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THE G-GRADED IDENTITIES OF THE GRASSMANN ALGEBRA

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Abstract. Let G be a finite abelian group with identity element 1_G and $L=\bigoplus_{g\in G}L^g$ be an infinite dimensional G-homogeneous vector space over a field of characteristic 0. Let $E=E(L)$ be the Grassmann algebra generated by L. It follows that E is a G-graded algebra. Let $|G|$ be odd, then we prove that in order to describe any ideal of G-graded identities of E it is sufficient to deal with G'-grading, where $|G'|\le|G|$, $\dim_F L^{1_{G'}}=\infty$ and $\dim_F L^{g'}<\infty$ if $g'\neq 1_{G'}$. In the same spirit of the case $|G|$ odd, if $|G|$ is even it is sufficient to study only those G-gradings such that $\dim_F L^g=\infty$, where $o(g)=2$, and all the other components are finite dimensional. We also compute graded cocharacters and codimensions of E in the case $\dim L^{1_G}=\infty$ and $\dim L^g<\infty$ if $g\neq 1_G$.

1. Introduction

All algebras we refer to are to be considered associative and unitary over a field of characteristic 0 unless explicitly written. Let F be a field and $X=\{x_1,x_2,\dots\}$ be a countable infinite set of variables and let $F\langle X\rangle$ be the free associative algebra freely generated by X. If A is an F-algebra, we say that $f(x_1,\dots,x_n)\in F\langle X\rangle$ is a *polynomial identity of* A if $f(a_1,\dots,a_n)=0$ for any $a_1,\dots,a_n\in A$. If A has a non-trivial polynomial identity we say that A is a *polynomial identity algebra* or *PI-algebra* and we denote by $T(A)$ the set of all polynomial identities satisfied by A. It is well known that $T(A)$ is an ideal of $F\langle X\rangle$ invariant under all endomorphisms of $F\langle X\rangle$, i.e., it is a T-ideal called the *T-ideal of* A. We say that the *variety* generated by the algebra A is the class

$$\mathcal{V}=\mathcal{V}(A)=\{B \text{ associative algebra } \mid T(A)\subseteq T(B)\}\,.$$

The Grassmann algebra E, generated by an infinite dimensional vector space and its identities, plays an important role in the structure theory of Kemer on varieties of associative algebras with polynomial identities [11, 10]. More precisely, Kemer proved that any associative PI-algebra over a field F of characteristic zero satisfies the same identities (is PI-*equivalent*) of the Grassmann envelope of a finite

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dimensional associative superalgebra, i.e., they have the same T-ideal. Moreover, the matrix algebras $M_n(F)$, $M_n(E)$ with entries in E and its subalgebras $M_{p,q}(E)$ $(p+q=n)$, generate the only non-trivial prime varieties.

Here $[a,b]=ab-ba$, and $[a,b,c]=[[a,b],c]$ for every a, b, $c\in F\langle X\rangle$. In [13] Latyshev proved that the T-ideal generated by the triple commutator $[x_1,x_2,x_3]$ is Spechtian, i.e., every proper subvariety of the variety generated by $[x_1,x_2,x_3]$ is finitely based. In [12] Krakowski and Regev proved that the polynomial $[x_1,x_2,x_3]$ forms a basis of the polynomial identities of E. Moreover, they found the codimension sequence of E. Later, Olsson and Regev determined the cocharacter sequence of E (see [14]). In [4] Di Vincenzo gave a different proof of the result of Krakowski and Regev and he also exhibited, for any k, finite bases of the identities of the Grassmann algebra generated by a k-dimensional vector space.

In light of this it seems a natural and interesting problem to investigate more closely the structure of the graded polynomial identities of the Grassmann algebra. For example, the structure of the $\mathbb{Z}_2$-graded identities of E with respect to its natural $\mathbb{Z}_2$-grading is well known, see for instance [9]. Recently, Di Vincenzo and da Silva gave in [6] a complete description of the $\mathbb{Z}_2$-graded polynomial identities of E with respect to any $\mathbb{Z}_2$-grading such that the generating space is $\mathbb{Z}_2$-homogeneous. This work has been generalized by the author for any infinite field of characteristic $p>2$ (see [2]). In [1] Anisimov constructed an algorithm to compute the exact value of the graded codimension of E for any $\mathbb{Z}_p$-grading of E, where p is a prime number.

In this paper we consider a finite abelian group G with identity element 1_G and an infinite dimensional G-homogeneous vector space L over the field F which generates the infinite dimensional Grassmann algebra $E=E(L)$. The latter inherits the structure of a G-graded algebra, hence we are allowed to study its G-graded identities. Let $|G|$ be odd, then we prove that in order to describe any ideal of G-graded identities of E it is sufficient to deal with a G'-grading, where $|G'|\leq|G|$, $\dim_F L^{1_{G'}}=\infty$ and $\dim_F L^{g'}<\infty$ if $g'\neq 1_{G'}$. In the same spirit of the case $|G|$ odd, if $|G|$ is even it is sufficient to study only those G-gradings such that $\dim_F L^g=\infty$ and $o(g)=2$, where $o(g)$ stands for the order of the group element g. Finally we give a complete description of $T_G(E)$, where $G=\mathbb{Z}_2\times\mathbb{Z}_2,\mathbb{Z}_4$ in some particular cases. We also compute graded cocharacters and codimensions of E in the case $\dim L^{1_G}=\infty$ and $\dim L^g<\infty$ if $g\neq 1_G$.

2. Free algebras, graded PI-algebras

We introduce the key tools for the study of graded polynomial identities. We start off with the following definition.

Definition 2.1. Let G be a group and A be an algebra over a field F. We say that the algebra A is *G-graded* if $A=\bigoplus_{g\in G}A^g$ and for all g, $h\in G$, one has $A^gA^h\subseteq A^{gh}$.

It is easy to note that if a is any element of A it can be uniquely written as a finite sum $a=\sum_{g\in G}a_g$, where $a_g\in A^g$. We shall call the subspaces A^g

the *G-homogeneous components of A*. Accordingly, an element $a \in A$ is called *G-homogeneous* if exists $g \in G$ such that $a \in A^g$. If $B \subseteq A$ is a subspace of A, B is G-graded if and only if $B = \bigoplus_{g\in G}(B \cap A^g)$. Analogously one can define G-graded *algebras, subalgebras, ideals, etc.*

Let $\{X^g \mid g \in G\}$ be a family of disjoint countable sets of indeterminates. Set $X = \bigcup_{g\in G} X^g$ and denote by $F\langle X \mid G\rangle$ the free associative algebra freely generated by X. An indeterminate $x \in X$ is said to be of *homogeneous G-degree g*, written $\|x\| = g$, if $x \in X^g$. We always write x^g if $x \in X^g$. The homogeneous G-degree of a monomial $m = x_{i_1} x_{i_2} \dots x_{i_k}$ is defined to be $\|m\| = \|x_{i_1}\| \cdot \|x_{i_2}\| \cdot \dots \cdot \|x_{i_k}\|$. For every $g \in G$, denote by $F\langle X \mid G\rangle^g$ the subspace of $F\langle X \mid G\rangle$ spanned by all monomials having homogeneous G-degree g. Notice that $F\langle X \mid G\rangle^g F\langle X \mid G\rangle^{g'} \subseteq F\langle X \mid G\rangle^{gg'}$ for all g, $g' \in G$. Thus

$$F\langle X \mid G\rangle = \bigoplus_{g\in G} F\langle X \mid G\rangle^g$$

is a G-graded algebra. The elements of the G-graded algebra $F\langle X \mid G\rangle$ are called *G-graded polynomials* or, simply, *graded polynomials.*

Definition 2.2. If A is a G-graded algebra, a G-graded polynomial

$$f(x_1, \dots, x_n)$$

is said to be a *graded polynomial identity* of A if

$$f(a_1, a_2, \dots, a_n) = 0$$

for all $a_1, a_2, \dots, a_n \in \bigcup_{g\in G} A^g$ such that $a_k \in A^{\|x_k\|}$, $k = 1, \dots, n$. We shall write $f \equiv 0$ in order to say that f is a graded polynomial identity for A.

