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SUBMANIFOLDS WITH HARMONIC MEAN CURVATURE IN PSEUDO-HERMITIAN GEOMETRY

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Abstract. We classify Hopf cylinders with proper mean curvature vector field in Sasakian 3-manifolds with respect to the Tanaka-Webster connection.

Introduction

The harmonicity equation $\Delta\mathbb{H} = 0$ for the mean curvature vector field $\mathbb{H}$ of an immersed submanifold $x\colon M^m \to \mathbb{E}^n$ in Euclidean n-space is equivalent to the biharmonicity of the immersion: $\Delta\Delta x = 0$, since $\Delta x = -m\mathbb{H}$.

A submanifold $x\colon M \to \mathbb{E}^n$ is said to be a *biharmonic submanifold* if $\Delta\mathbb{H} = 0$. In 1985, B. Y. Chen proved the nonexistence of proper biharmonic surfaces in Euclidean 3-space. Chen conjectured that biharmonic submanifolds in Euclidean space are harmonic, i.e., minimal. Some partial and positive answers have been obtained by several authors [7]–[9], [11]–[12].

The biharmonicity equation is regarded as a special case of the following condition:

$$\Delta\mathbb{H} = \lambda\mathbb{H}\,, \qquad \lambda \in \mathbb{R}\,.$$

Namely the mean curvature vector field is an eigenvector field of the Laplacian. Submanifolds satisfying the condition $\Delta\mathbb{H} = \lambda\mathbb{H}$ are called *submanifolds with proper mean curvature vector field*.

The study of Euclidean submanifolds with proper mean curvature vector field was initiated by Chen in 1988 (see [4]). It is known that submanifolds in $\mathbb{E}^n$ satisfying $\Delta\mathbb{H} = \lambda\mathbb{H}$ are either biharmonic ($\lambda = 0$), of 1-type or null 2-type. In particular all surfaces in $\mathbb{E}^3$ with $\Delta\mathbb{H} = \lambda\mathbb{H}$ are of constant mean curvature. Moreover a surface in $\mathbb{E}^3$ satisfies $\Delta\mathbb{H} = \lambda\mathbb{H}$ if and only if it is minimal, an open portion of a totally umbilical sphere or an open portion of a circular cylinder. I. Dimitrić [9] obtained some nonexistence theorem for biharmonic submanifolds in Euclidean space. Th. Hasanis and Th. Vlachos [12] obtained the nonexistence of proper biharmonic hypersurfaces in $\mathbb{E}^4$. F. Defever [7] gave an alternative proof to Hasanis–Vlachos' result.

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Defever [6] showed that hypersurfaces satisfying $\Delta\mathbb{H} = \lambda\mathbb{H}$ are of constant mean curvature. Note that Chen [2] studied submanifolds with $\Delta\mathbb{H} = \lambda\mathbb{H}$ in hyperbolic space. On the other hand, M. Barros and O. J. Garay [1] showed that Hopf cylinders in the unit 3-sphere S^3 with $\Delta\mathbb{H} = \lambda\mathbb{H}$ are Hopf cylinders over circles in the 2-sphere S^2. Thus the only Hopf cylinders with proper mean curvature vector field are Hopf tori of constant mean curvature. In particular, the only Hopf cylinders in S^3 with harmonic mean curvature vector field are Clifford tori.

A. Ferrández, P. Lucas and M. A. Meroño [10] studied Hopf cylinders with proper mean curvature in anti de Sitter 3-space H^3_1 with respect to the fibration $H^3_1 \to H^2(-4)$.

Here we would like to point out that the 3-sphere and anti de Sitter 3-space are typical examples of homogeneous contact semi-Riemannian manifolds. In particular both spaces are 3-dimensional semi-Riemannian Sasakian space forms.

A contact semi-Riemannian 3-manifold M is said to be regular if its characteristic vector field is complete and its flow acts simple transitively and isometrically on M. Then there exits a Riemannian fibration $\pi\colon M \to M/\xi$. By using this fibration, one can extend the notion of Hopf cylinder in S^3 and H^3_1 to that in regular contact semi-Riemannian 3-manifolds.

In [13], the first named author investigated curves and surfaces with proper mean curvature vector field in 3-dimensional Sasakian space forms with respect to the Levi-Civita connection. More precisely, Legendre curves and Hopf cylinders with proper mean curvature vector field in 3-dimensional Sasakian space forms.

On the other hand, contact Riemannian 3-manifolds admit strongly pseudo-convex pseudo-Hermitian structure associated to the contact Riemannian structure. From the viewpoint of pseudo-Hermitian structure, it is natural to use the Tanaka-Webster connection instead of Levi-Civita connection.

In [17], the second named author studied Legendre curves in contact Riemannian 3-manifolds whose mean curvature vector filed is proper with respect to the Tanaka-Webster connection.

As a continuation to the previous work [17], in the present paper, we classify Hopf cylinders with proper mean curvature vector field in regular Sasakian 3-manifolds with respect to the Tanaka-Webster connection.

