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GLOBAL SOLVABILITY CRITERIA FOR QUATERNIONIC RICCATI EQUATIONS

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Abstract. Some global existence criteria for quaternionic Riccati equations are established. Two of them are used to prove a completely non conjugation theorem for solutions of linear systems of ordinary differential equations.

1. Introduction

Let $a(t)$, $b(t)$, $c(t)$ and $d(t)$ be continuous quaternionic valued functions on $[t_0;+\infty)$, i.e.: $a(t) \equiv a_0(t) + ia_1(t) + ja_2(t) + ka_3(t)$, $b(t) \equiv b_0(t) + ib_1(t) + jb_2(t) + kb_3(t)$, $c(t) \equiv c_0(t) + ic_1(t) + jc_2(t) + kc_3(t)$, $d(t) \equiv d_0(t) + id_1(t) + jd_2(t) + kd_3(t)$, where $a_n(t)$, $b_n(t)$, $c_n(t)$, $d_n(t)$ $(n = \overline{0,3})$ are real valued continuous functions on $[t_0;+\infty)$, i, j, k are the imaginary unities satisfying the conditions

$$i^2 = j^2 = k^2 = ijk = -1\,, \quad ij = -ji = k\,. \tag{1.1}$$

Consider the quaternionic Riccati equation

$$q' + qa(t)q + b(t)q + qc(t) + d(t) = 0\,, \quad t \geq t_0\,. \tag{1.2}$$

Here $q = q(t)$ is the unknown continuously differentiable quaternionic valued function. Currently, there is a growing interest in quaternionic differential equations, in particular, in Eq. (1.2) in connection with their various applications (see e.g., [3]–[9]). Criteria for the existence of periodic (and, therefore, global) solutions of Eq. (1.2) with periodic coefficients were obtained in [1, 10]. Explicit global existence criteria for complex solutions of Eq. (1.2) in the case of its complex coefficients were obtained in [7].

In this paper some global existence criteria for scalar quaternionic Riccati equations are obtained. Two of them are used to prove a completely non conjugation theorem for solutions of linear systems of ordinary differential equations.

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2. Auxiliary propositions

Substituting $q = q_0 - iq_1 - jq_2 - kq_3$ in (1.2), where q_0 is the real and $-q_n(\overline{1,3})$ are the imaginary parts of q, and separating the real and imaginary parts we come to the following nonlinear system

$$
\left\{
\begin{aligned}
&q_0' + a_0(t)q_0^2 + \{b_0(t) + c_0(t) + 2[a_1(t)q_1 + a_2(t)q_2 + a_3(t)q_3]\}q_0 \\
&\qquad\qquad\qquad\qquad - P(t, q_1, q_2, q_3) = 0\,; \\
&q_1' + a_1(t)q_1^2 + \{b_0(t) + c_0(t) + 2[a_0(t)q_0 + a_2(t)q_2 + a_3(t)q_3]\}q_1 \\
&\qquad\qquad\qquad\qquad - Q(t, q_0, q_2, q_3) = 0\,; \\
&q_2' + a_2(t)q_2^2 + \{b_0(t) + c_0(t) + 2[a_0(t)q_0 + a_1(t)q_1 + a_3(t)q_3]\}q_2 \\
&\qquad\qquad\qquad\qquad - R(t, q_0, q_1, q_3) = 0\,; \\
&q_3' + a_3(t)q_3^2 + \{b_0(t) + c_0(t) + 2[a_0(t)q_0 + a_1(t)q_1 + a_2(t)q_2]\}q_3 \\
&\qquad\qquad\qquad\qquad - S(t, q_0, q_1, q_2) = 0\,;
\end{aligned}
\right. \tag{2.1}
$$

