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MINIMAL AND MAXIMAL SOLUTIONS OF FOURTH ORDER ITERATED DIFFERENTIAL EQUATIONS WITH SINGULAR NONLINEARITY

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Abstract. In this paper we are concerned with sufficient conditions for the existence of minimal and maximal solutions of differential equations of the form

$$L_4 y + f(t,y) = 0\,,$$

where $L_4 y$ is the iterated linear differential operator of order 4 and $f\colon [a,\infty)\times(0,\infty)\to(0,\infty)$ is a continuous function.

1. Introduction

The purpose of this paper is to study the existence of positive solutions with specific asymptotic behavior for differential equations of the form

$$L_4 y + f(t,y) = 0\,,$$

where $L_4 y$ is the iterated linear differential operator of fourth order defined below and $f\colon [a,\infty)\times(0,\infty)\to(0,\infty)$ is a continuous function, nonincreasing in the second variable.

A prototype of such equations is the equation with singular nonlinearity $f(t,y) = Q(t)y^{-\lambda}$, where $\lambda > 0$ and $Q\colon [a,\infty)\to(0,\infty)$ is continuous. Such equations of the second order were studied in [3], [4].

Differential equations with iterated linear differential operator were studied, for instance, in [5].

2. Iterated differential equations of the fourth order

If u and v are linearly independent solutions of

$$y'' + P(t)y = 0\,, \tag{A$_2$}$$

where $P\in C^2[a,\infty)$, then u, $v\in C^4[a,\infty)$ and the linearly independent functions

$$y_1(t) = u^3(t)\,,\quad y_2(t) = u^2(t)v(t)\,,\quad y_3(t) = u(t)v^2(t)\,,\quad y_4(t) = v^3(t)$$

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satisfy the fourth order linear differential equation

$$(\mathrm{A}_4) \qquad y^{IV} + 10P(t)y'' + 10P'(t)y' + [3P''(t) + 9P^2(t)]y = 0\,,$$

(see [1]). Differential equation (A_4) is called *iterated* linear differential equation of the fourth order.

We may suppose without loss of generality that

$$W[u,v](t) \equiv 1 \quad \text{for} \quad t \geq a\,,$$

where $W[u,v](t)$ denotes Wronskian of functions u and v. An elementary calculation shows that Wronskian of functions

$$y_1(t) = u^3(t)\,, \quad y_2(t) = u^2(t)v(t)\,, \quad y_3(t) = u(t)v^2(t)\,, \quad y_4(t) = v^3(t)$$

satisfies

$$W(u^3, u^2v, uv^2, v^3)(t) \equiv 12 \quad \text{for} \quad t \geq a\,.$$

We suppose that the equation (A_2) is nonoscillatory and the $u(t)$ (resp. $v(t)$) denote a principal (resp. nonprincipal) solution of (A_2) such that

$$\lim_{t\to\infty} \frac{u(t)}{v(t)} = 0$$

and

$$\int^\infty \frac{dt}{u^2(t)} = \infty \quad (\text{resp.} \quad \int^\infty \frac{dt}{v^2(t)} < \infty)\,.$$

We may assume that both $u(t)$ and $v(t)$ are eventually positive. Second, nonprincipal $v(t)$ of (A_2) is given by

$$v(t) = u(t)\int_{t_0}^t \frac{ds}{u^2(s)}\,, \quad t \geq t_0\,.$$

In this paper we are concerned with the behavior of solutions of differential equations of the form

$$L_4y + f(t,y) = 0\,,$$

where L_4y is the iterated linear differential operator of order 4 and $f\colon [a,\infty)\times(0,\infty) \to (0,\infty)$ is a continuous function.

From Pólya's factorization theory it follows that the operator L_4y can be written in the form

$$L_4y = a_4(t)\big(a_3(t)\big(a_2(t)\big(a_1(t)\big(a_0(t)y\big)'\big)'\big)'\big)',$$

where

$$a_0(t) = \frac{1}{u^3(t)}\,, \quad a_1(t) = u^2(t)\,, \quad a_2(t) = \frac{u^2(t)}{2}\,, \quad a_3(t) = \frac{u^2(t)}{3}\,, \quad a_4(t) = \frac{6}{u^3(t)}\,,$$

see [6].

