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ISOTROPIC ALMOST COMPLEX STRUCTURES AND HARMONIC UNIT VECTOR FIELDS

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Abstract. Isotropic almost complex structures $J_{\delta,\sigma}$ define a class of Riemannian metrics $g_{\delta,\sigma}$ on tangent bundles of Riemannian manifolds which are a generalization of the Sasaki metric. In this paper, some results will be obtained on the integrability of these almost complex structures and the notion of a harmonic unit vector field will be introduced with respect to the metrics $g_{\delta,0}$. Furthermore, the necessary and sufficient conditions for a unit vector field to be a harmonic unit vector field will be obtained.

1. Introduction

Let (M,g) be a Riemannian manifold and $\pi\colon TM \to M$ be its tangent bundle. furthermore, for a vector field X on M, let X^h, X^v be the horizontal and vertical lifts. Using natural lifts of the Riemannian metric g from the base manifold M to the total space TM of the tangent bundle, some new interesting geometric structures were studied (e.g. [1, 3, 6]). Maybe the best known Riemannian metric on the tangent bundle is the Sasaki metric introduced by Sasaki in 1958 (see [13]).

In [4], Aguilar defined a class of almost complex structures $J_{\delta,\sigma}$ on $\mathrm{T}M$, namely isotropic almost complex structures with definition

$$J_{\delta,\sigma}(X^h) = \alpha X^v + \sigma X^h\,, \qquad J_{\delta,\sigma}(X^v) = -\sigma X^v - \delta X^h\,,$$

for functions α, δ, $\sigma\colon \mathrm{T}M \to \mathbb{R}$ which satisfy $\alpha\delta - \sigma^2 = 1$. He showed that there exists an integrable isotropic almost complex structure on an open subset $\mathcal{A} \subseteq TM$ if and only if the sectional curvature of $(\pi(\mathcal{A}), g)$ is constant which is a good result rather than the classical case. Besides, he introduced special class of Riemannian metrics $g_{\delta,\sigma}$ constructed by the Liouville 1-form on $\mathrm{T}M$ together with the isotropic almost complex structures. They are generalizations of the Sasaki metric and in some cases, intersect the class of g-natural metrics.

Aguilar proved the existence part of his theorem for integrable structures in special cases. These special cases induce a class of g-natural metrics on TM. So, it is natural to ask the following question: **Is there any other integrable structures**

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on a space form? The authors asked this question in Mathoverflow and R. Bryant proved there are many other cases. Bryant characterized the integrable isotropic almost complex structures on $T\mathbb{R}^n$ and TS^n and then answered to the stated problem.

In this paper, the authors will represent other equivalents to the integrability of isotropic almost complex structures on $T\mathbb{R}^n$ and TS^n based on PDE's.

Let (M, g) be a compact Riemannian manifold and (TM, g_s) be its tangent bundle equipped with the Sasaki metric. Moreover, suppose $(S(M), i^*g_s)$ is the unit tangent bundle of (M, g) where $i\colon S(M) \longrightarrow TM$ is the inclusion map and i^*g_s is the induced Sasaki metric on the unit tangent bundle. Denote by $\Gamma(TM)$ the set of all smooth vector fields on M. Moreover, let ∇ be the Levi-Civita connection of g and $\Delta_g X$ be the rough Laplacian of vector field X with respect to metric g.

Since, every vector field defines a map from (M, g) to (TM, g_s), it is natural to investigate the harmonicity of maps defined by vector fields.

Nouhaud [9] deduced that a vector field X defines a harmonic map from (M, g) to (TM, g_s) if and only if X is a parallel vector field. She found the expression of the Dirichlet energy associated to the vector field X as

$$E(X) = \frac{n}{2}\,\mathrm{vol}(M) + \frac{1}{2}\int_M \|\nabla X\|^2 d\,\mathrm{vol}(g)\,,$$

where $\mathrm{vol}(M)$ is the volume of M with respect to the metric g and $\|\nabla X\|$ is the norm of ∇X as a $(1,1)$-tensor. She proved that parallel vector fields are the critical points of the Dirichlet energy defined from $C^\infty((M, g), (TM, g_s))$ to $\mathbb{R}^+$ by using the stated formula for $E(X)$.

Gil-Medrano [12] investigated the critical points of the energy functional $E\colon \Gamma(TM) \to \mathbb{R}^+$ where E is the restricted Dirichlet energy functional to the vector fields on a compact Riemannian manifold (M, g). She proved that such vector fields are again parallel vector fields i.e., $\nabla X = 0$.

Wood [9] introduced the notion of harmonic unit vector fields on a compact Riemannian manifold (M, g) by restricting the Dirichlet energy functional to the unit vector fields and called the critical points of that functional, harmonic unit vector fields. Recall that when the Dirichlet energy functional is restricted to the unit vector fields, the vanishing of ∇X ensures that the unit vector field X is a harmonic unit vector field. But the inverse is not true; harmonic vector fields need not be parallel. So, it is natural to investigate the harmonicity of unit vector fields. Wood demonstrated that a unit vector field X is a harmonic unit vector field if and only if $\Delta_g X = \|\nabla X\|^2 X$.

The contributions on the harmonicity of maps defined by vector fields is not limited to the tangent bundles equipped with the Sasaki metric. Abbassi et al. [1] and [2] studied the problem of determining of harmonicity of such maps with respect to the g-natural metrics. They proved that a unit vector field is harmonic if and only if its Laplacian is colinear with itself which is a similar result to the Wood's. Recently, Calvaruso [7, 8] has started investigating the harmonicity of vector fields on pseudo-Riemannian manifolds and this subject has been continued in [10] and [11]. The generalized metrics $g_{\delta,0}$ is of the form

$$\begin{aligned} g_{\delta,0}(X^h,Y^h) &= \frac{1}{\delta}g(X,Y)o\pi\,,\\ g_{\delta,0}(X^h,Y^v) &= 0\,,\\ g_{\delta,0}(X^v,Y^v) &= \delta g(X,Y)o\pi\,, \end{aligned}$$

and so it would be interesting to investigate the harmonicity of vector fields with respect to these metrics.

The rest of the paper is organized as follows: In Section 2, some propositions on integrability of isotropic almost complex structures are resulted. It is notable that one of them is based on R. Bryants answer on Mathoverflow[1 2]. Section 3 is devoted to achieve the necessary and sufficient conditions for a unit vector field to be a critical point of the Dirichlet energy functional when the variations pass through the unit vector fields.

2. Isotropic almost complex structures

Assume (M,g) is an n-dimensional Riemannian manifold and ∇ represents the Levi-Civita connection of g. Moreover, let $\pi\colon TM \longrightarrow M$ be its tangent bundle and $K\colon TTM \longrightarrow TM$ be the connection map with respect to ∇. The tangent bundle of $TM(TTM)$ can be split to vertical and horizontal vector sub-bundles $\mathcal{V}$ and $\mathcal{H}$, respectively, i.e., for every $v\in TM$, $T_vTM=\mathcal{V}_v\oplus\mathcal{H}_v$.