where

$$
\begin{aligned}
P(t, q_1, q_2, q_3) &\equiv a_0(t)[q_1^2 + q_2^2 + q_3^2] - (b_1(t) + c_1(t))q_1 - (b_2(t) + c_2(t))q_2 - (b_3(t) \\
&\quad + c_3(t))q_3 - d_0(t)\,; \\
Q(t, q_0, q_2, q_3) &\equiv a_1(t)[q_0^2 + q_2^2 + q_3^2] + (b_1(t) + c_1(t))q_0 + (b_3(t) - c_3(t))q_2 - (b_2(t) \\
&\quad - c_2(t))q_3 + d_1(t)\,; \\
R(t, q_0, q_1, q_3) &\equiv a_2(t)[q_0^2 + q_1^2 + q_3^2] + (b_2(t) + c_2(t))q_0 - (b_3(t) - c_3(t))q_1 + (b_1(t) \\
&\quad - c_1(t))q_3 + d_2(t)\,; \\
S(t, q_0, q_1, q_2) &\equiv a_3(t)[q_0^2 + q_1^2 + q_2^2] + (b_3(t) + c_3(t))q_0 + (b_2(t) - c_2(t))q_1 - (b_1(t) \\
&\quad - c_1(t))q_2 + d_3(t)\,;
\end{aligned}
$$

$t \geq t_0$. Consider the square matrices

$$
E \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad
I \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},
$$

$$
J \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad
K \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
$$

It is not difficult to check that $I^2 = J^2 = K^2 = IJK = -E$, $IJ = -JI = K$. Then by (1.1) there is an one to one correspondence between the quaternions $m \equiv m_0 + im_1 + jm_2 + km_3$ and the matrices of the form $M \equiv m_0E + m_1I +$

$m_2J + m_3K$:

$$m \equiv m_0 + im_1 + jm_2 + km_3 \leftrightarrow M \equiv \begin{pmatrix} m_0 & m_1 & m_2 & -m_3 \\ -m_1 & m_0 & -m_3 & -m_2 \\ -m_2 & m_3 & m_0 & m_1 \\ m_3 & m_2 & -m_1 & m_0 \end{pmatrix} \tag{2.2}$$

The matrix M corresponding to the quaternion m by the rule (2.2) we will call the symbol of the quaternion m and will denote by $\widehat{m}$.

Let $A(t)$, $B(t)$, $C(t)$ and $D(t)$ be the symbols of $a(t)$, $b(t)$, $c(t)$ and $d(t)$ respectively. Consider the matrix Riccati equation

$$Y' + YA(t)Y + B(t)Y + YC(t) + D(t) = 0\,, \quad t \geq t_0\,. \tag{2.3}$$

By (2.2) the solutions $q(t)$ of Eq. (1.2), existing on some interval $[t_1; t_2)$ $(t_0 \leq t_1 < t_2 \leq +\infty)$, are connected wit solutions $Y(t)$ of Eq. (2.3) by equalities

$$\widehat{q(t)} = Y(t)\,, \quad t \in [t_1; t_2),\ \widehat{q(t_1)} = Y(t_1)\,. \tag{2.4}$$

Along with Eq. (2.3) consider the system of matrix equations

$$\begin{cases} \Phi' = C(t)\Phi + A(t)\Psi\,; \\ \Psi' = -D(t)\Phi - B(t)\Psi\,, \quad t \geq t_0\,. \end{cases} \tag{2.5}$$

Here $\Phi \equiv \Phi(t)$, $\Psi \equiv \Psi(t)$ are the unknown continuously differentiable matrix functions of dimension 4×4 on $[t_0; +\infty)$. Let $Y_0(t)$ be a solution of Eq. (2.3) on $[t_1; t_2)$. The substitution

$$\Psi = Y_0(t)\Phi\,, \quad t \in [t_1; t_2)\,, \tag{2.6}$$

in (2.5) leads to the system

$$\begin{cases} \Phi' = [A(t)Y_0(t) + C(t)]\Phi\,; \\ [Y_0'(t) + Y_0(t)A(t)Y_0(t) + B(t)Y_0(t) + Y_0(t)C(t) + D(t)]\Phi = 0 \quad t \in [t_1; t_2). \end{cases}$$

Therefore $(\Phi_0(t), Y_0(t)\Phi_0(t))$ is a solution of the system (2.5) on $[t_1; t_2)$, where $\Phi_0(t)$ is a solution to the following matrix equation

$$\Phi' = [A(t)Y_0(t) + C(t)]\Phi\,, \quad t \in [t_1; t_2)\,. \tag{2.7}$$

Let $Y(t)$ $(q(t))$ be a solution to Eq. (2.3) (to Eq. (1.2)) on $[t_1; t_2)$.