Given an algebra A graded by a group G, we define

$$T_G(A) := \{f \in F\langle X \mid G\rangle \mid f \equiv 0 \text{ on } A\},$$

the set of G-graded polynomial identities of A.

Definition 2.3. An ideal I of $F\langle X \mid G\rangle$ is said to be a T_G-ideal if it is invariant under all F-endomorphisms $\varphi\colon F\langle X \mid G\rangle \to F\langle X \mid G\rangle$ such that $\varphi(F\langle X \mid G\rangle^g) \subseteq F\langle X \mid G\rangle^g$ for all $g \in G$.

Hence $T_G(A)$ is a T_G-ideal of $F\langle X \mid G\rangle$. On the other hand, it is easy to check that all T_G-ideals of $F\langle X \mid G\rangle$ are of this type. We shall denote by $\langle S\rangle^{T_G}$ the *T_G-ideal generated by the set S*, i.e., the smallest T_G-ideal containing S. In this case we say S is a basis for $\langle S\rangle^{T_G}$ or the elements of $\langle S\rangle^{T_G}$ *follow* from those of S.

From now on all the groups are assumed to be finite abelian. The theory of G-graded PI-algebras passes through the representation theory of the symmetric group. More precisely we study the following spaces.

Definition 2.4. Let

$$P_n^G = \operatorname{span}\langle x^{g_1}_{\sigma(1)} x^{g_2}_{\sigma(2)} \cdots x^{g_n}_{\sigma(n)} \mid g_i \in G,\ \sigma \in S_n\rangle,$$

then the elements in P_n^G are called *multilinear polynomials* of degree n of $F\langle X \mid G\rangle$.

It turns out that P_n^G is a left S_n-module under the natural left action of the symmetric group S_n. As a consequence the factor module $P_n^G(A) := P_n^G/(P_n^G \cap T_G(A))$ is an S_n-module, too. We observe that $P_n^G(A)$ affords a representation of the symmetric group S_n which naturally carries on a character of S_n (or S_n-character). Let us denote the S_n-character of $P_n^G(A)$ by $\chi_n^G(A)$, and by $c_n^G(A)$ its dimension over F. We say that

$$\left(\chi_n^G(A)\right)_{n\in\mathbb{N}} \text{ is the } G\text{-}\textit{graded cocharacter sequence} \text{ of } A$$

$$\left(c_n^G(A)\right)_{n\in\mathbb{N}} \text{ is the } G\text{-}\textit{graded codimension sequence} \text{ of } A\,.$$

Now, for $l_{g_1}, \dots, l_{g_r} \in \mathbb{N}$ let us consider the blended components of the multilinear polynomials in the indeterminates labeled as follows: $x_1^{g_1}, \dots, x_{l_{g_1}}^{g_1}$, then $x_{l_{g_1}+1}^{g_2}, \dots, x_{l_{g_1}+l_{g_2}}^{g_2}$ and so on. We denote this linear space by $P_{l_{g_1},\dots,l_{g_r}}^G$. Of course, this is a left $S_{l_{g_1}} \times \cdots \times S_{l_{g_r}}$-module. We shall denote by $\chi_{l_{g_1},\dots,l_{g_r}}^G(A)$ the character of the module $P_{l_{g_1},\dots,l_{g_r}}^G(A)/(P_{l_{g_1},\dots,l_{g_r}}^G(A) \cap T_G(A))$ and by $c_{l_{g_1},\dots,l_{g_r}}^G(A)$ its dimension.

Since the ground field F is infinite, a standard *Vandermonde-argument* yields that a polynomial f is a G-graded polynomial identity for A if and only if its homogeneous components (with respect to the ordinary $\mathbb{N}$-grading), are identities as well. Moreover, since $\mathrm{char}(F) = 0$, the well known multilinearization process shows that the T_G-ideal of a G-graded algebra A is determined by its multilinear polynomials, i.e. by the various $P_{l_{g_1},\dots,l_{g_r}}^G(A)$. We remark that, given the cocharacter $\chi_{l_{g_1},\dots,l_{g_r}}^G(A)$, the graded cocharacter $\chi_n^G(A)$ is known as well. More precisely, the following is due to Di Vincenzo (see [5, Theorem 2]).

Proposition 2.5. *Let A be a G-graded algebra with graded cocharacter sequences $\chi_{l_{g_1},\dots,l_{g_r}}^G(A)$. Then*

$$\chi_n^G(A) = \sum_{\substack{(l_{g_1},\dots,l_{g_r})\\ l_{g_1}+\cdots+l_{g_r}=n}} \chi_{l_{g_1},\dots,l_{g_r}}^G(A)^{\uparrow S_n}\,,$$

where $\chi_{l_{g_1},\dots,l_{g_r}}^G(A)^{\uparrow S_n}$ stands for the induced S_n-character of the $S_{l_{g_1}} \times \cdots \times S_{l_{g_r}}$-module $P_{l_{g_1},\dots,l_{g_r}}^G(A)$.

Moreover

$$c_n^G(A) = \sum_{\substack{(l_{g_1},\dots,l_{g_r})\\ l_{g_1}+\cdots+l_{g_r}=n}} \binom{n}{l_{g_1},\dots,l_{g_r}} c_{l_{g_1},\dots,l_{g_r}}^G(A)\,.$$

Let us consider the free algebra $F\langle Y \cup Z\rangle$ (where Y is the set of all indeterminates of G-degree 1_G and Z is the set of all the remaining indeterminates). The Y-proper polynomials (see [7, Section 2]) are the elements of the unitary F-subalgebra B of $F\langle X\rangle$ generated by the elements of Z and by all non-trivial commutators. More precisely, a polynomial $f \in F\langle Y \cup Z\rangle$ is *Y-proper* if all the $y \in Y$ occurring in f

appear in commutators only. Notice that if $f \in F\langle Z\rangle$, then f is Y-proper. It is well known (see, for instance, Lemma 1 Section 2 in [7]) that all the graded polynomial identities of a superalgebra A follow from the Y-proper ones. This means that the set $T_{\mathbb{Z}_2}(A) \cap B$ generates the whole $T_{\mathbb{Z}_2}(A)$ as a $T_{\mathbb{Z}_2}$-ideal. Similarly, for any finite abelian group G, all the G-graded polynomial identities of a G-graded algebra A follow from the Y-proper ones. This means that the set $T_G(A) \cap B$ generates the whole $T_G(A)$ as a T_G-ideal. Let us define $B(A) := B/(T_G(A) \cap B)$. We shall refer to $B(A)$ as *Y-proper relatively-free algebra* of A.

We shall denote by Γ_n^G the set of multilinear Y-proper polynomials of P_n^G. It is not difficult to see that Γ_n^G is a left S_n-submodule of P_n^G and the same holds for $\Gamma_n^G \cap T_G(A)$. Hence the factor module

$$\Gamma_n^G(A) := \Gamma_n^G / \big(\Gamma_n^G \cap T_G(A)\big)$$

is an S_n-submodule of $P_n^G(A)$. We denote the S_n-character of the factor module $\Gamma_n^G/(\Gamma_n^G \cap T_G(A))$ by $\xi_n^G(A)$, and by $\gamma_n^G(A)$ its dimension over F. We say:

$\big(\xi_n^G(A)\big)_{n\in\mathbb{N}}$ is the *G-graded proper cocharacter sequence* of A;

$\big(\gamma_n^G(A)\big)_{n\in\mathbb{N}}$ is the *G-graded proper codimension sequence* of A.