If we denote the vertical and horizontal lifts of $X\in\Gamma(TM)$ by X^v and X^h then the Lie bracket of the horizontal and vertical vector fields at $u\in TM$ are expressed as follows

$$[X^h,Y^h](u) = [X,Y]^h_u - (R(X,Y)u)^v_u\,, \tag{1}$$

$$[X^h,Y^v](u) = (\nabla_X Y)^v_u\,, \tag{2}$$

$$[X^v,Y^v](u) = 0(u)\,, \tag{3}$$

where 0 is the zero vector field on TM. Moreover, if we consider the vector field $X\colon M\longrightarrow TM$ as a map between manifolds, its derivative X_* at a point p in M is given by

$$X_{*p}(V) = V^h_{X(p)} + (\nabla_V X)^v_{X(p)} \qquad \forall\, V\in\Gamma(TM)\,. \tag{4}$$

Moreover, we have the following useful equations

$$X^h\big(g(Y,Z)o\pi\big) = \big(Xg(Y,Z)\big)o\pi\,, \tag{5}$$

$$X^v\big(g(Y,Z)o\pi\big) = 0\,. \tag{6}$$

Now, let (M,J) be an almost complex manifold and $T^C(M)$ be the Complexification of TM. For $x\in M$, define the spaces $T^{(0,1)}_x(M)$ and $T^{(1,0)}_x(M)$ of $T^C_x(M)$ as

$$T^{(0,1)}_x(M) = \big\{X_x+\sqrt{-1}J_xX_x \mid X(x)\in T_x(M)\big\}\,,$$

[1]link: solutions of equations characterizing a complex structure
[2]link: existence of non constant solutions for this equations

and

$$T_x^{(1,0)}(M) = \left\{X_x - \sqrt{-1}J_x X_x \mid X(x) \in T_x(M)\right\}.$$

Now, let $T^{(0,1)}(M) = \cup_x T_x^{(0,1)}(M)$ and $T^{(1,0)}(M) = \cup_x T_x^{(1,0)}(M)$. It is a well known fact that J is an integrable structure if and only if for all sections A, $B \in \Gamma(T^{(0,1)}(M))$ we have $[A, B] \in \Gamma(T^{(0,1)}(M))$; equivalently, for an arbitrary 1-form ζ of the dual space of $T^{(1,0)}(M)$ (where denoted by $T^{(1,0)}(M)^*$) we have

$$\mathrm{d}\zeta \in \wedge^2 T^{(1,0)}(M)^*.$$

Definition 1. Let $\eta, \eta^1, \ldots, \eta^n$ be 1-forms on a differentiable manifold M. We say that $\mathrm{d}\eta \equiv 0 \bmod \{\eta^1, \ldots, \eta^n\}$, if and only if $\mathrm{d}\eta = \sum_{i,j} f_{ij}\eta^i \wedge \eta^j$, for some functions f_{ij} on M.

So, if $\zeta^1, \ldots, \zeta^n$ are locally $(1,0)$-forms generating $\Gamma(T^{(1,0)}(M)^*)$, then J is integrable if and only if $\mathrm{d}\zeta \equiv 0 \bmod \{\zeta^1, \ldots, \zeta^n\}$, $\forall\, \zeta \in \Gamma(T^{(1,0)}(M)^*)$.

Isotropic almost complex structures are a generalized type of the natural almost complex structure $J_{1,0}\colon TTM \longrightarrow TTM$ given by

$$J_{1,0}(X^h) = X^v\,, \qquad J_{1,0}(X^v) = -X^h\,, \qquad \forall\, X \in \Gamma(TM)\,.$$

It is a well-known fact that the necessary condition for integrability of $J_{1,0}$ is that the base manifold is flat. Aguilar proved [4] that there is an integrable isotropic almost complex structure on an open subset $\mathcal{A} \subset TM$ if and only if $(\pi(\mathcal{A}), g)$ is of constant sectional curvature.

Definition 2 ([4])**.** An almost complex structure J on TM is said to be isotropic with respect to the Riemannian metric g on M, if there are smooth functions α, δ, $\sigma\colon TM \longrightarrow \mathbb{R}$ such that $\alpha\delta - \sigma^2 = 1$ and

$$JX^h = \alpha X^v + \sigma X^h\,, \quad JX^v = -\sigma X^v - \delta X^h\,, \quad \forall X \in \Gamma(TM)\,. \tag{7}$$

Hereafter, we will represent the isotropic almost complex structure associated to the maps α, δ and σ by $J_{\delta,\sigma}$.

Suppose (M, g) has constant sectional curvature k. Aguilar [4] proved that $J_{\delta,\sigma}$ is an integrable structure on an open subset $\mathcal{A} \subset TM$ if and only if the following equation holds

$$d\sigma + k\delta\Theta - \sqrt{-1}(1 - \sqrt{-1}\sigma)\delta^{-1}d\delta \equiv 0 \qquad \bmod\ \{\zeta^1, \ldots, \zeta^n\}\,, \tag{8}$$

where $\zeta^1, \ldots, \zeta^n$ are 1-forms generating the space of $(1,0)$-forms induced by $J_{\delta,\sigma}$ on $\mathcal{A} \subset TM$ and Θ is the Liouvill 1-form on TM. When α, δ and σ are functions of $E(u) = \frac{1}{2}g(u,u)$ then the above equation gives the following solutions for δ and σ [4]

$$\delta^{-1} = \sqrt{2kE + b}\,, \qquad \sigma = 0\,, \tag{9}$$

$$\delta^{-2} = \frac{1}{2}\{2kE + b + \sqrt{(2kE+b)^2 + 4a^2k^2}\} \qquad \sigma = ak\delta^2\,,\ a \neq 0\,, \tag{10}$$

where a, $b \in \mathbb{R}$.

Remark 3. When we work with Θ, it is convenient to work with a locally orthonormal frame field on (M,g) like $X_1,\dots,X_n$. Because, if we suppose that π, K are the natural projection from $\mathrm{T}M$ to M and the connection map, respectively and if we suppose θ^i is the dual 1-forms of X_i then

$$\mathrm{d}\Theta = \sum_{i=1}^{n} (\theta^i \circ K) \wedge (\pi^* \theta^i)\,,$$

where $\{\theta^i \circ K, \pi^*\theta^i\}$ is the dual basis of $\{X_i^v, X_i^h\}$.

When $k=0$ we prove that the equation (8) is equivalent to the n-complex equations in the following proposition.

Proposition 4. *Let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be the Euclidean space. Suppose $(x^1,\dots,x^n)$ and $(x^1,\dots,x^n,y^1,\dots,y^n)$ are the natural coordinate systems for $\mathbb{R}^n$ and $\mathbb{R}^{2n}=T\mathbb{R}^n$, respectively. Suppose $J_{\delta,\sigma}$ is an almost complex structure and let $z=u+iv$ be a complex function on $\mathbb{R}^{2n}=T\mathbb{R}^n$ with $v=\frac{1}{\delta}$ and $u=\frac{\sigma}{\delta}$. Then $J_{\delta,\sigma}$ is integrable if and only if*

$$\frac{\partial z}{\partial x^l} + z\frac{\partial z}{\partial y^l} = 0 \qquad \forall\, l,\ 1\le l\le n\,. \tag{11}$$

Proof. It is easy to check that the 1-forms $u^l=\sqrt{-1}\delta(dy^l - zdx^l)$, $1\le l\le n$, span the space of $(1,0)$-forms induced by $J_{\delta,\sigma}$. Let $\zeta^l = dy^l - zdx^l$ then $J_{\delta,\sigma}$ is integrable if and only if

$$d\zeta^l \equiv 0 \qquad \mathrm{mod}\{\zeta^1,\dots,\zeta^n\}\,. \tag{12}$$

Since $d\zeta^1 = -dz\wedge dx^l$, the equation (12) can only happen if

$$dz \equiv 0 \qquad \mathrm{mod}\{\zeta^1,\dots,\zeta^n\}\,. \tag{13}$$

But $dz=\sum_{l=1}^n \left(\frac{\partial z}{\partial x^l}dx^l + \frac{\partial z}{\partial y^l}dy^l\right)$ and so

$$dz \equiv \sum_{l=1}^{n} \Big(\frac{\partial z}{\partial x^l} + z\frac{\partial z}{\partial y^l}\Big)dx^l \qquad \mathrm{mod}\{\zeta^1,\dots,\zeta^n\}\,. \tag{14}$$

So the equation (13) holds if and only if $\frac{\partial z}{\partial x^l} + z\frac{\partial z}{\partial y^l} = 0,\quad \forall\, 1\le l\le n$ and the proof is completed. □

It is natural to think of **is there any other integrable structure $J_{\delta,\sigma}$ except the types given by (9) and (10)?**

The following arguments are based on the Bryant's answer in Mathoverflow. For more information we refer the reader to their web addresses mentioned in the introduction.

Let $\mathbb{R}^{n+1}$ be given its standard inner product (and extend it complex linearly to a complex inner product on $\mathbb{C}^{n+1}$, which will be used below). Then

$$S^n = \{u\in\mathbb{R}^{n+1} \mid u\cdot u = 1\}$$

and $TS^n = \{(u,v)\in\mathbb{R}^{2n+2} \mid u\cdot u = 1 \text{ and } u\cdot v = 0\}$.