Definition 2.1. The set $[t_1; t_2)$ is called the maximum existence interval for the solution $Y(t)$ of Eq. (2.3) (for the solution $q(t)$ of Eq. (1.2)), if $Y(t)$ $(q(t))$ cannot be continued to the right from t_2.

Lemma 2.1. *Let $Y(t)$ be a solution of Eq.* (2.3) *on* $[t_1; t_2)$ $(t_0 \leq t_1 < t_2 < +\infty)$. *Then $[t_1; t_2)$ is not the maximum existence interval for $Y(t)$ provided the function*

$$f_0(t) \equiv \int\limits_{t_0}^{t} \mathrm{tr}\,[A(\tau)Y(\tau)]d\tau\,, \quad t \in [t_1; t_2)\,,$$

is bounded from below on $[t_1; t_2)$.

Proof. Let $\Phi(t)$ be a solution to the matrix equation

$$\Phi' = [A(t)Y(t) + C(t)]\Phi\,, \quad t \in [t_1; t_2)\,, \quad \text{with}$$

$$\det \Phi(t_1) \neq 0\,. \tag{2.8}$$

By (2.6) and (2.7), $(\Phi(t), Y(t)\Phi(t))$ is a solution to the system (2.5) on $[t_1; t_2)$ which can be continued on $[t_0; +\infty)$ as a solution $(\Phi(t), \Psi(t))$ of the system (2.5). According to the Liouville's formula (see [8, p. 46, Theorem 1.2]) we have:

$$\det \Phi(t) = \det \Phi(t_1) \exp\Big\{ \int\limits_{t_0}^{t} \mathrm{tr}\,[A(\tau)Y(\tau) + C(\tau)]d\tau \Big\}, \quad t \in [t_1; t_2)\,.$$

From here from the conditions of lemma and from (2.8) it follows that $\det \Phi(t) \neq 0$, $t \in [t_1; t_3)$, for some $t_3 > t_2$. Then by (2.6) and (2.7) the matrix function $\widetilde{Y}(t) \equiv \Psi(t)\Phi^{-1}(t)$, $t \in [t_1; t_3)$, is a solution to Eq. (2.3) on $[t_1; t_3)$. Obviously $\widetilde{Y}(t)$ coincides with $Y(t)$ on $[t_1; t_2)$. Therefore $[t_1; t_2)$ is not the maximum existence interval for $Y(t)$.
The lemma is proved. □

Let $f(t)$, $g(t)$, $h(t)$, $f_1(t)$, $g_1(t)$, $h_1(t)$ be real valued continuous functions on $[t_0; +\infty)$. Consider the Riccati equations

$$y' + f(t)y^2 + g(t)y + h(t) = 0\,, \quad t \geq t_0\,; \tag{2.9}$$

$$y' + f_1(t)y^2 + g_1(t)y + h_1(t) = 0\,, \quad t \geq t_0\,. \tag{2.10}$$

and the differential inequalities

$$y' + f(t)y^2 + g(t)y + h(t) \geq 0\,, \quad t \geq t_0\,; \tag{2.11}$$

$$y' + f_1(t)y^2 + g_1(t)y + h_1(t) \geq 0\,, \quad t \geq t_0\,. \tag{2.12}$$

Remark 2.1. For $f(t) \geq 0$, $t \geq t_0$, every solution of the linear equation $y' + g(t)y + h(t) = 0$ on $[t_0; \tau_0)$ $(t_0 < \tau_0 \leq +\infty)$ is a solution of the inequality (2.11) on $[t_0; \tau_0)$.

Remark 2.2. Every solution of Eq. (2.10) on $[t_0; \tau_0)$ $(t_0 < \tau_0 \leq +\infty)$ is also a solution of the inequality (2.12) on $[t_0; \tau_0)$.