3. Classification of positive solutions

Consider the fourth order differential equation

$$L_4 y(t) + f\big(t, y(t)\big) = 0 \tag{A}$$

where $L_4 y$ is the iterated linear differential operator of order 4 and $f\colon [a,\infty) \times (0,\infty) \to (0,\infty)$ is continuous, and nonincreasing in the second variable.

We assume that the equation (A_2) is nonoscillatory and put

$$L_0 y(t) = \frac{y(t)}{u^3(t)}\,,$$

$$L_i y(t) = \frac{u^2(t)}{i}\frac{d}{dt}\big(L_{i-1} y(t)\big)\,, \qquad 1 \le i \le 3\,,$$

and

$$L_4 y = \frac{6}{u^3(t)}\Big(\frac{u^2(t)}{3}\Big(\frac{u^2(t)}{2}\Big(u^2(t)\Big(\frac{1}{u^3(t)}y\Big)'\Big)'\Big)'\Big)'.$$

The domain $\mathcal{D}(L_4)$ of the operator L_4 is defined to be the set of all continuous functions $y\colon [T_y,\infty) \to (0,\infty)$, $T_y \ge a$ such that $L_i y(t)$ for $0 \le i \le 3$ exist and are continuously differentiable on $[T_y,\infty)$.

Those functions which vanish in a neighborhood of infinity will be excluded from our consideration.

Our purpose here is to make a detailed analysis of the structure of the set of all possible positive solutions of the equation (A). We use the following lemma which is the special case of generalized Kiguradze's lemma (see [2]).

Lemma 1. *If $y(t)$ is a positive solution of* (A), *then either*

$$L_0 y(t) > 0\,, \quad L_1 y(t) > 0\,, \quad L_2 y(t) < 0\,, \quad L_3 y(t) > 0\,, \quad L_4 y(t) < 0\,, \tag{1}$$

or

$$L_0 y(t) > 0\,, \quad L_1 y(t) > 0\,, \quad L_2 y(t) > 0\,, \quad L_3 y(t) > 0\,, \quad L_4 y(t) < 0\,, \tag{2}$$

for all sufficiently large t.

Solutions satisfying (1) and (2) are called solutions of *Kiguradze's degree* 1 and 3, respectively. If we denote by P the set of all positive solution of (A) and by P_l the set of all solution of degree l, then we have:

$$P = P_1 \cup P_3\,.$$

Consider P_l for $l = \{1,3\}$. For any $y \in P_l$ the limits

$$\lim_{t\to\infty} L_l y(t) = c_l \qquad \text{(finite)},$$

$$\lim_{t\to\infty} L_{l-1} y(t) = c_{l-1} \qquad \text{(finite or infinite but not zero)}$$

both exist.

Solution $y \in P_l$ is called a *maximal in P_l*, if c_l is nonzero and a *minimal in P_l*, if c_{l-1} is finite. The set of all maximal solutions in P_l denote $P_l[\max]$ and the set of all minimal solutions in P_l denote $P_l[\min]$.

If $c_l = 0$ for a solutions $y \in P_l$ for $l \in \{1,3\}$, then y is called *intermediate in* P_l. The set of all intermediate solutions in P_l denote $P_l[\text{int}]$.
Then

$$P = P_1[\min] \cup P_1[\text{int}] \cup P_1[\max] \cup P_3[\min] \cup P_3[\text{int}] \cup P_3[\max]\,.$$

Our objective is to give sufficient conditions for the existence of maximal and minimal solutions in P_i for $i = 1, 3$.

Crucial role will be played by integral representations for those fourth types of solutions of (A) as derived below.
First we define: $I_0 = 1$ and

$$I_i(t,s;u) = \int_s^t \frac{1}{u^2(r)} I_{i-1}(r,s;u)\,dr\,, \quad 1 \le i \le 3\,.$$

