \geq -6x_p$. Therefore

$$x_{p+4} > a_p x_p^2 - 6x_p - x_p = a_p x_p^2 - 7x_p = (a_p x_p - 7)x_p \,.$$

Since $a_p x_p \geq a_p c \geq 8$, we obtain $a_p x_p - 7 \geq 1$. Hence $x_{p+4} > x_p$. This contradicts the fact that x is nonincreasing.

References

[1] Drozdowicz, A., Popenda, J., *Asymptotic behavior of the solutions of an n-th order difference equations*, Comment. Math. Prace Mat. **29** (2) (1990), 161–168.

[2] Gleska, A., Werbowski, J., *Comparison theorems for the asymptotic behavior of solutions of nonlinear difference equations*, J. Math. Anal. Appl. **226** (2) (1998), 456–465.

[3] Li, Wan-Tong, Agarwal, R. P., *Positive solutions of higher-order nonlinear delay difference equations*, Comput. Math. Appl. **45** (6-7) (2003), 1203–1211.

[4] Migda, J., *Asymptotic properties of solutions of higher order difference equations*, submitted.

[5] Migda, J., *Asymptotically linear solutions of second order difference equations*, submitted.

[6] Migda, J., *Asymptotic behavior of solutions of nonlinear difference equations*, Math. Bohem. **129** (4) (2004), 349–359.

[7] Migda, M., Migda, J., *On the asymptotic behavior of solutions of higher order nonlinear difference equations*, Nonlinear Anal. **47** (7) (2001), 4687–4695.

[8] Migda, M., Migda, J., *Asymptotic properties of solutions of second-order neutral difference equations*, Nonlinear Anal. **63** (2005), 789–799.

[9] Wang, Z., Sun, J., *Asymptotic behavior of solutions of nonlinear higher-order neutral type difference equations*, J. Differ. Equations Appl. **12** (2006), 419–432.

[10] Zafer, A., *Oscillatory and asymptotic behavior of higher order difference equations*, Math. Comput. Modelling **21** (4) (1995), 43–50.

[11] Zafer, A., *Necessary and sufficient condition for oscillation of higher order delay difference equations*, Comput. Math. Appl. **35** (10) (1998), 125–130.

[12] Zhang, B., Sun, Y., *Classification of nonoscillatory solutions of a higher order neutral difference equation*, J. Differ. Equations Appl. **8** (11) (2002), 937–955.

Faculty of Mathematics and Computer Science
A. Mickiewicz University
ul. Umultowska 87, 61-614 Poznań, Poland
E-mail: migda@amu.edu.pl