We shall denote by $\Gamma^G_{m_1,\dots,m_r}$ the set of multilinear Y-proper polynomials of $P^G_{m_1,\dots,m_r}$. We observe that $\Gamma^G_{m_1,\dots,m_r}$ is a left $S_{m_1} \times \cdots \times S_{m_r}$-submodule of $P^G_{m_1,\dots,m_r}$ and the same holds for $\Gamma^G_{m_1,\dots,m_r} \cap T_G(A)$. Hence the factor module

$$\Gamma^G_{m_1,\dots,m_r}(A) := \Gamma^G_{m_1,\dots,m_r} / \big(\Gamma^G_{m_1,\dots,m_r} \cap T_G(A)\big)$$

is an $S_{m_1} \times \cdots \times S_{m_r}$-submodule of $P_{m_1,\dots,m_r}(A)^G$. We denote the $S_{m_1} \times \cdots \times S_{m_r}$-character of the factor module $\Gamma^G_{m_1,\dots,m_r}/(\Gamma^G_{m_1,\dots,m_r} \cap T_G(A))$ by $\xi^G_{m_1,\dots,m_r}(A)$, and by $\gamma_{m_1,\dots,m_r}(A)$ its dimension over F. When we refer to A without any ambiguity, we shall use $\gamma_{m_1,\dots,m_r}$ instead of $\gamma_{m_1,\dots,m_r}(A)$.

Let L be an infinite dimensional vector space over F, a field of characteristic zero, then we indicate by $E = E(L)$ the Grassmann algebra generated by L. Let $(G, \cdot)$ be a finite abelian group and suppose E is G-graded. In this section we want to study the G-graded identities of E in the case when L is a G-homogeneous space.

Let $B_L = \{e_1, e_2, \dots\}$ be a linear basis of L, where for any $i \in \mathbb{N}$, e_i is a G-homogeneous element, so $B_E = \{e_{i_1} e_{i_2} \dots e_{i_n} \mid n \in \mathbb{N}, i_1 < i_2 < \cdots < i_n\}$ is a basis of E as a vector space over F. Notice that the existence of a homogeneous G-grading is equivalent to the existence of a map

$$\|\ \|\colon B_L \to G\,.$$

We have that the G-degree of the element $e_{i_1} e_{i_2} \dots e_{i_n}$ is

$$\|e_{i_1} e_{i_2} \dots e_{i_n}\| = \|e_{i_1}\|\,\|e_{i_2}\| \dots \|e_{i_n}\|\ .$$

In this case we say that the set

$$\{e_{i_1}, e_{i_2}, \dots, e_{i_n}\}$$

is the *support* of $e_{i_1} e_{i_2} \dots e_{i_n}$ and the non-negative integer n is its *length*.

For our purposes we shall pass from a fixed G-grading of E to the grading associated to the quotient group G/H, for some special subgroups H of G. More generally, let $A = \bigoplus_{g\in G} A^g$ be a G-graded algebra and let $H < G$, then for every coset $gH \in G/H$, we define $A^{gH} = \bigoplus_{f\in gH} A^f$. In particular, if T is a transversal set for H in G, then

$$A = \bigoplus_{t\in T} A^{tH} .$$

We observe that for every g, $g' \in G$ $A^{gH}A^{g'H} \subseteq A^{gg'H}$, so A inherits a structure of G/H-graded algebra and we shall call it *quotient grading of A*.

3. Graded identities of E

In what follows we shall denote by $Z(A)$ the center of the algebra A. Recall that in the case E is the infinite dimensional Grassmann algebra, then $Z(E) = E^0$, where E^0 is the $\mathbb{Z}_2$-component of degree 0 in the canonical $\mathbb{Z}_2$-grading of E. In what follows we shall use the following notation: if H is a normal subgroup of G and no confusion occurs, we shall denote by $\overline{g}$ the coset gH.

In order to investigate the relations between the graded identities of E with respect to G-gradings and to its quotient G/H-gradings, it is reasonable to consider the following homomorphism between free graded algebras

$$\pi\colon F\langle X \mid G\rangle \to F\langle Y \mid G/H\rangle ,$$

where Y is an infinite set of G/H-graded variables, such that for every $g \in G$ and for every $i \in \mathbb{N}$, $\pi(x_i^g) = y_i^{gH}$. For any G-graded algebra A and for any subgroup H of G we have the next result.

Lemma 3.1. *Let $f(x_1, \ldots, x_n) \in F\langle X \mid G\rangle$ be a multilinear polynomial. If $\pi(f) \in T_{G/H}(A)$, then $f \in T_G(A)$.*

Proof. Let $\varphi\colon x_i \mapsto a_i$ be a G-graded substitution, so $\|a_i\| = \|x_i\| = g_i \in G$ for some $g_i \in G$. Now we have that $a_i \in A^{g_i}$ and a_i is homogeneous of degree g_iH in the quotient grading. Then φ is a G/H-graded substitution too. Due to the fact that $\pi(f) = f(y_1, \ldots, y_n) \in T_{G/H}(A)$, we have $0 = f(a_1, \ldots, a_n)$ and $f \in T_G(A)$. □

Under opportune hypothesis, it is possible to invert this result. Above all, we give the following definition.

Definition 3.2. Let G be a finite abelian group and suppose E is G-graded. We say that the subgroup H of G has the *property $\mathcal{P}$* if for any $h \in H$, E^h has infinite elements of even length with pairwise disjoint support.

The interest of this property is given by the following proposition.

Proposition 3.3. *Let $H < G$ having the property $\mathcal{P}$ and let $f \in F\langle X \mid G\rangle$ be a multilinear polynomial. Then $f \in T_G(E)$ if and only if $\pi(f) \in T_{G/H}(E)$.*

Proof. We have to prove just the only if part. Let

$$f = f(x_1^{g_1}, x_2^{g_1}, \dots, x_{l_{g_1}}^{g_1}, \dots, x^{g_r}_{\sum_{i=1}^{r-1} l_{g_i}+1}, \dots, x^{g_r}_{\sum_{i=1}^{r} l_{g_i}}) \in T_G(E)$$

and let $F = \pi(f)$. Let φ be any G/H-graded substitution, hence $\varphi(y_j^{gH}) = \sum_{h\in H} a_j^{gh}$, and by the multilinearity of f, we can consider only substitutions φ such that $y_j^{gH} \mapsto a_j^{gh}$, for some $h \in H$ and for any j. Now observe that if every homogeneous component E^h has infinite elements of even length, then for every j and for every $h \in H$ there exists $b_j^{h^{-1}}$ of even length such that $\|b_j^{h^{-1}}\| = h^{-1}$. For every $h \in H$, $w_j^g = a_j^{gh} b_j^{h^{-1}}$ is a homogeneous element of degree g in the G-grading of E. Let us consider a new substitution ψ such that $x_j^g \mapsto w_j^g$. This is a G-graded substitution. Now, since $f \in T_G(E)$, $0 = f(w_1^{g_1}, \dots, w_{l_{g_r}}^{g_r}) = \prod_{h\in H, j} b_j^{h^{-1}} \cdot F(a_j^{gh})$ because the $b_j^{h^{-1}}$'s are in $Z(E)$ and this implies $F(a_j^{gh}) = 0$. □

We consider now some subgroups of G having the property $\mathcal{P}$.

Lemma 3.4. *Let $H = \langle g\rangle$ for some $g \in G$. If L^g is infinite dimensional and $|H| = n$ is odd, then H has the property $\mathcal{P}$.*

Proof. Let $\{v_1, v_2, \dots\}$ be a linear basis of L^g and let $h = g^t$ for some $t \in \mathbb{N}$. Notice that

$$\|v_{i_1} \dots v_{i_l}\| = g^l \,,$$

hence $v_{i_1} \cdots v_{i_l} \in E^h$ if and only if $g^l = g^t$ that is if and only if $l \equiv t \bmod n$. Now, if t is even, then the elements of E^h, $v_1 \dots v_t$, $v_{t+1} \dots v_{2t}, \dots, v_{kt+1} \dots v_{(k+1)t}, \dots$ have pairwise disjoint supports of even length. Similarly, if t is odd, an infinite subset of elements of E^h having pairwise disjoint supports is given by $v_1 \dots v_{t+n}$, $v_{t+n+1} \dots v_{2(t+n)}, \dots, v_{k(t+n)+1} \dots v_{(k+1)(t+n)}, \dots$ and we are done. □

Proposition 3.5. *Let G be a finite abelian group and*

$$H = \Big\langle g \mid \dim_F L^g = \infty \text{ and } o(g) \text{ is odd} \Big\rangle .$$

Then H has the property $\mathcal{P}$.