If the second, linearly independent solution $v(t)$ of (A_2) is given by

$$v(t) = u(t)\int_{t_0}^t \frac{ds}{u^2(s)} \quad \text{for} \quad t \ge t_0\,, \quad \text{then the set of positive functions}$$

$$\begin{aligned}
x_0(t) &= u^3(t)\,,\\
x_1(t) &= u^3(t)\int_{t_0}^t \frac{1}{u^2(s)}\,ds = u^2(t)\,v(t)\,,\\
x_2(t) &= u^3(t)\int_{t_0}^t \frac{1}{u^2(s)}\int_{t_0}^s \frac{2}{u^2(r)}\,dr\,ds = u(t)\,v^2(t)\,,\\
x_3(t) &= u^3(t)\int_{t_0}^t \frac{1}{u^2(s)}\int_{t_0}^s \frac{2}{u^2(r)}\int_{t_0}^r \frac{3}{u^2(\xi)}\,d\xi\,dr\,ds = v^3(t)
\end{aligned}$$

defined on $[t_0,\infty)$ form fundamental set of positive solutions for $L_4x = 0$ (i.e. (A_4), which are asymptotically ordered in the sense that

$$\lim_{t\to\infty} \frac{x_i(t)}{x_j(t)} = 0$$

for $0 \le i < j \le 3$, see [2]. It is useful to note that

$$I_i(t,a;u) = \frac{1}{i!}\Big(\frac{v(t)}{u(t)}\Big)^i \quad \text{for} \quad i = 1,2,3\,.$$

The solutions from the classes $P_3[\max]$, $P_3[\min]$, $P_1[\max]$ and $P_1[\min]$ satisfy the properties

$$\lim_{t\to\infty} \frac{y(t)}{v^3(t)} = \lambda_3\,, \quad \lim_{t\to\infty} \frac{y(t)}{u(t)v^2(t)} = \lambda_2\,, \quad \lim_{t\to\infty} \frac{y(t)}{u^2(t)v(t)} = \lambda_1$$

$$\text{and} \quad \lim_{t\to\infty} \frac{y(t)}{u^3(t)} = \lambda_0\,, \quad \text{respectively, where } 0 < \lambda_i < \infty,\ i = 1,2,3.$$

4. Integral representations for solutions

Now we can derive integral representations for types $P_3[\max]$, $P_3[\min]$, $P_1[\max]$ and $P_1[\min]$.

Let y be a solution of (A) such that $y(t) > 0$ for $t \geq T \geq a$. Integrating (A) from t to ∞ gives

$$L_3 y(t) = c_3 + \int_t^\infty \frac{u^3(s)}{6}\, f\big(s, y(s)\big)\, ds, \quad t \geq T\,, \tag{3}$$

where $c_3 = \lim_{t\to\infty} L_3 y(t) \geq 0$.

If $y \in P_3[\max]$, then we integrate (3) three times over $[T, t]$ to obtain

$$\begin{aligned} y(t) = {} & k_0 u^3(t) + k_1 u^3(t) \int_T^t \frac{1}{u^2(s)}\, ds + 2k_2 u^3(t) \int_T^t \frac{1}{u^2(s_1)} \int_T^{s_1} \frac{1}{u^2(s_2)} ds_2\, ds_1 \\ & + 6c_3 u^3(t) \int_T^t \frac{1}{u^2(s_1)} \int_T^{s_1} \frac{1}{u^2(s_2)} \int_T^{s_2} \frac{1}{u^2(s_3)}\, ds_3\, ds_2\, ds_1 \\ & + u^3(t) \int_T^t I_2(t, s; u) \frac{1}{u^2(s)} \int_s^\infty u^3(r)\, f\big(r, y(r)\big)\, dr\, ds\,, \end{aligned}$$

for $t \geq T$, where $k_i = L_i y(T)$ for $i = 0, 1, 2$ and we used Fubini theorem.

If y is a solution of type $P_3[min]$, then integrating (3) with $c_3 = 0$ from t to ∞ and then integrating the resulting equation twice from T to t, we have

$$\begin{aligned} y(t) = {} & k_0 u^3(t) + k_1 u^3(t) \int_T^t \frac{1}{u^2(s_1)}\, ds_1 + 2c_2 u^3(t) \int_T^t \frac{1}{u^2(s_1)} \int_T^{s_1} \frac{1}{u^2(s_2)}\, ds_2\, ds_1 \\ & - u^3(t) \int_T^t I_1(t, s; u) \frac{1}{u^2(s)} \int_s^\infty I_1(r, s; u)\, u^3(r)\, f\big(r, y(r)\big)\, dr\, ds\,, \end{aligned}$$

for $t \geq T$, where $c_2 = \lim_{t\to\infty} L_2 y(t)$.