Let $H_+ = \{x + iy \mid y \gneqq 0\} \subset \mathbb{C}$ be the upper-half line in $\mathbb{C}$. Define a mapping

$$\Phi\colon TS^n \times H_+ \longrightarrow \mathbb{C}^{n+1} \setminus \mathbb{R}^{n+1}$$

by $\Phi((u, v), z) = v - zu$ where any vector $w = (w_1, \dots, w_{n+1}) \in \mathbb{R}^{n+1}$ is considered as a vector in $\mathbb{C}^{n+1}$ like this vector $w = (w_1, 0, \dots, w_{n+1}, 0)$ and so $zu, v - zu$ can be done naturally. Φ is a diffeomorphism and establishes a foliation of $\mathbb{C}^{n+1} \setminus \mathbb{R}^{n+1}$ where the leaves of the foliation are the image of $\{(u, v)\} \times H_+$ for every $(u, v) \in TS^n$ under Φ.

The following proposition characterizes the integrable structures $J_{\delta,\sigma}$ when the base manifold is an sphere.

Proposition 5. *When $n \geq 2$, the almost complex structure $J_{\delta,\sigma}$ on an open subset $\mathcal{A} \subset TS^n$ is integrable if and only if the image of the mapping*

$$\Phi_z\colon \mathcal{A} \longrightarrow \mathbb{C}^{n+1} \setminus \mathbb{R}^{n+1}\,,$$

with the definition $\Phi_z(u, v) = \Phi((u, v), z(u, v))$ is a holomorphic hypersurface in $\mathbb{C}^{n+1} \setminus \mathbb{R}^{n+1}$, where $z\colon \mathcal{A} \longrightarrow H_+$ is a mapping defined by $z(u, v) = \frac{\sigma+i}{\delta}(u, v)$.

This proposition implies that any holomorphic hypersurface of $\mathbb{C}^{n+1} \setminus \mathbb{R}^{n+1}$ that is transverse to the half-line foliation determined by Φ and intersects each such half-line in at most one point introduces a complex structure $J_{\delta,\sigma}$. So, one can construct a complex structure $J_{\delta,\sigma}$ on TS^n by using certain holomorphic hypersurfaces of $\mathbb{C}^{n+1} \setminus \mathbb{R}^{n+1}$.

The following statements gives another equivalent to the integrability of $J_{\delta,\sigma}$ on an open subset $TU \subset TS^n$ where U is an open subset of the unit standard sphere (S^n, g).

Let $E_0\colon U \subset S^n \to \mathbb{R}^{n+1}$ denote the (vector-valued) inclusion mapping. Let $E_1, \dots, E_n\colon U \to \mathbb{R}^{n+1}$ be any (smooth) orthonormal tangential frame field extending E_0, i.e., $\langle E_a, E_b\rangle = \delta_{ab}$ for $0 \leq a,\ b \leq n$. Define functions $v_i\colon TU \to \mathbb{R}$ by $v_i(u, v) = E_i(u) \cdot v$ for $1 \leq i \leq n$, so that $v = \sum_{i=1}^n v_i E_i(u)$ for all $(u, v) \in TU$.

One can consider $\zeta^1, \dots, \zeta^n$ as a basis for the $(1, 0)$-forms on TU with respect to $J_{\delta,\sigma}$. With the above notifications, the almost complex structure $J_{\delta,\sigma}$ on TU is an integrable structure if and only if the following equation holds

$$d(z^2 + v_1^2 + \cdots + v_n^2) \equiv 0 \mod \{\zeta^1, \dots, \zeta^n\}\,. \tag{15}$$

One can conclude the following proposition using the above notations.

Proposition 6. *Let $((x^1, \dots, x^n), U)$ be the conformally flat coordinate system on $U \subset (S^n, g)$, $(x^1, \dots, x^n, y^1, \dots, y^n)$ be the associated coordinate system on its tangent bundle and moreover let $J_{\delta,\sigma}$ be an isotropic almost complex structure on TU. Then $J_{\delta,\sigma}$ is an integrable structure if and only if*

$$\sum_{i=1}^n \Big[\frac{\partial z}{\partial y^i}(y^{s_0}\mu_i - \mu_{s_0} y^i) - \frac{\partial z}{\partial y^{s_0}} y^i \mu_i\Big] = y^{s_0}\lambda^2 - \Big(\frac{\partial z}{\partial x^{s_0}} + z\frac{\partial z}{\partial y^{s_0}}\Big),$$

for all s_0 with $1 \leq s_0 \leq n$.

Proof. By considering λ as the conformal factor, the metric g on U can be written as follow

$$g = \lambda^2(dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n)\,. \tag{16}$$

So, one can define $E_i = \frac{1}{\lambda}\frac{\partial}{\partial x^i}$ for all $i = 1, \ldots, n$ and

$$v_i(u, y) = \lambda(u)y^i\,, \qquad \forall\,(u, y) \in TU \qquad \text{and} \qquad i = 1, \ldots, n\,.$$

It can be proved that in this coordinate system, $\zeta^1, \ldots, \zeta^n$ are given by

$$\zeta^k = dy^k + y^j(\delta^k_j \mu_l dx^l + \mu_j dx^k - \mu_k dx^j) - z dx^k\,,$$

where $\mu_i = \frac{1}{\lambda}\frac{\partial \lambda}{\partial x^i}$, $i = 1, \ldots, n$. Moreover, in this coordinate system, one can get

$$d_{(u,y)}(z^2 + v_1^2 + \cdots + v_n^2) = \sum_{i=1}^{n}\Big[\Big(2z\frac{\partial z}{\partial x^i} + 2\mu_i\|y\|^2\Big)dx^i + 2\Big(y^i\lambda^2 + z\frac{\partial z}{\partial y^i}\Big)dy^i\Big]\,,$$

and so

$$\begin{aligned} & d_{(u,y)}(z^2 + v_1^2 + \cdots + v_n^2) - 2\sum_{i=1}^{2}\Big(y^i\lambda^2 + z\frac{\partial z}{\partial y^i}\Big)\zeta^i(u, y) \\ & = 2z\sum_{s=1}^{n}\Big[\frac{\partial z}{\partial y^i}(y^s\mu_i - \mu_s y^i) + \frac{\partial z}{\partial y^s}(z - y^j\mu_j) - y^s\lambda^2 + \frac{\partial z}{\partial x^s}\Big]dx^s\,. \end{aligned} \tag{17}$$

Using the equations (15) and (17), one can get the conclusion. □

The following example is one of the integrable structures constructed by Bryant which is different from (9) and (10).

Example 7. Let $v = \frac{1}{\delta}$ and $u = \frac{\sigma}{\delta}$ and define

$$u(x, y) = \frac{x \cdot y}{1 + x \cdot x}\,,$$

and

$$v = (x, y) = \frac{\sqrt{(1 + x \cdot x)(1 + y \cdot y) - (x \cdot y)^2}}{1 + x \cdot x}\,,$$

where $x = (x^1, \ldots, x^n)$, $y = (y^1, \ldots, y^n)$ and "$\cdot$" denotes the standard product on $\mathbb{R}^n$. It is easy to check that $z = u + iv$ satisfies the Proposition 4 and so $J_{\delta,\sigma}$ is a complex structure on a certain open subset of $T\mathbb{R}^n$.

Now, the metric $g_{\delta,\sigma}$ induced by the isotropic almost complex structure $J_{\delta,\sigma}$ will be defined.

Definition 8 ([4])**.** Let (M, g) be a Riemannian manifold and $J_{\delta,\sigma}$ be an isotropic almost complex structure on TM. Then the $(0, 2)$-tensor

$$g_{\delta,\sigma}(A, B) = d\Theta(J_{\delta,\sigma}A, B)\,,$$

defines a Riemannian metric on TM if $\alpha > 0$, where A, $B \in \Gamma(TTM)$.

Let X, Y be local sections of TM. A simple calculation shows that

$$g_{\delta,\sigma}(X^h, Y^h) = \alpha g(X, Y) o\pi \,, \tag{18}$$

$$g_{\delta,\sigma}(X^h, Y^v) = -\sigma g(X, Y) o\pi \,, \tag{19}$$

$$g_{\delta,\sigma}(X^v, Y^v) = \delta g(X, Y) o\pi \,. \tag{20}$$

The following theorem [5] states the formulas of the Levi-Civita connection.