1. Pseudo-Hermitian geometry

1.1. **Contact Riemannian manifolds.** A smooth 3-manifold M is called a *contact manifold*, if it admits a global 1-form η such that $\eta \wedge \mathrm{d}\eta \neq 0$ everywhere on M. This 1-form η is called a *contact form* on M.

On a contact 3-manifold $M = (M, \eta)$ equipped with a contact form η, there exists a unique vector field ξ satisfying $\eta(\xi) = 1$ and $\mathrm{d}\eta(\xi, X) = 0$ for any vector field X. This vector field ξ is called the *characteristic vector field* of (M, η). Moreover there exits an endomorphism field φ and a Riemannian metric g on M satisfying

$$\eta(X) = g(X, \xi), \quad \mathrm{d}\eta(X, Y) = g(X, \varphi Y), \quad \varphi^2 X = -X + \eta(X)\xi, \tag{1.1}$$

for all X, $Y \in \mathfrak{X}(M)$. Here $\mathfrak{X}(M)$ is the Lie algebra of all smooth vector fields on M. From (1.1), it follows that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0\,, \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)\,.$$

A Riemannian 3-manifold (M, g) equipped with the structure tensors (η, ξ, φ) satisfying (1.1) is said to be a *contact Riemannian* 3*-manifold.* We denote it by $M = (M, \eta; \xi, \varphi, g)$.

Let us define an endomorphism field h on a contact Riemannian 3-manifold M by $h = \frac{1}{2}\$_\xi \varphi$, where $\$_\xi$ denotes Lie differentiation in the characteristic direction ξ.

Then we observe that h is self-adjoint with respect to g and satisfies

$$h\xi = 0, \quad h\varphi = -\varphi h\,,$$

$$\nabla_X \xi = -\varphi(h + I)X\,, \tag{1.2}$$

where ∇ is the Levi-Civita connection of (M, g) and I is the identity transformation.

Next, on a contact Riemannian 3-manifold M, one can define an almost complex structure J on the product manifold $M \times \mathbb{R}$ by

$$J\Big(X, f\frac{\mathrm{d}}{\mathrm{d}t}\Big) = \Big(\varphi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\Big)\,, \qquad X \in \mathfrak{X}(M)\,,$$

where t is the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, then the contact Riemannian 3-manifold M is said to be a *Sasakian* 3*-manifold.*

Proposition 1.1. *Let $(M, \eta; \xi, \varphi, g)$ be a contact Riemannian* 3*-manifold. Then the following three conditions are mutually equivalent*:

(1) *The characteristic vector field ξ is a Killing vector field,*

(2) $h = 0$*,*

(3) *M is Sasakian.*

On a Sasakian 3*-manifold, the covariant derivative $\nabla\varphi$ is given by*

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X\,, \qquad X, Y \in \mathfrak{X}(M)\,. \tag{1.3}$$

Take a tangent vector X in the tangent space T_pM of a Sasakian 3-manifold M which is orthogonal to ξ_p. Then the plane section $X \wedge \varphi X$ is called a *holomorphic section.* The sectional curvature $K(X \wedge \varphi X)$ is called a *holomorphic sectional curvature.* Sasakian 3-manifolds of constant holomorphic sectional curvature are called 3-dimensional *Sasakian space forms.*

1.2. **Pseudo-Hermitian structure and Tanaka-Webster connection.** On a contact Riemannian 3-manifold $(M, \eta; \xi, \varphi, g)$, the tangent space T_pM of M at a point $p \in M$ can be decomposed

$$T_pM = D_p \oplus \mathbb{R}\xi_p\,, \quad D_p = \{v \in T_pM \mid \eta(v) = 0\}$$

as a direct sum of linear subspaces. Then $D\colon p \longmapsto D_p$ defines a 2-dimensional distribution orthogonal to ξ, which is called the *contact distribution.* We see that the restriction $J = \varphi|_D$ of φ to D defines an almost complex structure on D. Define a complex vector subbundle $\mathcal{H}$ of the complexified tangent bundle $T^{\mathbb{C}}M$ by

$$\mathcal{H} = \{X - iJX \mid X \in D\}\,.$$

Then we see that each fiber $\mathcal{H}_p$ is of complex dimension 1, $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$, and $D \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{H} \oplus \overline{\mathcal{H}}$. This subbundle is called the *almost CR-structure* on M associated to the contact Riemannian structure (φ, ξ, η, g).

Furthermore, since $\dim M = 3$, the associated almost CR-structure is always *integrable*, that is the space $\Gamma(\mathcal{H})$ of all smooth sections of $\mathcal{H}$ satisfies the *integrability condition*:

$$[\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H})\,.$$

The *Levi form* L is defined by

$$L\colon \Gamma(D) \times \Gamma(D) \to \mathfrak{F}(M)\,, \quad L(X,Y) = -\mathrm{d}\eta(X, JY)\,,$$

where $\mathfrak{F}(M)$ denotes the algebra of smooth functions on M. Then we see that the Levi form is Hermitian and positive definite. We call the pair (η, L) a *strongly pseudo-convex pseudo-Hermitian structure* on M.