An integral representation for a solution y of type $P_1[\max]$ is derived by integrating (3) with $c_3 = c_2 = 0$ twice from t to ∞ and once on $[T, t]$

$$\begin{aligned} y(t) = {} & k_0 u^3(t) + c_1 u^3(t) \int_T^t \frac{1}{u^2(s_1)}\, ds_1 \\ & + u^3(t) \int_T^t \frac{1}{u^2(s)} \int_s^\infty I_2(r, s; u) u^3(r)\, f\big(r, y(r)\big)\, dr\, ds\,, \end{aligned}$$

for $t \geq T$, where $c_1 = \lim_{t\to\infty} L_1 y(t)$ and we used Fubini theorem.

If $y \in P_1[\min]$, then integrations of (3) with $c_3 = c_2 = c_1 = 0$ three times on (t, ∞) yield

$$y(t) = c_0 u^3(t) - u^3(t) \int_t^\infty I_3(s, t; u) u^3(s)\, f\big(s, y(s)\big)\, ds\,,$$

for $t \geq T$, where $c_0 = \lim_{t\to\infty} L_0 y(t)$.

5. Existence theorems

We are now prepared to discuss the existence of maximal and minimal solutions of equation (A) of type P_1 and P_3.

Theorem 1. *The equation* (A) *has a positive solution of types* $P_3[\max]$ *if*

$$\int^{\infty} u^3(t)\, f\big(t, cv^3(t)\big)\, dt < \infty\,, \tag{4}$$

for some $c > 0$.

Proof. We assume that (4) holds. Then there is $T \geq a$ such that

$$\int_T^{\infty} u^3(t)\, f\big(t, cv^3(t)\big)\, dt < c\,.$$

Let C denote locally convex space of all continuous functions $y\colon [T,\infty) \to R$ with the topology of uniform convergence on compact subintervals of $[T,\infty)$.

Define the subset Y_3 of $C[T,\infty)$ and mapping $\Phi_3\colon Y_3 \to C[T,\infty)$ by

$$Y_3 = \{y \in C[T,\infty) : cv^3(t) \leq y(t) \leq 2cv^3(t), t \geq T\}$$

and

$$\Phi_3 y(t) = cv^3(t) + u^3(t) \int_T^t I_2(t,s;u) \frac{1}{u^2(s)} \int_s^{\infty} u^3(r)\, f\big(r, y(r)\big)\, dr\, ds\,.$$

We will show that (i): Φ_3 maps Y_3 into Y_3, (ii): Φ_3 is continuous on Y_3, (iii): $\Phi_3(Y_3)$ is relatively compact.

(i) Since

$$\begin{aligned} 0 &\leq u^3(t) \int_T^t I_2(t,s;u) \frac{1}{u^2(s)} \int_s^{\infty} u^3(r)\, f\big(r, y(r)\big)\, dr\, ds \\ &\leq u^3(t) \int_T^t I_2(t,s;u) \frac{1}{u^2(s)} \int_s^{\infty} u^3(r)\, f\big(r, cv^3(r)\big)\, dr\, ds\,, \end{aligned}$$

then

$$\Phi_3 y(t) \geq cv^3(t)$$

and

$$\begin{aligned} \Phi_3 y(t) &\leq cv^3(t) + u^3(t) \int_T^t I_2(t,s;u) \frac{1}{u^2(s)} \int_s^{\infty} u^3(r)\, f\big(r, cv^3(r)\big)\, dr\, ds \\ &\leq cv^3(t) + cu^3(t) \int_T^t I_2(t,s;u) \frac{1}{u^2(s)}\, ds \\ &\leq cv^3(t) + cv^3(t) = 2cv^3(t)\,. \end{aligned}$$

And so $\Phi_3 y \in Y_3$.