Theorem 9. *Let $g_{\delta,\sigma}$ be a Riemannian metric on TM as before. Then the Levi-Civita connection $\bar{\nabla}$ of $g_{\delta,\sigma}$ at $(p, u) \in TM$ is given by*

$$\begin{aligned}
\bar{\nabla}_{X^h} Y^h = {} & (\nabla_X Y)^h - \frac{\sigma}{\alpha}\big(R(u, X)Y\big)^h + \frac{1}{2\alpha} X^h(\alpha) Y^h + \frac{1}{2\alpha} Y^h(\alpha) X^h \\
& - \frac{\sigma}{\delta}(\nabla_X Y)^v - \frac{1}{2}\big(R(X, Y)u\big)^v - \frac{1}{2\delta} X^h(\sigma) Y^v \\
& - \frac{1}{2\delta} Y^h(\sigma) X^v - \frac{1}{2} g(X, Y) \bar{\nabla}\alpha \,,
\end{aligned} \tag{21}$$

$$\begin{aligned}
\bar{\nabla}_{X^h} Y^v = {} & -\frac{\sigma}{\alpha}(\nabla_X Y)^h + \frac{\delta}{2\alpha}\big(R(u, Y)X\big)^h - \frac{1}{2\alpha} X^h(\sigma) Y^h \\
& + \frac{1}{2\alpha} Y^v(\alpha) X^h + (\nabla_X Y)^v + \frac{1}{2\delta} X^h(\delta) Y^v - \frac{1}{2\delta} Y^v(\sigma) X^v \\
& + \frac{1}{2} g(X, Y) \bar{\nabla}\sigma \,,
\end{aligned} \tag{22}$$

$$\begin{aligned}
\bar{\nabla}_{X^v} Y^h = {} & \frac{\delta}{2\alpha}\big(R(u, X)Y\big)^h + \frac{1}{2\alpha} X^v(\alpha) Y^h - \frac{1}{2\alpha} Y^h(\sigma) X^h \\
& - \frac{1}{2\delta} X^v(\sigma) Y^v + \frac{1}{2\delta} Y^h(\delta) X^v + \frac{1}{2} g(X, Y) \bar{\nabla}\sigma \,,
\end{aligned} \tag{23}$$

$$\begin{aligned}
\bar{\nabla}_{X^v} Y^v = {} & -\frac{1}{2\alpha} X^v(\sigma) Y^h - \frac{1}{2\alpha} Y^v(\sigma) X^h + \frac{1}{2\delta} X^v(\delta) Y^v \\
& + \frac{1}{2\delta} Y^v(\sigma) X^v + \frac{1}{2} g(X, Y) \bar{\nabla}\delta \,.
\end{aligned} \tag{24}$$

Proof. For the first one using the Koszul formula, we have

$$\begin{aligned}
2g_{\delta,\sigma}(\bar{\nabla}_{X^h} Y^h, Z^h) = {} & X^h g_{\delta,\sigma}(Y^h, Z^h) + Y^h g_{\delta,\sigma}(X^h, Z^h) - Z^h g_{\delta,\sigma}(X^h, Y^h) \\
& + g_{\delta,\sigma}\big([X^h, Y^h], Z^h\big) + g_{\delta,\sigma}\big([Z^h, X^h], Y^h\big) \\
& - g_{\delta,\sigma}\big([Y^h, Z^h], X^h\big) \,.
\end{aligned}$$

Using relations (1), (5) and (18) gives us

$$\begin{aligned}2g_{\delta,\sigma}(\bar\nabla_{X^h}Y^h,Z^h)=&\,X^h(\alpha)g(Y,Z)+\alpha Xg(Y,Z)+Y^h(\alpha)g(X,Z)\\&+\alpha Yg(X,Z)-Z^h(\alpha)g(X,Y)-\alpha Zg(X,Y)\\&+\alpha g\big([X,Y],Z\big)+\sigma g\big(R(X,Y)u,Z\big)+\alpha g\big([Z,X]Y\big)\\&+\sigma g\big(R(Z,X)u,Y\big)-\alpha g\big([Y,Z],X\big)\\&-\sigma g\big(R(Y,Z)u,X\big)\,.\end{aligned}$$

Using the properties of the Levi-Civita connection of g, one can get

$$\begin{aligned}2g_{\delta,\sigma}(\bar\nabla_{X^h}Y^h,Z^h)=&\,g\big(X^h(\alpha)Y,Z\big)+g\big(Y^h(\alpha)X,Z\big)-Z^h(\alpha)g(X,Y)\\&+2\alpha g(\nabla_XY,Z)+\sigma g\big(R(X,Y)u,Z\big)\\&+\sigma g\big(R(Z,X)u,Y\big)-\sigma g\big(R(Y,Z)u,X\big)\,.\end{aligned}$$

Using (18) and the Bianchi's first identity, we have

$$\begin{aligned}2g_{\delta,\sigma}(\bar\nabla_{X^h}Y^h,Z^h)=g_{\delta,\sigma}\Big(&\frac{1}{\alpha}X^h(\alpha)Y^h+\frac{1}{\alpha}Y^h(\alpha)X^h-g(X,Y)\bar\nabla\alpha\\&+2(\nabla_XY)^h-\frac{2\sigma}{\alpha}\big(R(u,X)Y\big)^h,Z^h\Big)\,,\end{aligned}$$

therefore, we have

$$\begin{aligned}g_{\delta,\sigma}\Big(2\bar\nabla_{X^h}Y^h&-\frac{1}{\alpha}X^h(\alpha)Y^h+\frac{1}{\alpha}Y^h(\alpha)X^h-g(X,Y)\bar\nabla\alpha\\&+2(\nabla_XY)^h-\frac{2\sigma}{\alpha}\big(R(u,X)Y\big)^h,Z^h\Big)=0\,,\end{aligned}$$

for all $Z\in TM$. So,

$$\begin{aligned}\text{Horizontal}\Big(2\bar\nabla_{X^h}Y^h&-\frac{1}{\alpha}X^h(\alpha)Y^h+\frac{1}{\alpha}Y^h(\alpha)X^h-g(X,Y)\bar\nabla\alpha\\&+2(\nabla_XY)^h-\frac{2\sigma}{\alpha}\big(R(u,X)Y\big)^h\Big)=0\,.\end{aligned}$$

Therefore, the horizontal part of $\bar\nabla_{X^h}Y^h$ is

$$\begin{aligned}h(\bar\nabla_{X^h}Y^h)=&\,\frac{1}{2\alpha}X^h(\alpha)Y^h+\frac{1}{2\alpha}Y^h(\alpha)X^h-\frac{1}{2}g(X,Y)h(\bar\nabla\alpha)+(\nabla_XY)^h\\&-\frac{\sigma}{\alpha}\big(R(u,X)Y\big)^h\,,\end{aligned}$$

where $\bar\nabla\alpha=h(\bar\nabla\alpha)+v(\bar\nabla\alpha)$ is the splitting of the gradient vector field of α with respect to $g_{\delta,\sigma}$ to horizontal and vertical components, respectively. Similarly the vertical component of $\bar\nabla_{X^h}Y^h$ can be computed as followings using the Koszul formula

$$\begin{aligned}2g_{\delta,\sigma}(\bar\nabla_{X^h}Y^h,Z^v)=&\,X^hg_{\delta,\sigma}(Y^h,Z^v)+Y^hg_{\delta,\sigma}(X^h,Z^v)-Z^vg_{\delta,\sigma}(X^h,Y^h)\\&+g_{\delta,\sigma}([X^h,Y^h],Z^v)+g_{\delta,\sigma}([Z^v,X^h],Y^h)-g_{\delta,\sigma}([Y^h,Z^v],X^h)\,.\end{aligned}$$

Now, from the definition of $g_{\delta,\sigma}$ and Lie bracket of horizontal and vertical vectors we will have

$$\begin{aligned}2g_{\delta,\sigma}(\bar{\nabla}_{X^h}Y^h, Z^v) = &-X^h(\sigma)g(Y,Z) - \sigma Xg(Y,Z) - Y^h(\sigma)g(X,Z) - \sigma Yg(X,Z)\\ &- Z^v(\alpha)g(X,Y) - \sigma g([X,Y],Z) - g_{\delta,\sigma}\big((R(X,Y)u)^v, Z^v\big)\\ &+ \sigma g(\nabla_X Z, Y) + \sigma g(\nabla_Y Z, X)\,.\end{aligned}$$