Now, we recall the *Tanaka-Webster connection* on a strongly pseudo-convex pseudo-Hermitian manifold $M = (M, \eta, L)$ with the associated contact Riemannian structure (η, ξ, φ, g) (see [21], [23]). The Tanaka-Webster connection $\hat{\nabla}$ is defined by

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all vector fields X, Y on M. Together with (1.2), $\hat{\nabla}$ may be rewritten as

$$\hat{\nabla}_X Y = \nabla_X Y + A(X)Y, \tag{1.4}$$

where we have put

$$A(X)Y = \eta(X)\varphi Y + \eta(Y)\varphi(I+h)X - g(\varphi(I+h)X, Y)\xi\,. \tag{1.5}$$

We see that the Tanaka-Webster connection $\hat{\nabla}$ has the torsion

$$\hat{T}(X,Y) = 2g(X, \varphi Y)\xi + \eta(Y)\varphi h X - \eta(X)\varphi h Y\,.$$

In particular, for Sasakian manifolds, (1.5) and the above equation are reduced to:

$$\begin{aligned} A(X)Y &= \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi, \\ \hat{T}(X,Y) &= 2g(X, \varphi Y)\xi\,. \end{aligned}$$

Furthermore, it was proved in [22] that

Proposition 1.2. *The Tanaka-Webster connection $\hat{\nabla}$ on a 3-dimensional contact Riemannian manifold $M = (M; \eta, \varphi, \xi, g)$ is the unique linear connection satisfying the following conditions*:

(1) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$, $\hat{\nabla}g = 0$, $\hat{\nabla}\varphi = 0$,

(2) $\hat{T}(X,Y) = -\eta([X,Y])\xi$, $X,\ Y \in \Gamma(D)$,

(3) $\hat{T}(\xi, \varphi Y) = -\varphi \hat{T}(\xi, Y)$, $Y \in \Gamma(D)$.

2. Submanifolds in pseudo-Hermitian geometry

2.1. **Curves in pseudo-Hermitian geometry.** Let $\gamma(s)\colon I \to (M, g, \hat{\nabla})$ be a unit speed curve in a contact Riemannian 3-manifold M equipped with Tanaka-Webster connection.

Since $\hat{\nabla}$ is a metrical connection, *i.e.*, $\hat{\nabla} g = 0$, there exits an orthonormal frame field $\hat{F} = (\hat{T}, \hat{N}, \hat{B})$ along γ such that $\hat{T} = \gamma'$ and satisfies the following Frenet-Serret equation:

$$\begin{cases} \hat{\nabla}_{\hat{T}}\hat{T} = \hat{\kappa}\hat{N} \\ \hat{\nabla}_{\hat{T}}\hat{N} = -\hat{\kappa}\hat{T} + \hat{\tau}\hat{B} \\ \hat{\nabla}_{\hat{T}}\hat{B} = -\hat{\tau}\hat{N}\,. \end{cases} \tag{2.1}$$

Here $\hat{\kappa} = |\hat{\nabla}_T T|$ and $\hat{\tau}$ are called the *pseudo-Hermitian curvature* and *pseudo-Hermitian torsion* of γ, respectively. A *pseudo-Hermitian helix* is a curve both of whose pseudo-Hermitian curvature and pseudo-Hermitian torsion are constants. In particular, curves with constant non-zero pseudo-Hermitian curvature and zero pseudo-Hermitian torsion are called *pseudo-Hermitian circles*. Geodesics with respect to $\hat{\nabla}$ are called *pseudo-Hermitian geodesics*. Pseudo-Hermitian geodesics are characterized as unit speed curves with zero pseudo-Hermitian curvature.

The *contact angle* $\theta(s)$ of a unit speed curve $\gamma(s)$ is defined by $\cos\theta(s) = \eta(\gamma'(s))$. A unit speed curve $\gamma(s)$ is said to be a *slant curve* if its contact angle is constant. Slant curves of contact angle $\pi/2$ are traditionally called *Legendre curves*. The characteristic flow (flow of ξ) is a slant curve of contact angle 0.

Let us consider the mean curvature vector field $\hat{\mathbb{H}}$ of a unit speed curve γ in a contact Riemannian 3-manifold with respect to $\hat{\nabla}$:

$$\hat{\mathbb{H}} = \hat{\nabla}_{\gamma'}\gamma' = \hat{\kappa}\hat{N}\,.$$

This vector field $\hat{\mathbb{H}}$ is called the *pseudo-Hermitian mean curvature vector field* of γ, [5]. Next, we denote by $\hat{\Delta}$ the *Laplace-Beltrami operator*

$$\hat{\Delta} = -\hat{\nabla}_{\gamma'}\hat{\nabla}_{\gamma'}$$

acting the space $\Gamma(\gamma^*TM)$ of the all smooth sections of the vector bundle γ^*TM induced by γ.