(ii) Suppose that $\{y_n\} \subset Y_3$ and $y \in Y_3$, and that $\lim_{n\to\infty} y_n = y$ in the topology of $C[T,\infty)$. We have

$$\begin{aligned}\big|\Phi_3 y_n(t) - \Phi_3 y(t)\big| &\le u^3(t)\int_T^t I_2(t,s;u)\frac{1}{u^2(s)} \\ &\qquad \times \int_s^\infty u^3(r)\,\big|f\big(r,y_n(r)\big) - f\big(r,y(r)\big)\big|\,dr\,ds \\ &\le 6v^3(t)\int_T^\infty u^3(s)\big|f\big(s,y_n(s)\big) - f\big(s,y(s)\big)\big|\,ds\end{aligned}$$

Because

$$\big|f\big(s,y_n(s)\big) - f\big(s,y(s)\big)\big| \le 2f\big(s,cv^3(s)\big)$$

and $\lim_{t\to\infty} |f(s,y_n(s)) - f(s,y(s))| = 0$ for $s \ge T$, then applying the Lebesgue convergence theorem, we have $|\Phi_3 y_n(t) - \Phi_3 y(t)| \to 0$ for $n \to \infty$ on every compact subinterval of $[T,\infty]$, which implies that $\Phi_3 y$ is continuous on Y_3.

(iii) If $y \in Y_3$, then we have for $t \in (T,\infty)$

$$\Big|\frac{d}{dt}\big(\frac{1}{u^3(t)}\Phi_3 y(t)\big)\Big| \le 2c\frac{1}{u^2(t)} I_2(t,a;u)\,.$$

This shows that the function $\frac{d}{dt}\big(\frac{1}{u^3(t)}\Phi_3 y(t)\big)$ is uniformly bounded on any compact subinterval of $[T,\infty)$, and so function $\frac{1}{u^3(t)}\Phi_3 y(t)$ is equicontinuous on (T,∞).

Now for $t_1, t_2 \in [T,\infty)$ we see that

$$\begin{aligned}\big|\Phi_3 y(t_2) - \Phi_3 y(t_1)\big| &\le \big|u^3(t_2) - u^3(t_1)\big|\Big|\frac{1}{u^3(t_2)}\Phi_3 y(t_2)\Big| \\ &\qquad + u^3(t_1)\Big|\frac{1}{u^3(t_2)}\Phi_3 y(t_2) - \frac{1}{u^3(t_1)}\Phi_3 y(t_1)\Big| \\ &\le 2c\frac{1}{u^3(t_2)} v^3(t_2)\big|u^3(t_2) - u^3(t_1)\big| \\ &\qquad + u^3(t_1)\Big|\frac{1}{u^3(t_2)}\Phi_3 y(t_2) - \frac{1}{u^3(t_1)}\Phi_3 y(t_1)\Big|,\end{aligned}$$

and hence $\Phi_3(Y_3)$ is equicontinuous at every point of $[T,\infty)$. Since $\Phi_3(Y_3)$ is clearly uniformly bounded on $[T,\infty)$, it follow from Ascoli-Arzèl theorem that $\Phi_3(Y_3)$ is relatively compact.

Therefore, by the Schauder-Tychonoff fixed point theorem, there exists a fixed element $y \in Y_3$ of Φ_3, i.e. $\Phi_3 y = y$, which satisfies the integral equation

$$y(t) = cv^3(t) + u^3(t)\int_T^t I_2(t,s;u)\frac{1}{u^2(s)}\int_s^\infty u^3(r)\,f\big(r,y(r)\big)\,dr\,ds\,.$$

A simple computation shows that this fixed point is a solution of (A) of type $P_3[\max]$. The proof of Theorem 1 is complete. □

Theorem 2. *The equation* (A) *has a positive solution of type* $P_3[\min]$ *if*

$$\int_a^\infty u^2(t)v(t)\,f\big(t,cu(t)v^2(t)\big)\,dt < \infty, \tag{5}$$

for some $c > 0$.

Proof. Suppose that (5) holds. Choose $T \geq a$ so that

$$\int_T^\infty u^2(t)v(t)\, f\big(t, cu(t)v^2(t)\big)\, dt < c\,. \tag{6}$$

Consider the set Y_2 functions $y \in C[T,\infty)$ and mapping $\Psi_3\colon Y_3 \to C[T,\infty)$ defined by

$$Y_2 = \{y \in C[T,\infty) : cu(t)v^2(t) \leq y(t) \leq 2cu(t)v^2(t), t \geq T\}$$

and

$$\Psi_3 y(t) = 2cu(t)v^2(t) - u^3(t)e \int_T^t I_1(t,s;u)\frac{1}{u^2(s)} \times \int_s^\infty I_1(r,s;u)u^3(r)\, f\big(r, y(r)\big)\, dr\, ds\,.$$