Using the compatibility of $\bar{\nabla}$ and $g_{\delta,\sigma}$ and using the equation $Z^v(\alpha) = \bar{g}(Z^v, \bar{\nabla}\alpha)$ we will get

$$\begin{aligned}2g_{\delta,\sigma}(\bar{\nabla}_{X^h}Y^h, Z^v) = &\frac{-X^h(\sigma)}{\delta}g_{\delta,\sigma}(Y^v, Z^v) - \frac{Y^h(\sigma)}{\delta}g_{\delta,\sigma}(X^v, Z^v)\\ &- g_{\delta,\sigma}\big(g(X,Y)\bar{\nabla}\alpha, Z^v\big) - g_{\delta,\sigma}\big((R(X,Y)u)^v, Z^v\big)\\ &- 2\frac{\sigma}{\delta}g_{\delta,\sigma}\big((\nabla_X Y)^v, Z^v\big)\,.\end{aligned}$$

So, the vertical part is

$$\begin{aligned}v(\bar{\nabla}_{X^h}Y^h) = &-\frac{1}{2\delta}X^h(\sigma)Y^v - \frac{1}{2\delta}Y^h(\sigma)X^v - \frac{1}{2}g(X,Y)v(\bar{\nabla}\alpha)\\ &- \frac{\sigma}{\delta}(\nabla_X Y)^v - \frac{1}{2}(R(X,Y)u)^v\,.\end{aligned}$$

Using the equation $(\bar{\nabla}_{X^h}Y^h) = h(\bar{\nabla}_{X^h}Y^h) + v(\bar{\nabla}_{X^h}Y^h)$, the proof of first part will be complete. For the second formula and its horizontal part we have

$$\begin{aligned}2g_{\delta,\sigma}(\bar{\nabla}_{X^h}Y^v, Z^h) = &X^h g_{\delta,\sigma}(Y^v, Z^h) + Y^v g_{\delta,\sigma}(X^h, Z^h) - Z^h g_{\delta,\sigma}(Y^v, X^h)\\ &+ g_{\delta,\sigma}([X^h, Y^v], Z^h) + g_{\delta,\sigma}([Z^h, X^h], Y^v)\\ &- g_{\delta,\sigma}([Y^v, Z^h], X^h)\,.\end{aligned}$$

After using the definition of $g_{\delta,\sigma}$ and substituting the equation

$$g(\nabla_Z Y, X) = Zg(X,Y) - g([Z,X] + \nabla_X Z, Y)$$

and putting $R(Z,X,u,Y) = -R(u,Y,X,Z)$ and $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, we get

$$\begin{aligned}2g_{\delta,\sigma}(\bar{\nabla}_{X^h}Y^v, Z^h) = &-\frac{X^h(\sigma)}{\alpha}g_{\delta,\sigma}(Y^h, Z^h) + \frac{Y^v(\alpha)}{\alpha}g_{\delta,\sigma}(X^h, Z^h)\\ &+ g_{\delta,\sigma}\big(g(X,Y)\bar{\nabla}\sigma, Z^h\big) - 2\frac{\sigma}{\alpha}g_{\delta,\sigma}\big((\nabla_X Y)^h, Z^h\big)\\ &+ \frac{\delta}{\alpha}g_{\delta,\sigma}\big((R(u,Y)X)^h, Z^h\big)\,,\end{aligned}$$

and so the horizontal part can be given by

$$\begin{aligned}\text{Horizontal}\,(\bar{\nabla}_{X^h}Y^v) = &-\frac{X^h(\sigma)}{2\alpha}Y^h + \frac{Y^v(\alpha)}{2\alpha}X^h\\ &+ \frac{1}{2}g(X,Y)h(\bar{\nabla}\sigma) - \frac{\sigma}{\alpha}(\nabla_X Y)^h + \frac{\delta}{2\alpha}\big(R(u,Y)X\big)^h\,.\end{aligned} \tag{25}$$

For the vertical part of $\bar{\nabla}_{X^h}Y^v$ we have

$$\begin{aligned}2g_{\delta,\sigma}(\bar{\nabla}_{X^h}Y^v, Z^v) = X^h g_{\delta,\sigma}(Y^v, Z^v) + Y^v g_{\delta,\sigma}(X^h, Z^v)\\ - Z^v g_{\delta,\sigma}(Y^v, X^h) + g_{\delta,\sigma}([X^h, Y^v], Z^v)\\ + g_{\delta,\sigma}([Z^v, X^h], Y^v) - g_{\delta,\sigma}([Y^v, Z^v], X^h)\,.\end{aligned}$$

Using the equations for Lie brackets and definition of $g_{\delta,\sigma}$ and (2), (3) we will get

$$\begin{aligned}2g_{\delta,\sigma}(\bar{\nabla}_{X^h}Y^v, Z^v) = X^h(\delta)g(Y, Z) + \delta X g(Y, Z) - Y^v(\sigma)g(X, Z)\\ Z^v(\sigma)g(X, Y) + g_{\delta,\sigma}\big((\nabla_X Y)^v, Z^v\big) - \delta g(\nabla_X Z, Y)\,.\end{aligned}$$

One can use $g_{\delta,\sigma}$ and the compatibility of g and ∇ and substituting the equation $Z^v(\sigma) = g_{\delta,\sigma}(Z^v, \bar{\nabla}\sigma)$ and ca get

$$2g_{\delta,\sigma}(\bar{\nabla}_{X^h}Y^v, Z^v) = g_{\delta,\sigma}\Big(\frac{X^h(\delta)}{\delta}Y^v + 2(\nabla_X Y)^v - \frac{Y^v(\sigma)}{\delta}X^v + g(X, Y)\bar{\nabla}\sigma, Z^v\Big)\,.$$

So, the vertical part is given by

$$\text{Vertical}(\bar{\nabla}_{X^h}Y^v) = \frac{X^h(\delta)}{2\delta}Y^v + (\nabla_X Y)^v - \frac{Y^v(\sigma)}{2\delta}X^v + \frac{1}{2}g(X, Y)v(\bar{\nabla}\sigma)\,. \tag{26}$$

Relations (25) and (26) complete the proof of second equation. The proof of third equation is a little different. We have

$$\bar{\nabla}_{X^v}Y^h = \bar{\nabla}_{Y^h}X^v + [X^v, Y^h]\,.$$

Using the equation (22) we get the result. The last one will be proved by the following process

$$\begin{aligned}2g_{\delta,\sigma}(\bar{\nabla}_{X^v}Y^v, Z^h) = X^v g_{\delta,\sigma}(Y^v, Z^h) + Y^v g_{\delta,\sigma}(X^v, Z^h) - Z^h g_{\delta,\sigma}(X^v, Y^v)\\ + g_{\delta,\sigma}([X^v, Y^v], Z^h) + g_{\delta,\sigma}([Z^h, X^v], Y^v) - g_{\delta,\sigma}([Y^v, Z^h], X^v)\,.\end{aligned}$$

From the definition of $g_{\delta,\sigma}$ and after substituting the equations $Zg(X, Y) = g(\nabla_Z X, Y) + g(\nabla_Z Y, X)$ and $Z^h(\delta) = g_{\delta,\sigma}(\bar{\nabla}\delta, Z^h)$ we get

$$2g_{\delta,\sigma}(\bar{\nabla}_{X^v}Y^v, Z^h) = g_{\delta,\sigma}\Big(\frac{-X^v(\sigma)}{\alpha}Y^h - \frac{Y^v(\sigma)}{\alpha}X^h - g(X, Y)\bar{\nabla}\delta, Z^h\Big)\,,$$

this shows that the horizontal part is

$$\text{Horizontal}(\bar{\nabla}_{X^v}Y^v) = \frac{-X^v(\sigma)}{2\alpha}Y^h - \frac{Y^v(\sigma)}{2\alpha}X^h - \frac{1}{2}g(X, Y)h(\bar{\nabla}\delta)\,.$$

The following step is the computation of the vertical part

$$\begin{aligned}2g_{\delta,\sigma}(\bar{\nabla}_{X^v}Y^v, Z^v) = X^v(\delta)g(Y, Z) + Y^v(\delta)g(X, Z) - Z^v(\delta)g(X, Y)\\ g_{\delta,\sigma}\Big(\frac{X^v(\delta)}{\delta}X^v + \frac{Y^v(\delta)}{\delta}X^v - g(X, Y)\bar{\nabla}\delta, Z^v\Big)\,.\end{aligned}$$

And so

$$\text{Vertical}(\bar{\nabla}_{X^v}Y^v) = \frac{X^v(\delta)}{2\delta}X^v + \frac{Y^v(\delta)}{2\delta}X^v - \frac{1}{2}g(X, Y)v(\bar{\nabla}\delta)\,.$$

□

Note that the Levi-Civita connection of $g_{\delta,0}$ can be easily computed from the above formulas by setting $\sigma = 0$. When we work with $g_{\delta,0}$, its connection will be denoted by $\bar{\nabla}$, too.