Theorem 2.1. *Let Eq.* (2.10) *has a real solution* $y_1(t)$ *on* $[t_0; \tau_0)$ $(\tau_0 \leq +\infty)$*, and let the following conditions be satisfied:* $f(t) \geq 0$,

$$\int\limits_{t_0}^{t} \exp\Big\{ \int\limits_{t_0}^{\tau} \big[f(s)(\eta_0(s) + \eta_1(s)) + g(s)\big]ds \Big\}$$
$$\times \big[(f_1(\tau) - f(\tau))y_1^2(\tau) + (g_1(\tau) - g(\tau))y_1(\tau) + h_1(\tau) - h(\tau)\big]d\tau \geq 0\,,$$
$$t \in [t_0; \tau_0)\,.$$

where $\eta_0(t)$ *and* $\eta_1(t)$ *are solutions of the inequalities* (2.11) *and* (2.12) *on* $[t_0; \tau_0)$ *such that* $\eta_j(t_0) \geq y_1(t_0)$*,* $j = 0, 1$*. Then for every* $\gamma_0 \geq y_1(t_0)$ *Eq.* (2.9) *has a* *solution $y_0(t)$ on $[t_0;\tau_0)$, satisfying the initial conditions $y_0(t_0)=\gamma_0$, and $y_0(t)\ge y_1(t)$, $t\in[t_0;\tau_0)$.*

This theorem is proved in [4] (see [4, Theorem 3.1]).

Let $t_0<t_1<\cdots$ be a finite or infinite sequence such that $t_m\in[t_0;\tau_0]$ $(t_0<\tau_0\le+\infty)$. We assume that if $\{t_m\}$ is finite then $\max\{t_m\}=\tau_0$ otherwise $\lim\limits_{m\to+\infty}t_m=\tau_0$. Denote:

$I_{g,h}(\xi,t)\equiv\int\limits_{\xi}^{t}\exp\big\{-\int\limits_{\tau}^{t}g(s)ds\big\}h(\tau)d\tau,\ t\ge\xi\ge t_0.$

Theorem 2.2. *Let $f(t)\ge 0$, $t\in[t_0;\tau_0)$, and*

$$\int\limits_{t_k}^{t}\exp\Big\{\int\limits_{t_k}^{\tau}\big[g(s)-f(s)I_{g,h}(t_k,s)\big]ds\Big\}h(\tau)d\tau\le 0\,,\quad t\in[t_k;t_{k+1}),k=1,2,\ldots.$$

Then for every $\gamma_0\ge 0$ Eq. (2.9) has a solution $y_0(t)$ on $[t_0;\tau_0)$ satisfying the initial condition $y_0(t_0)=\gamma_0$ and $y_0(t)\ge 0$, $t\in[t_0;\tau_0)$.

This theorem is proved in [5] (see [5, Theorem 4.1]).

Theorem 2.3. *Let $\alpha(t)$ and $\beta(t)$ be continuously differentiable on $[t_0;\tau_0)$ functions and $\alpha(t)>0$, $\beta(t)>0$, $t\in[t_0;\tau_0)$;*
A) *$0\le f(t)\le\alpha(t)$, $h(t)\le\beta(t)$, $t\in[t_0;\tau_0)$;*

B) *$g(t)\ge\frac{1}{2}\big[\frac{\alpha'(t)}{\alpha(t)}-\frac{\beta'(t)}{\beta(t)}\big]+2\sqrt{\alpha(t)\beta(t)}$, $t\in[t_0;\tau_0)$.*

Then for every $\gamma_0\ge-\sqrt{\frac{\beta(t_0)}{\alpha(t_0)}}$ Eq. (2.9) has a solution $y_0(t)$ on $[t_0;\tau_0)$ with $y_0(t_0)=\gamma_0$ and

$$y_0(t)\ge-\sqrt{\frac{\beta(t)}{\alpha(t)}}\,,\quad t\in[t_0;\tau_0)\,.$$

This theorem is proved in [6] (see [6, Theorem 8]).

Theorem 2.4. *Let $\alpha(t)$ and $\beta(t)$ be the same as in Theorem 2.3. If assumption* A *of Theorem 2.3 and the inequality*

D) *$g(t)\le\frac{1}{2}\big[\frac{\alpha'(t)}{\alpha(t)}-\frac{\beta'(t)}{\beta(t)}\big]-2\sqrt{\alpha(t)\beta(t)}$, $t\in[t_0;\tau_0)$,*

are valid, then for every $\gamma_0\ge\sqrt{\frac{\beta(t_0)}{\alpha(t_0)}}$ Eq. (2.9) has a solution $y_0(t)$ on $[t_0;\tau_0)$ with $y_0(t_0)=\gamma_0$ and

$$y_0(t)\ge\sqrt{\frac{\beta(t)}{\alpha(t)}}\,,\quad t\in[t_0;\tau_0)\,.$$

This theorem is proved in [6] (see [6, Theorem 7]).