2.2. **Legendre curves in pseudo-Hermitian geometry.** In this subsection we consider Legendre curves in a Sasakian 3-manifold equipped with Tanaka-Webster connection.

For a unit speed curve curve $\gamma(s)$ in a Sasakian 3-manifold M, from (1.4) and (1.6) we get

$$\hat{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + 2\eta(\dot\gamma)\varphi\dot\gamma\nabla_{\dot\gamma}\dot\gamma + 2\cos\theta(s)\varphi\gamma'\,. \tag{2.2}$$

The formula (2.2) implies that every Legendre curve $\gamma(s)$ in a Sasakian 3-manifold satisfies $\hat{\nabla}_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma'$. Thus every Legendre curve has zero pseudo-Hermitian torsion. In particular we have

Proposition 2.1. *Let γ be a Legendre curve in a Sasakian 3-manifold M, then γ is $\hat{\nabla}$-geodesic if and only if it is a geodesic.*

Here we compare pseudo-Hermitian invariants and Riemannian invariants of Legendre curves.

Let $\gamma(s)$ be a *Legendre curve* in a Sasakian 3-manifold M. Then we have Frenet frame field $F = (T, N, B)$ along γ. Here the tangent vector field T is defined by $T(s) = \gamma'(s)$. The curvature $\kappa(s)$ of $\gamma(s)$ is given by $\nabla_T T = \kappa N$. The unit vector field $N(s)$ is called the *principal normal vector field* of γ. One can see that the mean curvature vector field $\mathbb{H} = \nabla_{\gamma'}\gamma'$ coincides with the pseudo-Hermitian mean curvature vector field. Thus we have

$$\hat{N} = N = \varphi T\,, \quad \hat{\kappa} = \kappa\,.$$

Now, we study Legendre curves satisfying $\hat{\Delta}\mathbb{H} = \lambda\hat{\mathbb{H}}$ in Sasakian 3-manifolds.

Direct computations using (2.1) and (2.2) show that

$$\hat{\Delta}\hat{\mathbb{H}} = -3\hat{\kappa}\hat{\kappa}'\hat{T} + (\hat{\kappa}'' - \hat{\kappa}^3)\hat{N}\,.$$

Theorem 2.1 ([17])**.** *Let γ be a Legendre curve in a Sasakian 3-manifold. Then $\hat{\Delta}\mathbb{H} = \lambda\mathbb{H}$ if and only if γ is a $\hat{\nabla}$-geodesic $(\lambda = 0)$ or a pseudo-Hermitian circle $(\lambda \neq 0)$ satisfying $\hat{\kappa}^2 = \lambda$ for non-zero constant $\hat{\kappa}$.*

Next, let $T^{\perp}\gamma$ be the normal bundle of a Legendre curve γ in a Sasakian 3-manifold M. We denote by $\hat{\nabla}^{\perp}$ the connection on $T^{\perp}\gamma$ induced from the Tanaka-Webster connection of M. With respect to the Laplace-Beltrami operator $\hat{\Delta}^{\perp} = -\hat{\nabla}^{\perp}_{\gamma'}\hat{\nabla}^{\perp}_{\gamma'}$ of the normal bundle, we get the following result (*cf.* [17]).

Theorem 2.2. *Let γ be a Legendre curve in a Sasakian 3-manifold and suppose that λ is a non-zero constant. Then $\hat{\Delta}^{\perp}\hat{\mathbb{H}} = \lambda\hat{\mathbb{H}}$ if and only if γ has the pseudo-Hermitian curvature*

(1) $\hat{\kappa}(s) = as + b,\ a, b \in \mathbb{R},\ \lambda = 0,$

(2) $\hat{\kappa}(s) = a\cos(\sqrt{\lambda}s) + b\sin(\sqrt{\lambda}s),\ \lambda > 0,$ *or*

(3) $\hat{\kappa}(s) = a\exp(\sqrt{-\lambda}s) + b\exp(-\sqrt{-\lambda}s),\ \lambda < 0.$

Proof. With respect to the connection $\hat{\nabla}^{\perp}$, we have $\hat{\Delta}^{\perp}\hat{\mathbb{H}} = -\hat{\kappa}''\hat{N}$. Thus the result follows. □

3. Hopf cylinders in regular Sasakian 3-manifolds

3.1. **Boothby-Wang fibration.** Let M be a contact Riemannian 3-manifold. Then M is said to be *regular* if its characteristic vector field ξ is complete and its flow acts freely and isometrically on M. The fibration $\pi\colon M \to \overline{M}$ is called the *Boothby-Wang fibration* of M.