Proof. For any $h \in H$ there exist distinct elements $b_1, \dots, b_s \in H$ such that $h = b_1^{t_1} \dots b_s^{t_s}$ for some positive integers $t_1, \dots, t_s$. Then by Lemma 3.4 and its proof, for every $i = 1, \dots, s$, $E^{b_i^{t_i}}$ has infinite elements

$$w_1^i, w_2^i, \dots, w_m^i, \dots$$

of even length with pairwise disjoint supports, moreover these elements belong to the Grassmann algebra E_i generated by the subspace L^{b_i}. We set

$$u_m = w_m^1 w_m^2 \dots w_m^s \,,$$

for $m \geq 1$ and clearly $\{u_1, u_2, \dots, u_m, \dots\}$ is the required subset of E^h. □

As a consequence of Propositions 3.5 and 3.3, we have the following.

Theorem 3.6. *Let G be a finite abelian group of odd order and let*

$$H = \langle g \mid \dim_F L^g = \infty \rangle .$$

Then the following properties hold:

(1) *for any multilinear polynomial $f(x_1, \dots, x_n) \in F\langle X \rangle$ one has*

$$f \in T_G(E) \quad \textit{if and only if} \quad \pi(f) \in T_{G/H}(E) .$$

(2) *In the quotient grading of E, $L^{\overline{g}}$ is infinite dimensional if and only if $\overline{g} = 1_{G/H}$.*

If G is any finite abelian group, we have the following result.

Proposition 3.7. *Let G be a finite abelian group and $g \in G$ such that $\dim_F L^g = \infty$. Let $H = \langle g \rangle$ and $|H| = n$, an even number. Then $K = \langle g^2 \rangle$ has the property $\mathcal{P}$.*

Proof. Let $\{e_1, e_2, \dots\}$ be a linear basis of L^g and let $k = g^{2t} \in K$, then the elements of E^k $e_1 \dots e_{2t}$, $e_{2t+1} \dots e_{4t}, \dots$, $e_{s(2t)+1} \dots e_{(s+1)(2t)}, \dots$ have pairwise disjoint supports of even length. □

Now let us consider the following subsets of G:

$$\mathcal{I} = \{g \in G \mid \dim_F L^g = \infty\} ,$$

$$\mathcal{I}_1 = \{g \in \mathcal{I} \mid o(g) \ \text{ is odd}\} ,$$

$$\mathcal{I}_2 = \mathcal{I} - \mathcal{I}_1 \quad \text{and}$$

$$\mathcal{I}_3 = \{g^2 \mid g \in \mathcal{I}_2\} - \mathcal{I}_1 .$$

We have the following.

Theorem 3.8. *Let G be a finite abelian group and let $H = \langle g \mid g \in \mathcal{I}_1 \cup \mathcal{I}_3 \rangle$. Then the following properties hold:*

(1) *for any multilinear polynomial $f = f(x_1, \dots, x_n) \in F\langle X \rangle$ one has*

$$f \in T_G(E) \quad \textit{if and only if} \quad \pi(f) \in T_{G/H}(E) .$$

(2) *In the quotient grading of E, if $L^{\overline{g}}$ is infinite dimensional, then $\overline{g}^2 = \overline{1} \in G/H$.*

Proof. (1) Let $h \in H$, then there exist $a_1, \dots, a_r \in \mathcal{I}_1$, $b_1, \dots, b_s \in \mathcal{I}_3$ and positive integers be such that $h = a_1^{m_1} \dots a_r^{m_r} b_1^{m_{r+1}} \dots b_s^{m_{r+s}}$. Let $a_{r+1}, \dots, a_{r+s} \in \mathcal{I}_2$ such that $b_i = a_{r+i}^2$, then $\dim_F L^{a_i} = \infty$ for any $i = 1, \dots, r+s$. Let us denote by E_i the Grassmann algebra generated by the subspace $L^{a_i^{m_i}}$. As in the proof of Proposition 3.5, for any $i = 1, \dots, r+s$, E_i contains infinitely many elements

$$w_1^i, w_2^i, \dots, w_m^i, \dots$$

of even length with pairwise disjoint supports. Moreover, for all $m \geq 1$ we have that $\left\| w_m^i \right\| = a_i^{m_i}$ if $i = 1, \dots, r$ and $\left\| w_m^i \right\| = b_{i-r}^{m_i}$ for $i = r+1, \dots, r+s$. We consider in E^h the elements $u_m = w_m^1 \dots w_m^{r+s}$, $m \geq 1$; clearly the elements $\{u_m | m \geq 1\}$ have pairwise disjoint supports and they have even length. Now H has the property $\mathcal{P}$ and the assertion comes by Proposition 3.3.

(2) Let $\overline{g} = gH \in G/H$ be such that $L^{\overline{g}} = \bigoplus_{h \in H} L^{gh}$ is infinite dimensional. Since G is finite there exists $g' \in gH$ such that $L^{g'}$ is infinite dimensional. If $o(g')$ is odd, then $g' \in H$ and so $gH = g'H = 1_{G/H}$. If $o(g')$ is even, then $g'^2 \in H$ and so $(gH)^2 = (g'H)^2 = 1_{G/H}$. □

4. Graded codimensions and cocharacters of E

We shall study graded codimensions and graded cocharacters for E in the case $\dim_F L^{1_G}$ is infinite and all the other homogeneous components of L have finite dimension. We shall use the language of the representation theory of symmetric groups (see the book [15] by Sagan for more details).

Theorem 4.1. *Let $G = \{g_1, \ldots, g_r\}$ be a finite abelian group with $g_1 = 1_G$. Suppose that L^{g_1} has infinite dimension. Let*

$$l_{g_1}, l_{g_2}, \ldots, l_{g_r} \in \mathbb{N}$$

such that

$$l_{g_1} + l_{g_2} + \cdots + l_{g_r} = m\,.$$

Then $P_{l_{g_1},\ldots,l_{g_r}} \subseteq T_G(E)$ or for any $f \in P_{l_{g_1},l_{g_2},\ldots,l_{g_r}}$ one has

$$f\big(x_1^{g_1}, \ldots, x_{l_{g_1}}^{g_1}, \ldots, x^{g_r}_{\sum_{i=1}^{r-1} l_{g_i}+1}, \ldots, x^{g_r}_{\sum_{i=1}^{r} l_{g_i}}\big) \in T_G(E)$$

if and only if $f(x_1, \ldots, x_m) \in T(E)$.