Now, we want to investigate the general characteristics of $J_{\delta,\sigma}$ in the sense of $g_{\delta,\sigma}$. It is easy to show that there are examples of almost complex structures on the tangent bundle $\mathrm{T}M$ like J such that the $(0,2)$-tensor $\Lambda(A,B) = \mathrm{d}\Theta(JA,B)$ is not a symmetric tensor. Let

$$\bar{J}(X^h) = \alpha X^v + J_1(X^h)\,, \qquad \bar{J}(X^v) = J_2(X^v) - \delta X^h\,, \tag{27}$$

be an almost complex structure on the tangent bundle of (M,g) where $\alpha, \delta \colon \mathrm{T}M \to \mathbb{R}^+$ are mappings, $J_1 \colon \mathcal{H}\mathrm{T}M \to \mathcal{H}\mathrm{T}M$ and $J_2 \colon \mathcal{V}\mathrm{T}M \to \mathcal{V}\mathrm{T}M$ are linear bundle maps. Suppose $G(A,B) = \mathrm{d}\Theta(\bar{J}A,B)$ be a Riemannian metric on $\mathrm{T}M$. Then one can state the following

Proposition 10. *Let (M,g) be a Riemannian manifold and $\bar{J} \colon \mathrm{TT}M \longrightarrow \mathrm{TT}M$ be an almost complex structure on $\mathrm{T}M$ given by (27). Suppose $G(A,B) = \mathrm{d}\Theta(\bar{J}A,B)$ be a Riemannian metric on $\mathrm{T}M$. Then $\pi_* J_1 X^h = -KJ_2X^v$ and $\alpha\delta - 1 \geq 0$ and J_1 is symmetric with respect to G, i.e., $G(J_1X^h, Y^h) = G(X^h, J_1Y^h)$, and has at most two eigen-values $-\sqrt{\alpha\delta - 1} = -\sigma$, $\sqrt{\alpha\delta - 1} = \sigma$.*

Proof. Since G is a Riemannian metric, using the symmetric property of G for given horizontal and vertical vectors X^h, Y^v, i.e., $G(X^h, Y^v) = G(Y^v, X^h)$, gives us $\pi_* J_1 X^h = -KJ_2X^v$. Moreover, from $G(\bar{J}X^h, \bar{J}Y^v) = G(\bar{J}Y^v, \bar{J}X^h)$ and $\pi_* J_1 X^h = -KJ_2X^v$ one can get that J_1 is a symmetric linear bundle map. Finally, $\bar{J}^2 = -\,\mathrm{id}$ gives us the equation

$$J_1^2 = (\alpha\delta - 1)\,\mathrm{id}\,,$$

and since J_1 is symmetric it has at most two real eigenvalues $\sqrt{\alpha\delta - 1}$ and $-\sqrt{\alpha\delta - 1}$. □

3. Harmonic unit vector fields

We denote harmonic vector fields by HVF and harmonic unit vector fields by HUVF. In this section after calculating the tension field of a map defined by a unit vector field $X \colon (M,g) \to (TM, g_{\delta,\sigma})$, we shall compute its tension field as a map from (M,g) to the $(S(M), i^* g_{\delta,0})$ and finally deduce some results on HUVFs.

3.1. **Tension field of unit vector fields.** In this sub-section, the formula of the tension field associated to a map between Riemannian manifolds is retrieved from [14]. So, one can refer to [14] for more details.

Suppose (M,g) and (M',g') are two Riemannian manifolds, with M compact. The Dirichlet energy associated to the Riemannian manifolds (M,g) and (M',g') is defined by

$$\begin{array}{rcl} E\colon C^\infty(M,M') & \longrightarrow & \mathbb{R}^+ \\ f & \longmapsto & \frac{1}{2}\int_M ||df||^2 d\,\mathrm{vol}(g)\,, \end{array}$$

where $\|df\|$ is the Hilbert-Schmidt norm of df, i.e., $\|df\|^2 = \mathrm{tr}_g(f^*g')$ and $d\,\mathrm{vol}(g)$ is the Riemannian volume form on M with respect to g.

The critical points of E are defined as harmonic maps. It is proved [14] that a map $f\colon (M,g) \longrightarrow (M',g')$ is a harmonic map if and only if the tension field associated to f vanishes identically. Therefore, one can investigate the harmonicity of a map defined by a vector field by calculating the tension field associated to this map.

Suppose (M,g) is a compact Riemannian manifold and $g_{\delta,\sigma}$ be a given metric on TM defined as before and $W \in \Gamma(TM)$. Let $\{V_1,\dots,V_n\}$ be a local orthonormal basis for the vector fields on M, defined in a neighborhood of $p \in M$ such that $\nabla V_i = 0$ at p. The Dirichlet energy of the map $W\colon (M,g) \longrightarrow (TM, g_{\delta,\sigma})$ defined by W can be calculated as following

$$E(W) = \frac{1}{2}\int_M (n\alpha - 2\sigma \operatorname{div}(W) + \delta\|\nabla W\|^2) d\operatorname{vol}(g)\,.$$

The tension field associated to the map $X\colon (M,g) \longrightarrow (TM, g_{\delta,\sigma})$ is locally defined by $\tau_q(X) = \sum_{i=1}^n \{\bar{\nabla}_{X_*(V_i)} X_*(V_i) - X_*(\nabla_{V_i}V_i)\}(X(q))$ for every q in the domain of V_i, $i=1,\dots,n$, which can be expressed [5] as following,

$$\begin{aligned}\tau_p(X) = \frac{1}{\alpha}\Big\{&\Big(1-\frac{n\alpha}{2}\Big)X_1 - \frac{\alpha}{2}\|\nabla X\|^2 Y_1 + \alpha\operatorname{div}(X)Z_1 + \operatorname{tr}_g(\nabla_{\boldsymbol{\cdot}})^v(\alpha)\boldsymbol{\cdot}\\ &- \sigma\operatorname{Ric}(X) - \nabla_{\alpha Z_1 - \sigma Z_2}X - \operatorname{tr}_g(\nabla_{\boldsymbol{\cdot}}X)^v(\sigma)\nabla_{\boldsymbol{\cdot}}X\\ &- \sigma\operatorname{tr}_g(\nabla_{\boldsymbol{\cdot}}\nabla_{\boldsymbol{\cdot}}X) + \delta\operatorname{tr}_g R(X,\nabla_{\boldsymbol{\cdot}}X)\boldsymbol{\cdot}\Big\}^h\big(X(p)\big)\\ &+\frac{1}{\delta}\Big\{-\frac{n\delta}{2}X_2 - \frac{\delta}{2}\|\nabla X\|^2 Y_2 + \delta\operatorname{div}(X)Z_2 - \alpha Z_1 + \sigma Z_2\\ &- \operatorname{tr}_g(\nabla_{\boldsymbol{\cdot}}X)^v(\sigma)\boldsymbol{\cdot} + \nabla_{\alpha Y_1 - \sigma Y_2}X + \operatorname{tr}_g(\nabla_{\boldsymbol{\cdot}}X)^v(\delta)\nabla_{\boldsymbol{\cdot}}X\\ &+\delta\Delta_g X\Big\}^v\big(X(p)\big)\,,\end{aligned}\tag{28}$$

where $X_1 = \pi_*\bar{\nabla}\alpha\circ X$, $X_2 = K\bar{\nabla}\alpha\circ X$, $Y_1 = \pi_*\bar{\nabla}\delta\circ X$, $Y_2 = K\bar{\nabla}\delta\circ X$, $Z_1 = \pi_*\bar{\nabla}\sigma\circ X$ and $Z_2 = K\bar{\nabla}\sigma\circ X$.