The contact Riemannian structure $(\eta; \xi, \varphi, g)$ on M induces an almost Hermitian structure $(\overline{g}, J)$ on the orbit space $\overline{M}$. Since $\overline{M}$ is 2-dimensional, the induced almost complex structure J is integrable. Hence the resulting almost Hermitian 2-manifold $(\overline{M}, \overline{g}, J)$ is a real 2-dimensional Kähler manifold.

The regularity of ξ implies that ξ is a Killing vector field. Hence regular contact Riemannian 3-manifolds are automatically Sasakian. Moreover, the natural projection $\pi\colon (M,g) \to (\overline{M},\bar{g})$ is a Riemannian submersion [20].

Let $\overline{X}_{\overline{p}}$ be a tangent vector of the orbit space $\overline{M}$ at $\overline{p}=\pi(p)$. Then there exists a tangent vector $\overline{X}^*_p$ of M at p which is orthogonal to ξ such that $\pi_{*p}\overline{X}^*_p = \overline{X}_{\overline{p}}$. The tangent vector $\overline{X}^*_p$ is called the *horizontal lift* of $\overline{X}_{\overline{p}}$ to M at p. The horizontal lift operation $*\colon \overline{X}_{\overline{p}} \mapsto \overline{X}^*_p$ is naturally extended to vector fields.

The complex structure J on the orbit space $\overline{M}$ is related to φ by

$$J\overline{X} = \pi_*(\varphi\overline{X}^*),\ \ \overline{X} \in \mathfrak{X}(\overline{M})\,. \tag{3.1}$$

Let us denote by $\overline{\nabla}$ the Levi-Civita connection of $\overline{M}$. Then, by using the fundamental equations for Riemannian submersions due to B. O'Neill [20], we have the following formula.

Lemma 3.1 ([19]). *Let M be a regular contact Riemannian 3-manifold. Then for any $\overline{X},\overline{Y} \in \mathfrak{X}(\overline{M})$:*

$$\nabla_{\overline{X}^*}\overline{Y}^* = (\overline{\nabla}_{\overline{X}}\overline{Y})^* - g(\overline{X}^*,\varphi\overline{Y}^*)\xi\,. \tag{3.2}$$

Now let us denote by $M^3(c)$ a complete and simply connected 3-dimensional Sasakian space form of constant holomorphic sectional curvature c. Then $M^3(c)$ is regular and the orbit space M/ξ is of constant curvature $c+3$ (see [19], [20]).

3.2. **Hopf cylinders.** Let $\pi : M \to \overline{M}$ be a Boothby-Wang fibration of a regular Sasakian 3-manifold discussed before. Let $\overline{\gamma}(s)$ be a unit speed curve in $\overline{M}$ with signed curvature $\overline{\kappa}(s)$. We take the inverse image $\Sigma = \Sigma_{\overline{\gamma}} := \pi^{-1}\{\overline{\gamma}\}$ of $\overline{\gamma}$ in M and call it the *Hopf cylinder* over $\overline{\gamma}$.

Let us denote by $\overline{F} = (\bar{\boldsymbol{t}},\overline{\boldsymbol{n}})$ the Frenet frame field of $\overline{\gamma}$ in $(\overline{M},\bar{g})$. By using the complex structure J of $\overline{M}$, $\overline{\boldsymbol{n}}$ is given by $\overline{\boldsymbol{n}} = J\bar{\boldsymbol{t}}$. Then the Frenet-Serret formula of $\overline{\gamma}$ is given by

$$\overline{\nabla}_{\overline{\gamma}'}\overline{F} = \overline{F}\begin{bmatrix} 0 & -\overline{\kappa} \\ \overline{\kappa} & 0 \end{bmatrix}.$$

Let $\boldsymbol{t} := \bar{\boldsymbol{t}}^*$ be the horizontal lift of $\bar{\boldsymbol{t}}$ with respect to the Boothby-Wang fibration. Then $\{\boldsymbol{t},\xi\}$ gives an orthonormal frame field of Σ. The horizontal lift $\boldsymbol{n} := (\overline{\boldsymbol{n}})^*$ is a unit normal vector field of Σ in M. Since $\overline{\boldsymbol{n}} = J\bar{\boldsymbol{t}}$, we have $\boldsymbol{n} = \varphi\boldsymbol{t}$. In fact,

$$(\overline{\boldsymbol{n}})^* = (J\bar{\boldsymbol{t}})^* = \varphi(\bar{\boldsymbol{t}})^* = \varphi\boldsymbol{t}\,.$$

Let us denote by ∇^Σ the Levi-Civita connection of Σ. Then the *second fundamental form* α of Σ derived from $\boldsymbol{n}$ is defined by the *Gauss formula*:

$$\nabla_X Y = \nabla^\Sigma_X Y + \alpha(X,Y)\boldsymbol{n},\ \ X,Y \in \mathfrak{X}(\Sigma)\,. \tag{3.3}$$