Proof. It is sufficient to prove that if $P_{l_{g_1},\ldots,l_{g_r}} \nsubseteq T_G(E)$, then any element of $P_{l_{g_1},\ldots,l_{g_r}} \cap T_G(E)$ is an ordinary polynomial identity for E. Let us suppose

$$P_{l_{g_1},\ldots,l_{g_r}} \nsubseteq T_G(E)\,,$$

then there exists a graded monomial with a non-zero graded evaluation of elements $a_1, \ldots, a_m$ of the basis B_L of E. Any other monomial of $P_{l_{g_1},\ldots,l_{g_r}}$ is non-zero with respect to the same evaluation. Since L^{g_1} is infinite dimensional, we can always suppose $a_1, \ldots, a_m$ are of even length multiplying them by some e_i's of degree g_1. Now let us consider the elements of the basis of L^{g_1} which are not involved in the expression of the given elements $a_1, \ldots, a_m$, to say v_i's. Clearly, the latter generate an infinite dimensional Grassmann algebra E', hence $T(E) = T(E')$. Let $f = f\big(x_1^{g_1}, \ldots, x_{l_{g_1}}^{g_1}, \ldots, x^{g_r}_{\sum_{i=1}^{r-1} l_{g_i}+1}, \ldots, x^{g_r}_{\sum_{i=1}^{r} l_{g_i}}\big) \in T_G(E)$ and let φ be any substitution such that $x_i \mapsto v_i \in E'$ for any i. Let us consider a new substitution ψ such that $x_i^{g_j} \mapsto v_i a_i$. This is a G-graded substitution on E. Now, since $f \in T_G(E)$, $0 = f(v_1 a_1, \ldots, v_m a_m) = a_1 \cdots a_m f(v_1, \ldots, v_m)$ because the a_i's are in $Z(E)$ and this implies $f(v_1, \ldots, v_m) = 0$ because the supports of $v_1, \ldots, v_m$ are distinct from those of $a_1, \ldots, a_m$ and $a_1 \ldots a_m \neq 0$ by hypothesis, then $f \in T(E') = T(E)$ and we are done. □

Theorem 4.2. *Let $G = \{g_1, \ldots, g_r\}$ be a finite abelian group with $g_1 = 1_G$. Let L be a G-homogeneous vector space over L such that $\dim_F L^{g_1} = \infty$ and $\dim_F L^{g_i} = k_i < \infty$, if $i \neq 1$. If $E = E(L)$ is the Grassmann algebra generated by L, then $T_G(E)$ is generated as a T_G-ideal by the following polynomials:*

(1) $[u_1, u_2, u_3]$ *for any choice of the G-degree of the variables* u_1, u_2, u_3,
(2) *monomials of* $P_{0,t_2,\dots,t_r}$ *such that* $\sum_{i=2}^r t_i = 1 + \sum_{i=2}^r k_i$,
(3) *monomials of* $P_{0,t_2,\dots,t_r}$ *such that* $\sum_{i=2}^r t_i < 1 + \sum_{i=2}^r k_i$ *and* $P_{0,t_2,\dots,t_r} \subseteq T_G(E)$.

Proof. In light of Theorem 4.1, we have that $T_G(E)$ is generated by the polynomials from (1) of the claim and by all monomials of $P_{l_1,\dots,l_r}$ such that $P_{l_1,\dots,l_r} \subseteq T_G(E)$. Notice that a graded monomial w is surely a graded polynomial identity when the sum of the numbers of its indeterminates of G-degree different from g_1 is strictly greater than $\sum_{i=2}^r k_i$. Moreover, for any $l_1, \dots, l_r \in \mathbb{N}$, any monomial in $P_{l_1,\dots,l_r}$ is in the T_G-ideal generated by the monomials in $P_{0,l_2,\dots,l_r}$. Now we have just to observe that the monomials in $P_{0,l_2,\dots,l_r}$ follow from the monomials in $P_{0,l_2-1,\dots,l_r}, \dots, P_{0,l_2,\dots,l_i-1,\dots,l_r}$ for $l_i \geq 1$ due to the Young rule. Hence if $l_2, \dots, l_r \in \mathbb{N}$ are such that $l_2 + \cdots + l_r > 1 + \sum_{i=2}^r k_i$, then $P_{0,l_2,\dots,l_r}$ is in the T_G-ideal generated by the monomials of $P_{0,t_2,\dots,t_r}$ such that $t_2 + \cdots + t_r = 1 + \sum_{i=2}^r k_i$ and the claim follows. □

We have the following corollary which proof repeats verbatim the one of Proposition 5 of [6].

Corollary 4.3. *Let* $G = \{g_1, \dots, g_r\}$ *be a finite abelian group with* $g_1 = 1_G$. *If* L^{g_1} *has infinite dimension and* $l_{g_1}, l_{g_2}, \dots, l_{g_r} \in \mathbb{N}$ *are such that* $l_{g_1} + l_{g_2} + \ldots + l_{g_r} = m$, *then*

$$c_{l_{g_1},\dots,l_{g_r}}(E) = 0 \quad \text{or} \quad c_{l_{g_1},\dots,l_{g_r}}(E) = 2^{m-1}$$

and in the latter case, $P_{l_{g_1},\dots,l_{g_r}}(E)$ *and* $P_m(E)$ *are isomorphic* $S_{l_{g_1}} \times \cdots \times S_{l_{g_r}}$*-modules.*

If G is a finite abelian group and L is a vector space with basis $B_L = \{e_1, e_2, \dots\}$, let

$$\varphi\colon B_L \to G$$

be any map. As we said before, φ induces a G-grading on E. Let us consider now the set

$$S(\varphi) = \left\{(l_{g_1}, l_{g_2}, \dots, l_{g_r}) \in \mathbb{N}^r \mid P_{l_{g_1},l_{g_2},\dots,l_{g_r}} \subseteq T_G(E)\right\} .$$

We note that if L^{1_G} is the only homogeneous subspace of L such that $\dim_F L^{1_G} = \infty$, then $S(\varphi) \neq \emptyset$.

$S(\varphi)$ allows us to give the complete description of the sequence of the graded cocharacters and codimensions of E. In fact, we have the following proposition.

Proposition 4.4. *Let* $G = \{g_1, \dots, g_r\}$ *be a finite abelian group and* L *be a* G*-homogeneous vector space with linear basis* $\{e_1, e_2, \dots\}$. *Let* $\varphi\colon B_L \to G$ *be a map such that* $|\varphi^{-1}(1_G)| = \infty$ *and consider* E, *the* G*-graded Grassmann algebra obtained by* φ. *Then*

$$\chi^G_{l_{g_1},\dots,l_{g_r}}(E) = 2^{|G|-1} \sum_{a_1=0}^{l_{g_1}-1} \sum_{a_2=0}^{l_{g_2}-1} \cdots \sum_{a_r=0}^{l_{g_r}-1} \lambda_{a_1} \otimes \lambda_{a_2} \otimes \cdots \otimes \lambda_{a_r}$$

if $(l_{g_1}, \dots, l_{g_r}) \notin S(\varphi)$, *where* λ_{a_i} *is the hook partition of leg* a_i *and arm* $l_{g_i} - a_i + 1$.

Moreover

$$c_n^G(E) = 2^{n-1} \sum_{\substack{(l_{g_1},\dots,l_{g_r}) \notin S(\varphi) \\ l_{g_1}+\dots+l_{g_r}=n}} \binom{n}{l_{g_1},\dots,l_{g_r}}.$$