That $\Psi_3(Y_2) \subset Y_2$ is an immediate consequence of (6). Since the continuity of Ψ_3 and the relative compactness of $\Psi_3(Y_2)$ can be proved as in the proof of Theorem 1, there exists an element $y \in Y_2$ such that $\Psi_3 y = y$, which satisfies

$$y(t) = 2cu(t)v^2(t) - u^3(t) \int_T^t I_1(t,s;u)\frac{1}{u^2(s)} \times \int_s^\infty I_1(r,s;u)u^3(r)\, f\big(r, y(r)\big)\, dr\, ds$$

for $t \geq T$. It is easy to verify that this fixed point is a solution of degree 3 of (A) such that $\lim\limits_{t\to\infty} L_2 y(t) = c_2$ exists and is finite and nonzero. This completes the proof. □

Theorem 3. *The equation* (A) *has a positive solution of type* $P_1[\max]$ *if*

$$\int_a^\infty u(t)v^2(t)\, f\big(t, cu^2(t)v(t)\big)\, dt < \infty\,, \tag{7}$$

for some $c > 0$.

Proof. Suppose that (7) holds. Take $T \geq a$ so large that

$$\int_T^\infty u(t)v^2(t)\, f\big(t, cu^2(t)v(t)\big)\, dt < c\,.$$

Consider a closed convex subset Y_1 of $C[T,\infty)$ defined by a

$$Y_1 = \{y \in C[T,\infty) : cu^2(t)v(t) \leq y(t) \leq 2cu^2(t)v(t), t \geq T\}\,.$$

Define the operator $\Phi_1\colon Y_1 \to C[T,\infty)$by the following formula

$$\Phi_1 y(t) = cu^2(t)\, v(t) + u^3(t) \int_T^t \frac{1}{u^2(s)} \int_s^\infty I_2(r,s;u)u^3(r)\, f\big(r, y(r)\big)\, dr\, ds\,.$$

Again we can show that (i) $\Phi_1(Y_1) \subset Y_1$, (ii) Φ_1 is a continuous operator and (iii) $\Phi_1(Y_1)$ is relatively compact.

Therefore, Φ_1 has a fixed point $y \in Y_1$, which gives rise to a type $P_1[\max]$ solution of (A) since it satisfies

$$y(t) = cu^2(t)\,v(t) + u^3(t)\int_T^t \frac{1}{u^2(s)}\int_s^\infty I_2(r,s;u)u^3(r)\,f\big(r,y(r)\big)\,dr\,ds$$

for $t \ge T$. Note that $\lim_{t\to\infty} L_1 y(t) = c$. The proof is thus complete. □

Theorem 4. *The equation* (A) *has positive solution of type* $P_1[\min]$ *if*

$$\int_a^\infty v^3(t)\,f\big(t, cu^3(t)\big)\,dt < \infty\,, \tag{8}$$

for some $c > 0$.

Proof. Suppose now that (8) holds. There exists a constant $T \ge a$ such that

$$\int_T^\infty v^3(t)\,f\big(t, cu^3(t)\big)\,dt < c\,.$$

Define the mapping Ψ_1 by

$$\Psi_1 y(t) = 2cu^3(t) - u^3(t)\int_t^\infty I_3(s,t;u)u^3(s)\,f\big(s,y(s)\big)\,ds\,.$$

Then, it can be verified without difficulty that Ψ_1 has a fixed element y in the set

$$Y_0 = \{y \in C[T,\infty) : cu^3(t) \le y(t) \le 2cu^3(t), t \ge T\}\,.$$

This fixed point gives rise to a required positive solution of (A), since it satisfies

$$y(t) = 2cu^3(t) - u^3(t)\int_t^\infty I_3(s,t;u)u^3(s)\,f\big(s,y(s)\big)\,ds\,.$$

Note that $\lim_{t\to\infty} L_0 y(t) = 2c$. This completes the proof. □

6. Special case and example

We consider equation (A) with special function $f(t,y) = Q(t)y^{-\lambda}$

$$L_4 y(t) + Q(t)y^{-\lambda} = 0\,, \tag{B}$$

where $\lambda > 0$ and $Q\colon [a,\infty) \to (0,\infty)$ is continuous. The objective of this section is to use above theorems to establish sufficient conditions for equation (B) to have solutions $x_i(t)$, $i = 1,2,3,4$ defined in some neighborhood of infinity with the same asymptotic behavior as

$$x_i(t) = u^{4-i}(t)v^{i-1}(t)\,, \qquad 1 \le i \le 4\,,$$

respectively, as $t \to \infty$. We write they as corollaries, where the symbol $\sim$ is used to denote the asymptotic equivalence

$$f(t) \sim g(t) \quad \text{as} \quad t \to \infty \quad \Leftrightarrow \quad \lim_{t\to\infty}\frac{f(t)}{g(t)} = 1\,.$$