By using (3.2)

$$\nabla_{\boldsymbol{t}}\boldsymbol{t} = (\overline{\nabla}_{\bar{\boldsymbol{t}}}\,\bar{\boldsymbol{t}})^* - g(\boldsymbol{t},\varphi\boldsymbol{t})\xi = (\overline{\kappa}\circ\pi)\boldsymbol{n}\,.$$

Hence $\nabla^\Sigma_{\boldsymbol{t}}\boldsymbol{t} = 0$. Since ξ is Killing, we have $\nabla^\Sigma_{\boldsymbol{t}}\xi = \nabla^\Sigma_\xi\xi = 0$. Thus $\Sigma_{\overline{\gamma}}$ is flat. The second fundamental form α is described as

$$\alpha(\boldsymbol{t},\boldsymbol{t}) = \overline{\kappa}\circ\pi\,, \quad \alpha(\boldsymbol{t},\xi) = -1\,, \quad \alpha(\xi,\xi) = 0\,.$$

The mean curvature function is $H = (\overline{\kappa} \circ \pi)/2$ and the mean curvature vector field $\mathbb{H}$ is $\mathbb{H} = H\boldsymbol{n}$.

3.3. Let us denote by ι the inclusion map of a Hopf cylinder $\Sigma \subset M$ in a regular Sasakian 3-manifold M. The inclusion map ι induces a vector bundle ι^*TM over Σ. Moreover the Levi-Civita connection ∇ of M induces a connection ∇^ι on ι^*TM. Then $(\iota^*TM, \iota^*g, \nabla^\iota)$ is a Riemannian vector bundle over Σ. The *rough Laplacian* Δ acting on the space $\Gamma(\iota^*TM)$ of all smooth sections of ι^*TM is given by

$$\Delta = -\nabla^\iota_{\boldsymbol{t}} \nabla^\iota_{\boldsymbol{t}} - \nabla^\iota_\xi \nabla^\iota_\xi\,,$$

since (Σ, ι^*g) is flat.

Next, let $T^\perp\Sigma$ be the normal bundle of Σ in M. Denote by $g^\perp$ the restriction of g to $T^\perp\Sigma$. With respect to the normal connection $\nabla^\perp$ of Σ, $(T^\perp\Sigma, g^\perp, \nabla^\perp)$ is a Riemannian vector bundle. The rough Laplacian $\nabla^\perp$ of $T^\perp\Sigma$ acting on the space $\Gamma(T^\perp M)$ of all smooth sections of the normal bundle is given by

$$\Delta^\perp = -\nabla^\perp_{\boldsymbol{t}} \nabla^\perp_{\boldsymbol{t}} - \nabla^\perp_\xi \nabla^\perp_\xi\,.$$

The first named author classified submanifolds with proper mean curvature vector field in regular Sasakian 3-manifolds with respect to the Levi-Civita connection ∇ as follows:

Theorem 3.1 ([13]). *A Hopf cylinder $\Sigma_{\overline{\gamma}}$ in a regular Sasakian 3-manifold satisfies $\Delta\mathbb{H} = \lambda\mathbb{H}$ if and only if $\overline{\gamma}$ is a geodesic ($\lambda = 0$) or a Riemannian circle ($\lambda \neq 0$). In case that $\lambda \neq 0$, the eigenvalue λ is $\lambda = 4H^2 + 2 > 2$.*

Theorem 3.2 ([13]). *A Hopf cylinder $\Sigma_{\overline{\gamma}}$ satisfies $\Delta^\perp\mathbb{H} = \lambda\mathbb{H}$ if and only if $\overline{\gamma}$ is defined by one of the following natural equations*:

(1) $\overline{\kappa}(s) = as + b,\ a, b \in \mathbb{R},\ \lambda = 0,$

(2) $\overline{\kappa}(s) = a\cos(\sqrt{\lambda}s) + b\sin(\sqrt{\lambda}s),\ \lambda > 0$ *or*

(3) $\overline{\kappa}(s) = a\exp(\sqrt{-\lambda}s) + b\exp(-\sqrt{-\lambda}s),\ \lambda < 0.$

Corollary 3.1 ([13]). *A Hopf cylinder $\Sigma_{\overline{\gamma}}$ satisfies $\Delta^\perp\mathbb{H} = 0$ if and only if $\overline{\gamma}$ is one of the following*:

(1) *a geodesic,*

(2) *a Riemannian circle or*

(3) *a Riemannian clothoid* (*Cornu spiral*).