Now, one can make the harmonicity discussion on the unit tangent bundle.

Definition 11. The unit tangent bundle $S(M)$ on a Riemannian manifold (M,g) is a fiber bundle on (M,g) which its fibers at every point $p \in M$ is the set

$$S_p(M) = \{v \in T_pM \mid g(v,v) = 1\}\,.$$

The tangent space of $S(M)$ is $\mathcal{H}\oplus\bar{\mathcal{V}}$, where $\mathcal{H}$ is the horizontal sub-bundle of TTM with respect to the Levi-Civita connection ∇ of g and $\bar{\mathcal{V}}$ is a vector bundle on $S(M)$ such that at $(p,u)\in S(M)$ is defined by

$$\begin{aligned}\bar{\mathcal{V}}_{(p,u)} &= \{Y_p^v \in \mathcal{V}_u \mid g(Y_p,u) = 0\,,\ \forall Y_p \in T_pM\}\\ &= \{Y_p^v - g(Y_p,u)u^v \mid Y_p \in T_pM\}\,,\end{aligned}\tag{29}$$

where $Y_p^v \in \mathcal{V}_u$ is the vertical lift of Y_p to $\mathcal{V}_u$.

Let now $g_{\delta,\sigma}$ be the Riemannian metric on TM defined as before. Assume $N(p,u) = \sqrt{\alpha}(\frac{\sigma}{\alpha}u^h + u^v)$ is a vector field on TM. One can simply derive N is normal unit vector field to $TS(M)$ with respect to the metric $g_{\delta,\sigma}$.

We equip $S(M)$ with the induced metric $i^*g_{\delta,0}$ and represent its Levi-Civita connection by $\tilde{\nabla}$.

Lemma 12. *Let $\tau_1(X)$ be the tension field of the unit vector field $X\colon (M,g) \to (S(M), i^*g_{\delta,0})$ and $\tau(X)$ be its tension field considered as the map $X\colon (M,g) \to (TM, g_{\delta,0})$. Then*

$$\tau_1(X) = \tan\ \tau(X)\,,$$

with respect to $g_{\delta,0}$.

Proof. By using the Gauss formula for the Levi-Civita connections $\tilde{\nabla}$ of $i^*g_{\delta,0}$ and $\bar{\nabla}$ of $g_{\delta,0}$, one can get the result. □

The following proposition calculates the formula of $\tau_1(X)$.

Proposition 13. *The tension field $\tau_1(X)$ can be expressed as follows*

$$\begin{aligned}(\tau_1)_p(X) = {} & \frac{1}{\alpha}\Big\{\Big(1 - \frac{n\alpha}{2} + \frac{1}{2\alpha}\|\nabla X\|^2\Big)X_1 + \mathrm{tr}_g\left((\nabla_{\bullet} X)^v(\alpha)\bullet\right) \\ & + \frac{1}{\alpha}\,\mathrm{tr}_g\, R(X, \nabla_{\bullet} X)\bullet\Big\}^h\big(X(p)\big) \\ & + \alpha\Big\{\frac{-1}{\alpha}\nabla_{X_1}X - \frac{1}{\alpha^2}\,\mathrm{tr}_g\left((\nabla_{\bullet} X)^v(\alpha)\nabla_{\bullet} X\right) \\ & + \frac{1}{\alpha}\Delta_g X + \Big(\frac{1}{2\alpha^3}\|\nabla X\|^2 - \frac{n}{2\alpha}\Big)X_2 - \Big[\frac{1}{\alpha}g(\Delta_g X, X) \\ & + \Big(\frac{1}{2\alpha^3}\|\nabla X\|^2 - \frac{n}{2\alpha}\Big)g(X_2, X)\Big]X\Big\}^v\big(X(p)\big)\,. \end{aligned} \tag{30}$$

Proof. Substituting $\sigma = 0$, $\delta = \frac{1}{\alpha}$, $Y_1 = -\frac{1}{\alpha^2}X_1$, $Y_2 = -\frac{1}{\alpha^2}X_2$, $Z_1 = 0$ and $Z_2 = 0$ in (28) and using the last lemma we get the result. □

The condition $\tau_1(X) = 0$ for the special vector fields can be reduced to a simple equation. Specially, for a parallel unit vector field X, we have the following corollary.

Corollary 14. *Let $(S(M), i^*g_{\delta,0})$ be the unit tangent bundle equipped with the induced metric $i^*g_{\delta,0}$ for a compact Riemannian manifold (M,g). Then, a map $X : (M,g) \longrightarrow (S(M), i^*g_{\delta,0})$ defined by a parallel unit vector field X on M is a harmonic map if and only if*

$$(1 - \frac{n\alpha}{2})X_1 = 0\,, \quad \textit{and} \quad X_2 = \|X_2\|X\,.$$

3.2. **Variations through unit vector fields and HUVF.** Let (M,g) be a compact Riemannian manifold and let

$$\begin{aligned} E\colon C^\infty((M,g), (S(M), i^*g_{\delta,0}) & \longrightarrow \mathbb{R}^+ \\ f & \longmapsto \tfrac{1}{2}\int_M \|df\|^2 d\,\mathrm{vol}(g)\,, \end{aligned} \tag{31}$$

be the Dirichlet energy functional where $\|df\|^2 = \mathrm{tr}_g(f^*(i^*g_{\delta,0}))$. The first variation formula of E is

$$\frac{d}{dt}\Big|_{t=0}E(U_t) = -\int_M i^*g_{\delta,0}\big(\mathcal{V}, \tau_1(f)\big)d\,\mathrm{vol}(g)\,, \tag{32}$$

where U_t is a variation along f for $|t| < \varepsilon$ which $U_0(x) = f(x)$ and $U_t(x) \in S(M)$ for every $x \in M$ and $\mathcal{V}(x) = \frac{d}{dt}|_{t=0}\{t \longmapsto U_t(x)\}$ is the variation vector field. Following proposition is useful for proving the main theorems.

Proposition 15 ([9])**.** *Let X be a unit vector field on M and $\mathcal{U}\colon M \times (-\varepsilon, \varepsilon) \longrightarrow S(M)$ be a smooth 1-parameter variation of X through unit vector fields. Then the variation vector field $\mathcal{V}$ associated to this variation is of the form $\mathcal{V}(x) = V^v_{X(x)}$ for some perpendicular vector field V to X.*

Next proposition proves the inverse of the Proposition 15.

Proposition 16 ([9])**.** *Let X be a unit vector field on M and let*

$$\mathcal{S} = \{V \in \Gamma(TM) \mid g(V, X) = 0\}\,.$$

Then for every $V \in \mathcal{S}$ there exists a smooth variation along X through unit vector fields whose variation vector field is V^v. Indeed let V be an arbitrary element of $\mathcal{S}$ and let us set $W_t = X + tV$, $U_t = \|W_t\|^{-1}W_t$, $|t| < \epsilon$. It is not hard to check that U_t is a variation along X through unit vector fields with variation vector field V^v.

The following definition is analogous to the definition of harmonic unit vector fields with respect to the Sasaki metric and g-natural metrics.

Definition 17. Let (M, g) be a compact Riemannian manifold and $(S(M), i^*g_{\delta,0})$ be its unit tangent bundle equipped with the Riemannian metric $i^*g_{\delta,0}$. A unit vector field $X \in \Gamma(S(M))$ is called a HUVF if and only if the equation

$$\frac{d}{dt}\{E(U_t)\}\,|_{t=0} = -\int_M g_{\delta,0}\big(V^v, \tau_1(X)\big)\, d\,\mathrm{vol}(g)\,, \tag{33}$$

vanishes for all vector field $V \in \mathcal{S}$.