Corollary 1. *A sufficient condition for* (B) *to have a positive solution* $y_4(t)$ *which satisfies*

$$y_4(t) \sim m v^3(t)$$

for some $m > 0$ *is that*

$$\int_a^\infty u^3(t)\, v^{-3\lambda}(t)\, Q(t)\, dt < \infty \,.$$

Corollary 2. *A sufficient condition for* (B) *to have a positive solution* $y_3(t)$ *which satisfies*

$$y_3(t) \sim m u(t)\, v^2(t)$$

for some $m > 0$ *is that*

$$\int_a^\infty u^{2-\lambda}(t)\, v^{1-2\lambda}(t)\, Q(t)\, dt < \infty \,.$$

Corollary 3. *A sufficient condition for* (B) *to have a positive solution* $y_2(t)$ *which satisfies*

$$y_2(t) \sim m u^2(t)\, v(t)$$

for some $m > 0$ *is that*

$$\int_a^\infty u^{1-2\lambda}(t)\, v^{2-\lambda}(t)\, Q(t)\, dt < \infty \,.$$

Corollary 4. *A sufficient condition for* (B) *to have a positive solution* $y_1(t)$ *which satisfies*

$$y_1(t) \sim 2m u^3(t)$$

for some $m > 0$ *is that*

$$\int_a^\infty u^{-3\lambda}(t)\, v^3(t)\, Q(t)\, dt < \infty \,.$$

We present here an example which illustrates theorems proved above and the corollaries.

Example. Consider the nonoscillatory linear differential equation of the second order

$$x'' + \frac{1}{4t^2} x = 0 \,, \quad t \geq 1 \,.$$

We know, that this equation has principal solution

$$u(t) = t^{\frac{1}{2}}$$

and nonprincipal solution

$$v(t) = t^{\frac{1}{2}} \ln t$$

and that the iterated equation

$$x^{IV} + \frac{5}{2t^2} x'' - \frac{5}{t^3} x' + \frac{81}{16t^4} x = 0 \,,$$

has independent solutions in the form

$$x_1(t) = t^{\frac{3}{2}} \,, \quad x_2(t) = t^{\frac{3}{2}} \ln t \,, \quad x_3(t) = t^{\frac{3}{2}} \ln^2 t \,, \quad x_4(t) = t^{\frac{3}{2}} \ln^3 t \,.$$

Then equation

$$y^{IV} + \frac{5}{2t^2}y'' - \frac{5}{t^3}y' + \frac{81}{16t^4}y + Q(t)y^{-\lambda} = 0\,, \qquad t \geq 1\,,$$

where $\lambda > 0$ and $Q\colon [1,\infty) \to (0,\infty)$ is continuous, has positive regular solution

a) $y_1(t)$ satisfying $y_1(t) \sim mt^{\frac{3}{2}}$ if $\int\limits_a^\infty t^{-\frac{3}{2}(\lambda-1)}(\ln t)^3 Q(t)\,dt < \infty$,

b) $y_2(t)$ satisfying $y_2(t) \sim mt^{\frac{3}{2}}\ln t$ if $\int\limits_a^\infty t^{-\frac{3}{2}(\lambda-1)}(\ln t)^{2-\lambda} Q(t)\,dt < \infty$,

c) $y_3(t)$ satisfying $y_3(t) \sim mt^{\frac{3}{2}}\ln^2 t$ if $\int\limits_a^\infty t^{-\frac{3}{2}(\lambda-1)}(\ln t)^{1-2\lambda} Q(t)\,dt < \infty$,

d) $y_4(t)$ satisfying $y_4(t) \sim mt^{\frac{3}{2}}\ln^3 t$ if $\int\limits_a^\infty t^{-\frac{3}{2}(\lambda-1)}(\ln t)^{-3\lambda} Q(t)\,dt < \infty$

for some $m > 0$.

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