Proof. By Corollary 4.3, the spaces $P_n(E)$ and $P^G_{l_{g_1},\dots,l_{g_r}}(E)$ are $S_{l_{g_1}} \times \cdots \times S_{l_{g_r}}$-isomorphic modules. Hence the result follows using the decomposition of $\chi_n(E) = \sum_{i=0}^{n-1}(n-i, 1^i)$ and the representation theory of symmetric groups. More precisely, it follows by Branching Rule that when we restrict the irreducible representation $\nu_i = (n-i, 1^i)$ of S_n to its subgroup $S_{l_{g_1}} \times \cdots \times S_{l_{g_r}}$ then its $S_{l_{g_1}} \times \cdots \times S_{l_{g_r}}$-irreducible components are $\lambda_{a_1} \otimes \lambda_{a_2} \otimes \cdots \otimes \lambda_{a_r}$ for some $\lambda_{a_i} = (l_{g_i} - a_i, 1^{a_i})$. By Frobenius Reciprocity Law the multiplicity of $\lambda_{a_1} \otimes \lambda_{a_2} \otimes \cdots \otimes \lambda_{a_r}$ in the decomposition of ν_i equals the multiplicity of ν_i in the induced representation $(\lambda_{a_1} \otimes \lambda_{a_2} \otimes \cdots \otimes \lambda_{a_r})^{\uparrow S_n}$. We argue only for $r = 2$ because the other cases are treated similarly. By the Littlewood-Richardson Rule, if $c^i_{a,b}$ is the multiplicity of ν_i in the induced representation $(\lambda_a \otimes \lambda_b)^{\uparrow S_n}$, $c^i_{a,b}$ is the number of semistandard tableau T such that T has shape ν_i/λ_a, content λ_b and the row word of T is a reverse lattice permutation. Since ν_i and λ_a are both hook partitions, then the skew shape ν_i/λ_a has at most two connected components. The first one is a row of length $n - i - (l_{g_1} - a) = l_{g_2} - (i - a)$, the second is a column of height $i - a$. By the previous conditions on the semistandard tableau T, we obtain that the entries in the column constitute a standard tableau T'. If 1 does not appear in T' then $1 + b = l_{g_2} - (i - a)$, on the other hand if one entry of T' is 1 then $b = l_{g_2} - (i - a)$. Therefore $c^i_{a,b}$ is non-zero if and only if either $i - a + b = l_{g_2} - 1$ or $i - a + b = l_{g_2}$, in both cases one has $c^i_{a,b} = 1$ since the semistandard tableau T is uniquely determined. Then there exist exactly two hook partitions in the decomposition of $(\lambda_a \otimes \lambda_b)^{\uparrow S_n}$. Repeating this process, we have that the total multiplicity of the hook partitions appearing in the decomposition of $(\lambda_{a_1} \otimes \lambda_{a_2} \otimes \cdots \otimes \lambda_{a_r})^{\uparrow S_n}$ is $2^{r-1} = 2^{|G|-1}$. Due to the fact that all of these partitions are components of $\chi_n(E)$, we have that the multiplicity of $\lambda_{a_1} \otimes \lambda_{a_2} \otimes \cdots \otimes \lambda_{a_r}$ in $\chi_{l_{g_1},\dots,l_{g_r}}$ is exactly $2^{|G|-1}$.

Finally we have just to use Proposition 2.5, while Corollary 4.3 says that

$$c^G_{l_{g_1},\dots,l_{g_r}}(A) = c_n(E) = 2^{n-1}$$

and the assertion follows. □

In light of the previous results, we can give a new proof of an Anisimov's result (see [1]). Let p be a prime odd number and let $G = \mathbb{Z}_p$, then we have the following:

Proposition 4.5. *If there exists $k \in G$, $k \neq 0$ such that $\dim_F L^k = \infty$, then for any $m \in \mathbb{N}$,*

$$c_m(E) = p^m 2^{m-1}.$$

If for any $k \in \mathbb{Z}_p - \{0\}$ $\dim_F L^k < \infty$, *then for any* $m \in \mathbb{N}$,

$$c_m(E) = 2^{m-1} \sum_{\substack{(m_0,\dots,m_{p-1}) \notin S(\varphi) \\ \sum_{i=0}^{p-1} m_i = m}} \binom{m}{m_0,\dots,m_{p-1}}.$$

Proof. If exists $k \in \mathbb{Z}_p - \{0\}$ such that $\dim_F L^k = \infty$, then $\langle k\rangle$ has the property $\mathcal{P}$. In particular, $\mathbb{Z}_p$ has this property. The quotient grading on E is the trivial one and in light of Proposition 3.3, every G-graded polynomial identity of E is an ordinary polynomial identity of E. Then, for any $m \in \mathbb{N}$, we have

$$c_m(E) = \sum_{m_0+\dots+m_{p-1}=m} \binom{m}{m_0,\dots,m_{p-1}} c_{m_0,\dots,m_{p-1}}(E)$$
$$= 2^{m-1} \sum_{m_0+\dots+m_{p-1}=m} \binom{m}{m_0,\dots,m_{p-1}} = p^m 2^{m-1}.$$

If for any $k \in \mathbb{Z}_p - \{0\}$ $\dim_F L^k < \infty$, then $\dim_F L^0 = \infty$. By Corollary 4.3 and Proposition 4.4, for any $m \in \mathbb{N}$, we have

$$c_m(E) = \sum_{\substack{(m_0,\dots,m_{p-1}) \notin S(\varphi) \\ \sum_{i=0}^{p-1} m_i = m}} \binom{m}{m_0,\dots,m_{p-1}} c_{m_0,\dots,m_{p-1}}(E)$$
$$= 2^{m-1} \sum_{\substack{(m_0,\dots,m_{p-1}) \notin S(\varphi) \\ \sum_{i=0}^{p-1} m_i = m}} \binom{m}{m_0,\dots,m_{p-1}}.$$

and we are done. □

Notice that the case $p = 2$ has been completely solved in [3] and in [6].

5. Two examples of gradings by groups of order 4

5.1. **$\mathbb{Z}_4$-grading on E.** The group $G = \mathbb{Z}_4 = \{0,1,2,3\}$ is the first cyclic group such that its order is not prime. In light of Theorem 3.8 we have that $T_G(E)$ "behaves" as $T_{\mathbb{Z}_2}(E)$ in the quotient grading if $\dim_F L^1 = \infty$ or $\dim_F L^3 = \infty$. Moreover, because of Theorem 4.1 the only cases to be studied are the ones for which $\dim_F L^2 = \infty$. We study a particular case of G-grading when $\dim_F L^2 = \infty$.

Let L be a vector space with basis $B_L = \{e_1, e_2, \dots\}$ and let us consider the following map:

$$\varphi\colon B_L \to G$$

such that $\varphi(e_1) = 1$, $\varphi(e_2) = 3$ and $\varphi(e_i) = 2$ for any $i \neq 1, 2$. Then φ induces a G-grading on E such that $\dim L^2 = \infty$. In particular, it is easy to see that:

- $E^0 = \operatorname{span}\langle e_1^k e_2^t e_{i_1} \dots e_{i_s} | s \equiv 0 \text{mod } 2 \text{ and} (k,t) \in \{(1,1),(0,0)\}\rangle$;
- $E^1 = \langle e_1^k e_2^t e_{i_1} \dots e_{i_s} | s \equiv 0 \bmod 2 \text{ and } (k,t) = (1,0)$
  or $s \equiv 1 \bmod 2$ and $(k,t) = (0,1)\rangle$;

- $E^2 = \operatorname{span}\langle e_1^k e_2^t e_{i_1} \dots e_{i_s} | s \equiv 1 \bmod 2$ and $(k,t) \in \{(1,1),(0,0)\}\rangle$;
- $E^3 = \operatorname{span}\langle e_1^k e_2^t e_{i_1} \dots e_{i_s} | s \equiv 0 \bmod 2$ and $(k,t) = (0,1)$ or $s \equiv 1 \bmod 2$ and $(k,t) = (1,0)\rangle$.

From the previous description of the G-graded homogeneous components of E one easily has the following.

Proposition 5.1. *The following monomials are G-graded polynomial identities of E:*

$$x_1^1x_2^1x_3^1,\ x_1^3x_2^3x_3^3,\ x_1^1x_2^1x_3^3,\ x_1^1x_2^3x_3^1,\ x_1^3x_2^1x_3^1,\ x_1^1x_2^3x_3^3,\ x_1^3x_2^1x_3^3,\ x_1^3x_2^3x_3^1\,.$$

Proof. We argue only for the monomial $x_1^1x_2^1x_3^1$ because the other cases are treated similarly. From the previous observations it follows that if we want to evaluate one variable of G-homogeneous degree 1, we shall deal with a word which contains at least one of the basis elements $e_1,\ e_2$. Now the proposition follows because any evaluation of three variables of G-degree 1 repeats twice one between e_1 or e_2 and we are done. □

We have not only monomial graded identities.