Theorem 2.5. *Let $\alpha_m(t)$ and $\beta_m(t)$, $m=1,2$, be continuously differentiable functions on $[t_0;\tau_0)$, and let $(-1)^m\alpha_m(t)>0$, $(-1)^m\beta_m(t)>0$, $t\in[t_0;\tau_0)$, $m=1,2$. If:*

E) $\alpha_1(t) \le f(t) \le \alpha_2(t)$, $\beta_1(t) \le h(t) \le \beta_2(t)$, $t \in [t_0; \tau_0)$;

F) $g(t) \ge \frac{1}{2}\big(\frac{\alpha'_m(t)}{\alpha_m(t)} - \frac{\beta'_m(t)}{\beta_m(t)}\big) + 2(-1)^m\sqrt{\alpha_m(t)\beta_m(t)}$, $t \in [t_0; \tau_0)$, $m = 1, 2$,

then for any $y_{(0)} \in \Big[-\sqrt{\frac{\beta_2(t_0)}{\alpha_2(t_0)}}; \sqrt{\frac{\beta_1(t_0)}{\alpha_1(t_0)}}\Big]$ *Eq.* (2.9) *has a solution* $y_0(t)$ *on* $[t_0; \tau_0)$ *satisfying the initial condition* $y_0(t_0) = y_{(0)}$, *and*

$$-\sqrt{\frac{\beta_2(t)}{\alpha_2(t)}} \le y_0(t) \le \sqrt{\frac{\beta_1(t)}{\alpha_1(t)}}, \quad t \in [t_0; \tau_0)\,.$$

This theorem is proved in [5] (see [5, Theorem 4.2]).

Let p, q, r, s, l be real numbers and let $\varepsilon > 0$.

Definition 2.2. The ordered fiver (p, q, r, s, l) is called ε-semi definite positive if:

1) $p > 0$, $l > 0$;
2) $\max\{q, r, s\} \ge \sqrt{l + \varepsilon}$ or
$0 \le \min\{q, r, s\} \le \max\{q, r, s\} \le \sqrt{l+\varepsilon}$ and
$q^2 + r^2 + s^2 \ge l + \varepsilon$.

Remark 2.3. From the geometrical point of view the relations 1) and 2) mean that the ball of radius $\sqrt{l+\varepsilon}$ with its center in the point (q, r, s) may be located in any such position in the space of coordinates x, y, z, that its intersection with the octant $x > 0$, $y > 0$, $z > 0$ is empty.

Consider the quadratic form

$$W(x, y, z) \equiv p\Big[\Big(x + \frac{q}{2p}\Big)^2 + \Big(y + \frac{r}{2p}\Big)^2 + \Big(z + \frac{s}{2p}\Big)^2\Big] - \frac{l}{4p}, \quad x, y, z \in (-\infty; +\infty)\,.$$

Lemma 2.2. *If for some* $\varepsilon > 0$ *the ordered fiver* (p, q, r, s, l) *is* ε*-semi definite positive then for every* $x \ge 0$, $y \ge 0$, $z \ge 0$ *the inequality*

$$W(x, y, z) \ge \varepsilon/4p$$

is satisfied.

Proof. For every $x \ge 0$, $y \ge 0$, $z \ge 0$ we have: if $\max\{q, r, s\} \ge \sqrt{l+\varepsilon}$, then $W(x, y, z) \ge p\frac{l+\varepsilon}{4p^2} - \frac{l}{4p} = \frac{\varepsilon}{4p}$, and if $0 \le \min\{q, r, s\} \le \max\{q, r, s\} \le \sqrt{l+\varepsilon}$, then since $q \ge 0$, $r \ge 0$, $s \ge 0$, we will get: $W(x, y, z) \ge p\big(\frac{q^2}{4p^2} + \frac{r^2}{4p^2} + \frac{s^2}{4p^2}\big) - \frac{l}{4p} \ge \frac{l+\varepsilon}{4p} - \frac{l}{4p} = \frac{\varepsilon}{4p}$.
The lemma is proved. □