3.4. We study Hopf cylinders with proper *pseudo-Hermitian mean curvature vector field*. Let Σ be a Hopf cylinder in a regular Sasakian 3-manifold M and $\iota : \Sigma \subset M$ the inclusion map as before. Then the Tanaka-Webster connection $\hat{\nabla}$ of M induces a connection $\hat{\nabla}^\iota$ on ι^*M and $\hat{\nabla}^\perp$ on the normal bundle $T^\perp\Sigma$, respectively. Denote by $\hat{\Delta}^\Sigma$ and $\hat{\Delta}^\perp$ the rough Laplacian on the Riemannian vector bundles $(\iota^*M, \hat{\nabla}^\iota, \iota^*g)$ and $(T^\perp\Sigma, \hat{\nabla}^\perp, g^\perp)$, respectively. Then, since (Σ, ∇^Σ) is flat, these rough Laplacians are given by

$$\hat{\Delta} = -\hat{\nabla}_{\boldsymbol{t}} \hat{\nabla}_{\boldsymbol{t}} - \hat{\nabla}_\xi \hat{\nabla}_\xi\,, \quad \hat{\Delta}^\perp = -\hat{\nabla}^\perp_{\boldsymbol{t}} \hat{\nabla}^\perp_{\boldsymbol{t}} - \hat{\nabla}^\perp_\xi \hat{\nabla}^\perp_\xi\,.$$

Remark 1 ([5])**.** Let $\overline{\gamma}(s)$ be a unit speed curve in $\overline{M}$ and denote by $\overline{\gamma}^*(s)$ the horizontal lift of $\overline{\gamma}(s)$ with respect to the Boothby-Wang fibration. Then the Frenet frame field of $\overline{\gamma}^*(s)$ with respect to the Levi-Civita connection is given by $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}) = (\overline{\boldsymbol{t}}^*, \overline{\boldsymbol{n}}^*, \pm\xi)$. Hence the horizontal lift is a Legendre curve with curvature $\kappa = \overline{\kappa} \circ \pi$ and torsion ± 1.

With respect to the Tanaka-Webster connection, the Hopf cylinder Σ satisfies [5]

$$\hat{\nabla}_{\boldsymbol{t}}\boldsymbol{t} = 2H\boldsymbol{n}\,, \quad \hat{\nabla}_{\boldsymbol{t}}\xi = \hat{\nabla}_{\xi}\boldsymbol{t} = 0\,, \quad \hat{\nabla}_{\xi}\xi = 0\,. \tag{3.4}$$

The pseudo-Hermitian mean curvature vector field $\hat{\mathbb{H}}$ with respect to $\hat{\nabla}$ coincides with $\mathbb{H}$. Hence $\hat{\mathbb{H}} = \mathbb{H} = H\boldsymbol{n} = \kappa\boldsymbol{n}/2$ with $\kappa = \overline{\kappa} \circ \pi$.

Proposition 3.1 ([5])**.** *Let Σ be a Hopf cylinder in a regular Sasakian 3-manifold equipped with the Tanaka-Webster connection, then the mean curvature vector field $\mathbb{H}$ satifies*

$$\hat{\nabla}_{\boldsymbol{t}}\mathbb{H} = -\frac{1}{2}\kappa^2\boldsymbol{t} + \frac{1}{2}\kappa'\boldsymbol{n}\,, \tag{3.5}$$

$$\hat{\nabla}_{\xi}\mathbb{H} = 0\,, \tag{3.6}$$

$$\hat{\Delta}\mathbb{H} = \frac{3}{2}\kappa\kappa'\boldsymbol{t} - \frac{1}{2}(\kappa'' - \kappa^3)\boldsymbol{n}\,. \tag{3.7}$$

By using (3.6), we get the following result.

Proposition 3.2. *If Σ is a Hopf cylinder with mean curvature vector field $\mathbb{H}$ in a regular Sasakian 3-manifold M equipped with the Tanaka-Webster connection, then*

$$\hat{\nabla}^{\perp}_{\boldsymbol{t}}\mathbb{H} = \frac{1}{2}\kappa'\boldsymbol{n}\,, \quad \hat{\nabla}^{\perp}_{\xi}\mathbb{H} = 0\,, \quad \hat{\Delta}^{\perp}\mathbb{H} = -\frac{1}{2}\kappa''\boldsymbol{n}\,. \tag{3.8}$$

From these results, we obtain

Theorem 3.3. *A Hopf cylinder $\Sigma_{\overline{\gamma}}$ in a regular Sasakian 3-manifold equipped with the Tanaka-Webster connection satisfies $\hat{\Delta}\mathbb{H} = \lambda\mathbb{H}$ if and only if the base curve $\overline{\gamma}$ is a geodesic ($\lambda = 0$) or a Riemannian circle ($\lambda > 0$). In case that $\lambda > 0$, the eigenvalue λ is $\lambda = \bar{\kappa}^2 > 0$.*

Proof. The Hopf cylinder $\Sigma_{\overline{\gamma}}$ satisfies $\hat{\Delta}\mathbb{H} = \lambda\mathbb{H}$ if and only if $\overline{\gamma}$ satisfies $\bar{\kappa} = 0$ or $\bar{\kappa}^2 - \lambda = 0$. Thus the result follows. □

Remark 2. Hopf cylinders in 3-dimensional Sasakian space forms satisfying $\hat{\Delta}\mathbb{H} = 0$ are minimal (with respect to ∇). This fact was already obtained in our previous paper [5].