Proposition 5.2. *The following polynomials are G-graded polynomial identities of E:*

$$x_1^2x_2^2 + x_2^2x_1^2,\ [x_1^1, x_2^1],\ [x_1^3, x_2^3],\ [x_1^0, x_1^g]\,,$$

for any $g \in G$.

Proof. The fact that $x_1^2x_2^2 + x_2^2x_1^2$ and $[x_1^0, x_1^g]$ are graded identities follows directly from the description of E^0, E^2. For, the elements of E^0 have even length so they are in the center of E. On the other hand, the elements of E^2 have odd length.

Let us argue for $[x_1^1, x_2^1]$. If we evaluate the variable x_1^1 with a G-degree 1 element of E of odd length, we are dealing with a word containing e_1. Hence the evaluation of x_2^1 lies in the center of E and the commutator vanishes, otherwise e_1 appears twice. We argue analogously for $[x_1^3, x_2^3]$ and we are done. □

We are now ready to compute $T_G(E)$. For this purpose, let

$$I_1 = \big\langle [u_1,u_2,u_3],\ x_1^2x_2^2 + x_2^2x_1^2,\ [x_1^1, x_2^1],\ [x_1^3, x_2^3],\ x_1^2x_2^2 + x_2^2x_1^2,$$
$$[x_1^0, x_1^g],\ x_1^1x_2^1x_3^1,\ x_1^3x_2^3x_3^3,\ x_1^1x_2^1x_3^3,\ x_1^1x_2^3x_3^3\big\rangle^{T_G}\,,$$

for any $g \in G$. Observe that modulo I the identity $[x_1^1x_2^3, x_3^g]$ equals the polynomial $x_1^1[x_2^3, x^g] + x_2^3[x_1^1, x^g]$ that is

(1) $$x_1^1[x_2^3, x^g] \equiv -x_2^3[x_1^1, x^g] \pmod I\,.$$

Analogously we have

(2) $$x_1^1[x_2^2, x_3^3] \equiv -x_2^2[x_1^1, x_3^3] \pmod I\,,$$

(3) $$x_1^3[x_2^2, x_3^1] \equiv +x_2^2[x_1^1, x_3^3] \pmod I\,,$$

(4) $$[x_1^1, x_2^2]x_3^3 \equiv -[x_1^1, x_3^3]x_2^2 \pmod I\,.$$

Then we have the following.

Theorem 5.3. *$I_1 = T_G(E)$.*

Proof. The Propositions 5.1 and 5.2 give the inclusion $I_1 \subseteq T_G(E)$. We shall use the method of Y-proper polynomials. In light of Proposition 5.1, we have that the only non-trivial subspaces of multilinear Y-proper polynomials are: $\Gamma_{0,1,l,1}$, $\Gamma_{0,1,l,0}$, $\Gamma_{0,0,l,1}$, $\Gamma_{0,0,l,0}$, $\Gamma_{0,2,l,0}$, and $\Gamma_{0,0,l,2}$ for any $l \in \mathbb{N}$. Let us argue only for $\Gamma_{0,1,l,1}$ because the other cases are treated similarly. Let w be any non-zero element in $\Gamma_{0,1,l,1}$, then w can be written as a linear combination of the following polynomials

$$x^1 x_1^2 \dots x_l^2 x^3 \,,$$

$$x_1^2 \dots x_l^2 [x^1, x^3] \,,$$

$$x^1 x_1^2 \dots \widehat{x_i^2} \dots x_l^2 [x_i^2, x^3] \,,$$

$$x_1^2 \dots \widehat{x_i^2} \dots x_l^2 x^3 [x^1, x_i^2] \,.$$

The Equations (1) and (2) give us

$$x^1 x_1^2 \dots \widehat{x_i^2} \dots x_l^2 [x_i^2, x^3] + \alpha x_1^2 \dots x_l^2 [x^1, x^3] \equiv x_1^2 \dots \widehat{x_i^2} \dots x_l^2 x^3 [x_i^2, x^1] \,.$$

Analogously it can be shown $x^1 x_1^2 \dots \widehat{x_i^2} \dots x_l^2 [x_i^2, x^3]$ is a linear combination of $x_1^2 \dots x_l^2 [x^1, x^3]$ and $x^1 x_1^2 \dots x_l^2 x^3$. Finally, any non-trivial polynomial of $\Gamma_{0,1,l,1}$ is a linear combination of the following polynomials:

$$w_1 = x^1 x_1^2 \dots x_l^2 x^3 \,,$$
$$w_2 = x_1^2 \dots x_l^2 [x^1, x^3] \,.$$

Now it suffices to show that w_1, w_2 are linearly independent modulo $T_G(E)$. Suppose by contradiction they are linearly dependent, then there exist $\alpha_1, \alpha_2 \in F$ such that $\sum_{i=1}^2 \alpha_i w_i \in T_G(E)$. Let us consider the following substitution φ:

$$\varphi(x^1) = e_2 e_3 \,,$$

$$\varphi(x^3) = e_1 e_4 \,,$$

$$\varphi(x_i^2) = e_{i+4} \quad \text{for any} \quad i = 1, \dots, l \,.$$

Then

$$\varphi(w_2) = 0$$

but

$$\varphi(w_1) = e_2 e_3 e_5 \dots e_{l+4} e_1 e_4 \neq 0 \,,$$

a contradiction and the proof is complete. □

According to [6], it seems that f is a multilinear $\mathbb{Z}_4$-graded identity of E if and only if $\gamma(f)$ is a $\mathbb{Z}_2$-graded identity of E_{2^*} for some special function γ.

5.2. $\mathbb{Z}_2\times\mathbb{Z}_2$**-gradings on** E**.** Theorem 3.8 is useful in order to reduce the order of the grading group if G has non-trivial squares. This is not the case of finite powers of $\mathbb{Z}_2$. In this section we shall deal with some special cases of $\mathbb{Z}_2\times\mathbb{Z}_2$-grading of E.

Let us suppose firstly that L is a G-homogeneous vector space over F such that

$$\dim_F L^{(0,1)} = \dim_F L^{(1,0)} = \dim_F L^{(1,1)} = \infty\,,\ \dim_F L^{(0,0)} < \infty\,,$$

and let $E = E(L)$ be the Grassmann algebra generated by L. Let $B_1 = \{e_1, e_2, \ldots\}$ be a basis of $L^{(1,0)}$, $B_2 = \{e'_1, e'_2, \ldots\}$ be a basis of $L^{(0,1)}$ and $B_3 = \{e''_1, e''_2, \ldots\}$ be a basis of $L^{(1,1)}$ as vector spaces. Let us consider the map

$$\varphi\colon B_1\cup B_2\cup B_3\colon\ \to G$$

associated to the G-grading over E. It is such that $\varphi(e_i) = (1,0)$ for any $i = 1,2,\ldots$, $\varphi(e'_j) = (0,1)$ for any j and $\varphi(e''_s) = (1,1)$ for any s. We have the following.

Lemma 5.4. *G has the property $\mathcal{P}$.*

Proof. The pairwise disjoint sets of elements $\{e_{2k+1}e'_{2k+1} \mid k \geq 0\}$, $\{e_{2k}e''_{2k} \mid k \geq 1\}$, $\{e'_{2k}e''_{6k+1} | k \geq 1\}$, and $\{e''_{6k+3}e''_{6k+5} \mid k \geq 1\}$ belong respectively to $E^{(1,1)}$, $E^{(0,1)}$, $E^{(1,0)}$, $E^{(0,0)}$ and the proof is complete. □

In light of the Proposition 3.3, we have the following result.

Theorem 5.5. *Let L be a G-homogeneous vector space over F such that*

$$\dim_F L^{(0,1)} = \dim_F L^{(1,0)} = \dim_F L^{(1,1)} = \infty\,,\ \dim_F L^{(0,0)} < \infty\,,$$

and let $E = E(L)$ the Grassmann algebra generated by L. Let f be a multilinear polynomial in $F\langle X \mid G\rangle$. Then $f \in T_G(E)$ if and only if $\pi(f) \in T(E)$.