Now, the necessary and sufficient conditions for a unit vector field to be a HUVF can be resulted by the following theorems.

Theorem 18. *Let (M, g) be a compact Riemannian manifold and X be a unit vector field on M. Then, $X\colon (M, g) \longrightarrow (S(M), i^*g_{\delta,0})$ is a HUVF if and only if the vertical part of $\tau_1(X)$ is zero.*

Proof. $(\Longrightarrow)$ Let X be a HUVF. Suppose

$$(\tau_1)_p(X) = \zeta X^h_{X(p)} + \lambda X^v_{X(p)} + V^v_{X(p)} + W^h_{X(p)}\,, \qquad \forall\, p \in M\,, \tag{34}$$

for some vector fields V and W perpendicular to X and for some functions λ, ζ on $X(M) \subseteq S(M)$. We will show that $V = 0$ and $\lambda = 0$. From (34), we have

$$\|V^v_{X(p)}\|^2 = g_{\delta,0}\big((\tau_1)_p(X), V^v_{X(p)}\big)\,, \qquad \forall\, p \in M\,, \tag{35}$$

and then from the Proposition 16 and the Proposition 15 and the Definition 17 we have $\int_M \|V^v\|^2 d\,\mathrm{vol}(g) = \int_M g_{\delta,0}(\tau_1(X), V^v) d\,\mathrm{vol}(g) = 0$. This shows that $V = 0$, and the equation

$$(\tau_1)_p(X) = \tau_p(X) - \alpha\big(X(p)\big) g_{\delta,0}\big(\tau_p(X), X^v_{X(p)}\big) X^v_{X(p)}\,, \tag{36}$$

shows that $\tau_1(X)$ hasn't any component in direction of X^v, i.e., $\lambda = 0$. From (34) and $V = 0$ and $\lambda = 0$, one can get $K(\tau_1(X)) = 0$.

($\Longleftarrow$) Let $K(\tau_1(X)) = 0$, we will show that X is a HUVF. If $\mathcal{U}_t$ is an arbitrary variation along X through unit vector fields and $\mathcal{V} = V^v$ is its variation vector field then $K(\tau_1(X)) = 0$ shows that

$$\frac{d}{dt}\{E(\mathcal{U}_t)\} \mid_{t=0} = -\int_M g_{\delta,0}\big(V^v, \tau_1(X)\big)\, d\,\mathrm{vol}(g) = 0\,. \tag{37}$$

Note that the vertical and the horizontal sub-bundles are perpendicular to each other with respect to $g_{\delta,0}$. The equation (37) completes the proof. □

The next theorem gives an equation characterizing the HUVFs.

Theorem 19. *A vector field $X\colon (M,g) \longrightarrow (S(M), i^*g_{\delta,0})$ is a HUVF if and only if*

$$\begin{aligned}\Delta_g X = {} & \Big[\|\nabla X\|^2 + \Big(\frac{1}{2\alpha^2}\|\nabla X\|^2 - \frac{n}{2}\Big)g(X_2, X)\Big]X \\ & + \Big(\frac{n}{2} - \frac{1}{2\alpha^2}\|\nabla X\|^2\Big)X_2 + \nabla_{X_1}X \\ & + \frac{1}{\alpha}\,\mathrm{tr}_g\,\big((\nabla_{.}X)^v(\alpha)\nabla_{.}X\big)\end{aligned} \tag{38}$$

Proof. We know that X is HUVF if and only if $K(\tau_1(X)) = 0$. From (30) we have

$$\begin{aligned}K\big(\tau_1(X)\big) = \alpha\Big\{ & \frac{-1}{\alpha}\nabla_{X_1}X - \frac{1}{\alpha^2}\,\mathrm{tr}_g\,\big((\nabla_{\bullet}X)^v(\alpha)\nabla_{\bullet}X\big) \\ & + \frac{1}{\alpha}\Delta_g X + \Big(\frac{1}{2\alpha^3}\|\nabla X\|^2 - \frac{n}{2\alpha}\Big)X_2 - \Big[\frac{1}{\alpha}g(\Delta_g X, X) \\ & + \Big(\frac{1}{2\alpha^3}\|\nabla X\|^2 - \frac{n}{2\alpha}\Big)g(X_2, X)\Big]X\Big\}\,.\end{aligned} \tag{39}$$

Using $g(\Delta_g X, X) = \frac{1}{2}\Delta(\|\nabla X\|^2) + \|\nabla X\|^2 = \|\nabla X\|^2$ and $K(\tau_1(X)) = 0$, we get

$$\begin{aligned}\Delta_g X = {} & \Big[\|\nabla X\|^2 + \Big(\frac{1}{2\alpha^2}\|\nabla X\|^2 - \frac{n}{2}\Big)g(X_2, X)\Big]X \\ & + \Big(\frac{n}{2} - \frac{1}{2\alpha^2}\|\nabla X\|^2\Big)X_2 + \nabla_{X_1}X \\ & + \frac{1}{\alpha}\,\mathrm{tr}_g\,\big((\nabla_{\bullet}X)^v(\alpha)\nabla_{\bullet}X\big)\,.\end{aligned} \tag{40}$$

($\Longleftarrow$) Let (38) holds. Substituting (38) in (39) gives us, $K(\tau_1(X)) = 0$, i.e., the vertical part of $\tau_1(X)$ is zero which implies that X is a HUVF. □

Remark 20. Let g_s be the Sasaki metric. It is proved [9] that a unit vector field $X\colon (M,g) \longrightarrow (S(M), i^*g_s)$ is a harmonic unit vector field with respect to i^*g_s if and only if $\Delta_g X = \|\nabla X\|^2 X$.

The following corollary gives an analogous result to the Sasaki metric.

Corollary 21. *If we suppose that (M,g) is a Riemannian manifold of constant sectional curvature k and $g_{\delta,0}$ is defined by $\alpha = \delta^{-1} = \sqrt{2kE+b}$ and $\sigma = 0$ (mappings defined in the equation (9)) then $X\colon (M,g) \to (S(M), i^*g_{\delta,0})$ is a HUVF if and only if*

$$\Delta_g X = \|\nabla X\|^2 X\,. \tag{41}$$

Proof. Since, every space form is locally isomorphic to the one of $\mathbb{E}^n, \mathbb{S}^n$ or $\mathbb{H}^n$ then one can consider there exists a locally coordinate system $(x^1, \dots, x^n)$ on M such that the metric g is of the form $g = \lambda^2 \sum_{i=1}^n dx^i \otimes dx^i$ where λ is the conformal factor. Let $X_i = \frac{1}{\lambda} \frac{\partial}{\partial x^i}$ for $i = 1, \dots, n$, be the locally orthonormal vector fields on M. Furthermore, suppose

$$\theta^i = \lambda dx^i\,, \quad i = 1, \dots, n\,,$$

be thier dual 1-forms. Suppose $(x^1, \dots, x^n, y^1, \dots, y^n)$ is the associated locally coordinate system on TM and $\xi^i = \lambda dy^i$, $i = 1, \dots, n$. From [4], we know that

$$dE(A_v) = \sum_{i=1}^n \theta^i(v)\xi^i(A_v)\,, \quad A_v \in T_vTM$$

and so for the given α, one can deduce that $d\alpha = -\frac{\lambda k}{\alpha} \sum_{i=1}^n \theta^i dy^i$. This implies that the gradient vector field of α with respect to the given metric $g_{\delta,0}$ is given by

$$(\bar{\nabla}\alpha)V = -\frac{k}{\lambda(\pi o V)} V^v_V\,, \tag{42}$$

for all vectors $V \in TM$. The equation (42) shows that the vector fields X_1, X_2 stated in the theorem 19 are $X_1 = 0$ and $X_2(p) = -\frac{k}{\lambda(p)} X(p)$ for all $p \in M$. Since X is a unit vector field and $(\bar{\nabla}\alpha)X(p) = -\frac{k}{\lambda(p)} X^v_{X(p)}$ then

$$(\nabla_Y X)^v(\alpha) = 0\,,$$

for all $Y \in TM$. Now, using the stated properties the equation (38) can be easily reduced to this equation:

$$\Delta_g X = \|\nabla X\|^2 X\,.$$

$\square$

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