3. Global solvability criteria

In this section we study the global solvability conditions of Eq. (1.2) in the case when $a_n(t) \ge 0$, $t \ge t_0$, $n = \overline{0,3}$. The cases when $(-1)^{m_n} a_n(t) \ge 0$, $t \ge t_0$, $m_n = 0, 1$, $n = \overline{0,3}$, $m_0 + m_1 + m_2 + m_3 > 0$ are reducible to the studying one by the simple transformations $q \to -q$, $q \to \overline{q}$, $q \to iq$, $q \to jq$, $q \to kq$ and their combinations in (1.2). Denote:

$p_{0,m}(t) \equiv b_m(t) + c_m(t)$, $m = \overline{1,3}$, $p_{1,1}(t) \equiv b_1(t) + c_1(t)$, $p_{1,2}(t) \equiv b_2(t) - c_2(t)$, $p_{1,3}(t) \equiv b_3(t) - c_3(t)$, $p_{2,1}(t) \equiv b_1(t) - c_1(t)$, $p_{2,2}(t) \equiv b_2(t) + c_2(t)$, $p_{2,3}(t) \equiv b_3(t) - c_3(t)$, $p_{3,m}(t) \equiv b_m(t) - c_m(t)$, $m = \overline{1,3}$, $t \geq t_0$.

$$D_0(t) \equiv \begin{cases} \sum\limits_{m=1}^{3} p_{0,m}^2(t) + 4a_0(t)d_0(t), & \text{if} \quad a_0(t) \neq 0\,; \\ 4d_0(t) & \text{if} \quad a_0(t) = 0\,, \end{cases}$$

$$D_n(t) \equiv \begin{cases} \sum\limits_{m=1}^{3} p_{n,m}^2(t) - 4a_n(t)d_n(t)\,, & \text{if} \quad a_n(t) \neq 0\,; \\ -4d_n(t) & \text{if} \quad a_n(t) = 0\,, \end{cases} \qquad n = \overline{1,3},\ t \geq t_0\,.$$

Let $\mathfrak{S}$ be a nonempty subset of the set $\{0,1,2,3\}$ and let $\mathfrak{O}$ be its complement, i.e., $\mathfrak{O} = \{0,1,2,3\} \backslash \mathfrak{S}$.

Theorem 3.1. *Assume $a_n(t) \geq 0$, $n \in \mathfrak{S}$ and if $a_n(t) = 0$ then $p_{n,m}(t) = 0$, $m = \overline{1,3}$, $n \in \mathfrak{S}$; $a_n(t) \equiv 0$, $n \in \mathfrak{O}$, $D_n(t) \leq 0$, $n \in \mathfrak{S}$, $t \geq t_0$.*

Then for every $\gamma_n \geq 0$, $n \in \mathfrak{S}$, $\gamma_n \in (-\infty;+\infty)G$, $n \in \mathfrak{O}$, Eq. (1.2) *has a solution $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ on $[t_0;+\infty)$ with $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$ and*

$$q_n(t) \geq 0\,, \qquad n \in \mathfrak{S}\,, \qquad t \geq t_0\,. \tag{3.1}$$

Moreover if for some $n \in \mathfrak{S}$, $\gamma_n > 0$, then also $q_n(t) > 0$.

Proof. Let $[t_0;T)$ be the maximum existence interval for the solution $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ of Eq. (1.2) satisfying the initial conditions $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$ (existence of $[t_0;T)$ follows from the theory of normal systems of ordinary differential equations and from (2.1)). Show that

$$q_n(t) \geq 0\,, \qquad t \in [t_0;T)\,,\ n \in \mathfrak{S}\,. \tag{3.2}$$

Let us prove the theorem in the case when $0 \in \mathfrak{S}$. The proof of the theorem for other nonempty $\mathfrak{S}$ can be proved by analogy. Consider the Riccati equations

$$\begin{aligned} &x' + a_0(t)x^2 + \big\{b_0(t) + c_0(t) + 2\big[a_1(t)q_1(t) + a_2(t)q_2(t) + a_3(t)q_3(t)\big]\big\}x \\ &\qquad - P(t, q_1(t), q_2(t), q_3(t)) = 0\,, \qquad t \in [t_0;T)\,, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &x' + a_0(t)x^2 + \big\{b_0(t) + c_0(t) + 2\big[a_1(t)q_1(t) + a_2(t)q_2(t) + a_3(t)q_3(t)\big]\big\}x \\ &\qquad = 0\,, \qquad\qquad\qquad\qquad t \in [t_0;T)\,. \end{aligned} \tag{3.4}$$