Proof. By Lemma 5.4 G has the property $\mathcal{P}$ and we are done because of Proposition 3.3. □

Suppose now that L is a G-homogeneous vector space over F such that

$$\dim_F L^{(0,1)} = \dim_F L^{(1,0)} = \infty\,,\ \dim_F L^{(0,0)} < \infty\,,\ \dim_F L^{(1,1)} < \infty$$

and let $E = E(L)$ the Grassmann algebra generated by L. Let $B_1 = \{e_1, e_2, \ldots\}$ be a basis of $L^{(1,0)}$, $B_2 = \{f_1, f_2, \ldots\}$ be a basis of $L^{(0,1)}$ as vector spaces. Let us consider the map

$$\varphi\colon B_1\cup B_2\colon\ \to G$$

associated to the G-grading over E. It is such that $\varphi(e_i) = (1,0)$ for any $i = 1,2,\ldots$, $\varphi(f_j) = (0,1)$ for any j. We have the analog of Lemma 5.4. Let $H = \langle g\rangle$, where $g = (1,1)$. Notice that $H \equiv \mathbb{Z}_2$. Then we obtain the next result.

Lemma 5.6. *H has the property $\mathcal{P}$.*

Theorem 5.7. *Let L be a G-homogeneous vector space over F such that*

$$\dim_F L^{(0,1)} = \dim_F L^{(1,0)} = \infty\,,\quad \dim_F L^{(0,0)} < \infty\,,\quad \dim_F L^{(1,1)} < \infty$$

and let $E = E(L)$ the Grassmann algebra gnerated by L. Let f be a multilinear polynomial in $F\langle X \mid G\rangle$. Then $f \in T_G(E)$ if and only if $\pi(f) \in T_{\mathbb{Z}_2}(E)$, where in the $\mathbb{Z}_2$-grading, the dimension of the G-homogeneous underlying vector space L^0 is finite.

Proof. By Lemma 5.6, we have that H has the property $\mathcal{P}$. In light of Proposition 3.3, we have that $f \in T_G(E)$ if and only if $\pi(f) \in T_{G/H}(E)$. It is easy to see that $G/H \cong \mathbb{Z}_2$. Now, by the definition of quotient grading, we have that the new $\mathbb{Z}_2$-grading is such that

$$E^0 = E^{(0,0)} \oplus E^{(1,1)}$$

$$E^1 = E^{(0,1)} \oplus E^{(1,0)} .$$

By hypothesis we have that L^0 is finite dimensional and we are done. □

We shall deal now with G-gradings such that there exists one and only one $g \in G$ $g \neq (0,0)$ such that $\dim_F L^g = \infty$.

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and L a G-homogeneous vector space over F such that $\dim_F L^{(0,1)} = \infty$, and $\dim_F L^{(1,0)} = k$. Let $E = E(L)$ be the Grassmann algebra generated by L. Let $B_1 = \{e_1, e_2, \dots, e_k\}$ be a basis of $L^{(1,0)}$ as a vector space and let $B_2 = \{f_1, f_2, \dots\}$ be a basis of $L^{(0,1)}$ as a vector space. Let us consider the map:

$$\varphi \colon B_L \to G ,$$

associated to the G-grading of E. It is such that $\varphi(e_i) = (1,0)$ for any $i = 1, 2, \dots, k$ and $\varphi(f_j) = (0,1)$ for any j. It is easy to see that:

- $E^{(0,0)} = \operatorname{span}\langle e_{i_1} e_{i_2} \dots e_{i_t} f_{j_1} \dots f_{i_s} \mid s \equiv t \equiv 0 \bmod 2\rangle$;
- $E^{(0,1)} = \operatorname{span}\langle e_{i_1} e_{i_2} \dots e_{i_t} f_{j_1} \dots f_{i_s} \mid s \equiv 1 \bmod 2 \text{ and } t \equiv 0 \bmod 2\rangle$;
- $E^{(1,0)} = \operatorname{span}\langle e_{i_1} e_{i_2} \dots e_{i_t} f_{j_1} \dots f_{i_s} \mid s \equiv 0 \bmod 2 \text{ and } t \equiv 1 \bmod 2\rangle$;
- $E^{(1,1)} = \operatorname{span}\langle e_{i_1} e_{i_2} \dots e_{i_t} f_{j_1} \dots f_{i_s} \mid s \equiv t \equiv 1 \bmod 2\rangle$.

Let r, $s \in \mathbb{N}$ and w be a monomial in the variables of G-degree $(1,0)$ and $(1,1)$ only. We say that $w \in W_{r,s}$ if the number of variables appearing in w having G-degree $(1,0)$ is exactly r and the number of variables appearing in w having G-degree $(1,1)$ is s. From the previous description of the G-graded homogeneous components of E one easily has the following:

Proposition 5.8. *The following monomials are G-graded polynomial identities of E:*

$$\bigcup_{r+s \geq k+1} W_{r,s} .$$

Proof. In light of the fact that any element of G-degree $(1,0)$ and $(1,1)$ may contain at least one element among $\{e_1, \dots, e_k\}$, we have that if φ is any graded substitution of $w_{r,s}$, one of the k basis elements repeats at least twice and the Proposition follows. □

We have not only monomial graded identities.

Proposition 5.9. *The following polynomials are G-graded polynomial identities of E:*

$$x_1^{(0,1)} x_2^{(0,1)} + x_2^{(0,1)} x_1^{(0,1)} , \quad x_1^{(0,1)} x_2^{(1,0)} + x_2^{(1,0)} x_1^{(0,1)} , \quad x_1^{(1,0)} x_2^{(1,0)} + x_2^{(1,0)} x_1^{(1,0)} ,$$

$$[x_1^{(0,0)}, x_2^g] , \quad [x_1^{(1,1)}, x_2^g] , \quad \textit{for any} \quad g \in G .$$

Proof. It follows directly from the description of the various E^g. □

We are now ready to compute $T_G(E)$. For this purpose, let I_2 the T_G ideal generated by

$$[u_1, u_2, u_3], \bigcup_{r+s=k+1} W_{r,s}\,,\ \ x_1^{(0,1)}x_2^{(0,1)} + x_2^{(0,1)}x_1^{(0,1)}\,,\ \ x_1^{(0,1)}x_2^{(1,0)} + x_2^{(1,0)}x_1^{(0,1)}\,,$$

$$x_1^{(1,0)}x_2^{(0,1)} + x_2^{(1,0)}x_1^{(1,0)}\,,\ \ [x_1^{(0,0)}, x_2^{g}]\,,\ \ [x_1^{(1,1)}, x_2^{g}]\,,\quad \text{for any}\quad g \in G\,.$$

We have the following:

Theorem 5.10. *$I_2 = T_G(E)$.*

Proof. The Propositions 5.8 and 5.9 give the inclusion $I_2 \subseteq T_G(E)$. We shall use the method of Y-proper polynomials once again. The only non-trivial subspaces of Y-proper polynomials are $\Gamma_{0,t,r,s}$, such that $r+s \leq k$. Due to the anticommutativity of the variables of G-degree $(0,1), (1,0)$ and the commutativity of the variables of G-degree $(1,1)$, as in the previous proposition, we can write any polynomial in $\Gamma_{0,t,r,s}$ as linear combination of polynomials

$$x_1^{(0,1)} \dots x_t^{(0,1)} x_{t+1}^{(1,0)} \dots x_{t+r}^{(1,0)} x_{t+r+1}^{(1,1)} \dots x_{t+r+s}^{(1,1)}\,,$$

such that $r + s \leq k$ which are clearly linearly independent modulo $T_G(E)$. The conclusion follows as in the proof of Theorem 5.3. □

Again, according to [6], it seems that f is a multilinear $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded identity of E if and only if $\gamma(f)$ is a $\mathbb{Z}_2$-graded identity of E_{2^*} for some special function γ.

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