Next, we have

$$\hat{\Delta}^{\perp}\hat{\mathbb{H}} = -\frac{1}{2}\kappa''\boldsymbol{n}\,.$$

Thus we have the following result.

Theorem 3.4. *Let M be a regular Sasakian 3-manifold equipped with Tanaka-Webster connection and $\Sigma_{\overline{\gamma}}$ a Hopf cylinder. Then $\Sigma_{\overline{\gamma}}$ satisfies $\hat{\Delta}^{\perp}\mathbb{H} = \lambda\mathbb{H}$ if and only if $\Sigma_{\overline{\gamma}}$ satisfies $\Delta^{\perp}\mathbb{H} = \lambda\mathbb{H}$ with respect to the Levi-Civita connection.*

3.5. E. Loubeau and S. Montaldo introduced the notion of biminimal immersion [18]. Let (N^n, h) and (M^m, g) be Riemannian manifolds and $\phi : N \to M$ isometric immersion. The *bienergy* $E_2(\phi)$ of ϕ is defined by

$$E_2(\phi) = \frac{n^2}{2} \int |\mathbb{H}|^2 \, \mathrm{d}v_h \,,$$

where $\mathbb{H}$ is the mean curvature vector field of ϕ.

An isometric immersion ϕ is said to be *biminimal* if it is a critical point of the bienergy with respect to all normal variations with compact support. The Euler-Lagrange equation of the biminimality is

$$\big(\Delta^{\phi}\mathbb{H} - \mathrm{tr}\, R(\mathbb{H}, \mathrm{d}\phi)\mathrm{d}\phi\big)^{\perp} = 0\,.$$

Here the superscript $\perp$ means the normal component, Δ^{ϕ} is the rough Laplacian acting on $\Gamma(\phi^* TM)$ and R is the Riemannian curvature of (M, g).

More generally, an isometric immersion $\phi : (N, h) \to (M, g)$ is said to be λ-*biminimal* if

$$\big(\Delta^{\phi}\mathbb{H} - \mathrm{tr}\, R(\mathbb{H}, \mathrm{d}\phi)\mathrm{d}\phi\big)^{\perp} = -\lambda\mathbb{H}$$

for some constant λ. In particular, 0-biminimal immersions are biminimal immersions.

In our previous paper [14], we have shown that a Hopf cylinder in a Sasakian space form $M^3(c)$ of constant holomorphic sectional curvature c is biminimal if and only if its base curve is $(c+3)$-biminimal. Note that the S^3-case was proved in [18].

In addition, in [5] we showed that a Hopf cylinder in $M^3(c)$ is λ-biminimal with respect to *Tanaka-Webster connection* $\hat{\nabla}$, i.e.,

$$\big(\hat{\Delta}^{\perp}\hat{\mathbb{H}} - \mathrm{tr}\, \hat{R}(\hat{\mathbb{H}}, \mathrm{d}\iota)\mathrm{d}\iota\big)^{\perp} = -\lambda\hat{\mathbb{H}}$$

if and only if the base curve is λ-biminimal with respect to Levi-Civita connection.

Motivated by Loubeau-Montaldo's paper, we study Hopf cylinders satisfying $(\hat{\Delta}\hat{\mathbb{H}})^{\perp} = \lambda\hat{\mathbb{H}}$.

From (3.6) the condition $(\hat{\Delta}\hat{\mathbb{H}})^{\perp} = \lambda\hat{\mathbb{H}}$ gives the following natural equation

$$\bar{\kappa}'' - \bar{\kappa}^3 + \lambda\bar{\kappa} = 0 \tag{3.9}$$

of the base curve $\bar{\gamma}$. Multiplying $2\bar{\kappa}'$ to (3.9), we get

$$(\bar{\kappa}')^2 - \frac{1}{2}\bar{\kappa}^4 + \lambda\bar{\kappa}^2 = c$$

for some constant c. The above equation implies

$$\int \frac{\mathrm{d}\bar{\kappa}}{\sqrt{\bar{\kappa}^4 - 2\lambda\bar{\kappa}^2 + 2c}} = \pm \int \frac{\mathrm{d}s}{\sqrt{2}} = \pm\frac{s - s_0}{\sqrt{2}}\,. \tag{3.10}$$

The left hand side is an elliptic integral of the first kind. Thus the signed curvature of the base curve is given explicitly by Jacobi's elliptic functions.

Theorem 3.5. *A Hopf cylinder $\Sigma_{\bar{\gamma}}$ in a regular Sasakian* 3*-manifold M satisfies* $(\hat{\Delta}\hat{\mathbb{H}})^{\perp} = \lambda\hat{\mathbb{H}}$ *if and only if its base curve has the signed curvature $\kappa(s)$ which is a solution to* (3.10)*.*

In our previous papers [15]–[16], we gave explicit formulas for the ordinary differential equation (3.10) in terms of Jacobi's elliptic functions.

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