From the conditions of the theorem it follows that $P(t, q_1(t), q_2(t), q_3(t)) \geq 0$, $t \in [t_0;T)$. Then using Theorem 2.1 to the equations (3.3) and (3.4) we conclude that the solution $x(t)$ of Eq. (3.3) with $x(t_0) = \gamma_0 \geq 0$ exists on $[t_0;T)$ and is non negative (since $x_1(t) \equiv 0$ is a solution to Eq. (3.4) on $[t_0;T)$). Obviously $q_0(t)$ is a solution of Eq. (3.3). Hence $q_0(t) = x(t) \geq 0$, $t \in [t_0;T)$. By analogy can be proved the remaining inequalities (3.2). By (2.4) $Y(t) \equiv \widehat{q(t)}$, $t \in [t_0;T)$, is a solution of Eq. (2.3) on $[t_0;T)$. Then it is not difficult to verify that $\mathrm{tr}[A(t)Y(t)] = \sum\limits_{n=0}^{3} a_n(t)q_n(t) = \sum\limits_{n\in\mathfrak{S}} a_n(t)q_n(t)$, $t\in[t_0;T)$. From here and from (3.2) we have:

$$\mathrm{tr}\,[A(t)Y(t)] \ge 0\,, \qquad t\in[t_0;T)\,. \tag{3.5}$$

Show that

$$T = +\infty\,. \tag{3.6}$$

Suppose $T < +\infty$. Then by virtue of Lemma 2.1 from (3.5) it follows that $[t_0;T)$ is not the maximum existence interval for $Y(t)$. Therefore $[t_0;T)$ is not the maximum existence interval for $q(t)$. The obtained contradiction proves (3.6). From (3.6) and (3.2) it follows (3.1). Assume $\gamma_0 > 0$. By already proven the solution $\widetilde{x}(t)$ of Eq. (3.3) with $\widetilde{x}(t_0) = 0$ exists on $[t_0;+\infty)$ and is nonnegative. Then by virtue of Theorem 2.1 the solution $x(t)$ of Eq. (3.3) with $x(t_0) = \gamma_0 > 0$ exists on $[t_0;+\infty)$ and $x(t) \ne \widetilde{x}(t),\ t\ge t_0$. Therefore $x(t) > 0,\ t \ge t_0$. Obviously $x(t)\equiv q_0(t),\ t\ge t_0$. Hence $q_0(t) > 0,\ t\ge t_0$. By analogy it can be shown that if $\gamma_n > 0$ for some other $n\in\mathfrak{S}$, then also $q_n(t) > 0$, $t \ge t_0$.
The theorem is proved. $\square$

Remark 3.1. Theorem 3.1 is a generalization of Theorem 3.1 of work [7].

Set:
$$\begin{aligned}
\mathcal{L}_0(t) &\equiv \big(a_0(t), -b_1(t)-c_1(t), -b_2(t)-c_2(t), -b_3(t)-c_3(t),\ D_0(t)\big)\,;\\
\mathcal{L}_1(t) &\equiv \big(a_1(t),\ b_1(t)+c_1(t), -b_2(t)+c_2(t),\ b_3(t)-c_3(t),\ D_1(t)\big)\,;\\
\mathcal{L}_2(t) &\equiv \big(a_2(t),\ b_1(t)-c_1(t),\ b_2(t)+c_2(t),\ b_3(t)-c_3(t),\ D_2(t)\big)\,;\\
\mathcal{L}_3(t) &\equiv \big(a_3(t), -b_1(t)+c_1(t),\ b_2(t)-c_2(t),\ b_3(t)+c_3(t),\ D_3(t)\big)\,.
\end{aligned}$$

Theorem 3.2. *Let for some $\varepsilon > 0$ and for every $t \ge t_0$ the ordered fivers $\mathcal{L}_n(t)$, $n = \overline{0,3}$ be ε-semi definite positive. Then for every $\gamma_n > 0$, $n = \overline{0,3}$, Eq. (1.2) has a solution $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ on $[t_0;+\infty)$ with $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$, and*

$$q_n(t) > 0, \quad t\ge t_0\,, \qquad n=\overline{0,3}\,. \tag{3.7}$$

Proof. Let $[t_0;T)$ be the maximum existence interval for the solution $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ of Eq. (1.2) satisfying the initial conditions $q_n(t_0) = \gamma_n$ $n = \overline{0,3}$. Show that

$$q_n(t) \ge 0\,, \quad t\in[t_0;T) \;\; n=\overline{0,3}\,. \tag{3.8}$$

Set: $T_1 \equiv \sup\{t\in[t_0;T) : q_n(t)\ge 0,\ \ t\in[t_0;T)\ n=\overline{0,3}\}$. Suppose (3.8) is not true. Then (obviously $T_1 > t_0$)

$$T_1 < T\,. \tag{3.9}$$

On the other hand from the conditions of the theorem it follows that

$$P\big(t, q_1(t),\ q_2(t), q_3(t)\big) \ge \frac{\varepsilon}{4a_0(t)}\,, \quad Q\big(t, q_0(t), q_2(t), q_3(t)\big) \ge \frac{\varepsilon}{4a_1(t)}\,,$$

$$R\big(t, q_0(t), q_1(t), q_3(t)\big) \ge \frac{\varepsilon}{4a_2(t)}\,, \quad S\big(t, q_0(t), q_1(t), q_2(t)\big) \ge \frac{\varepsilon}{4a_3(t)},\ t\in[t_0;T_1)\,.$$

By the continuity property of the functions P, Q, R, S, q_0, q_1, q_2 and q_3 it follows that for some $T_2 > T_1$ ($T_2 < T$) the following inequalities are fulfilled:

$$\begin{cases} P\big(t, q_1(t), q_2(t), q_3(t)\big) \geq 0\,; \qquad Q\big(t, q_0(t), q_2(t), q_3(t)\big) \geq 0\,, \\ R\big(t, q_0(t), q_1(t), q_3(t)\big) \geq 0\,; \qquad S\big(t, q_0(t), q_1(t), q_2(t)\big) \geq 0\,, \end{cases} \tag{3.10}$$

for all $t \in [t_0; T_2)$. Consider on $[t_0; T_2)$ the Riccati equations

$$\begin{aligned} x' + a_0(t)x^2 + \{b_0(t) + c_0(t) + 2[a_1(t)q_1(t) + a_2(t)q_2(t) + a_3(t)q_3(t)]\}x \\ - P\big(t, q_1(t), q_2(t), q_3(t)\big) = 0\,; \end{aligned} \tag{3.11}$$

$$\begin{aligned} x' + a_1(t)x^2 + \{b_0(t) + c_0(t) + 2[a_0(t)q_0(t) + a_2(t)q_2(t) + a_3(t)q_3(t)]\}x \\ - Q\big(t, q_0(t), q_2(t), q_3(t)\big) = 0\,; \end{aligned} \tag{3.12}$$

$$\begin{aligned} x' + a_2(t)x^2 + \{b_0(t) + c_0(t) + 2[a_0(t)q_0(t) + a_1(t)q_1(t) + a_3(t)q_3(t)]\}x \\ - R\big(t, q_0(t), q_1(t), q_3(t)\big) = 0\,; \end{aligned} \tag{3.13}$$

$$\begin{aligned} x' + a_3(t)x^2 + \{b_0(t) + c_0(t) + 2[a_0(t)q_0(t) + a_1(t)q_1(t) + a_2(t)q_2(t)]\}x \\ - S\big(t, q_0(t), q_1(t), q_2(t)\big) = 0\,. \end{aligned} \tag{3.14}$$

Let $x_0(t)$, $x_1(t)$, $x_2(t)$ and $x_3(t)$ be the solutions of the equations (3.11), (3.12), (3.13) and (3.14) respectively with $x_n(t_0) = 0$, $n = \overline{0,3}$. By virtue of Theorem 2.1 from (3.10) it follows that $x_n(t)$, $n = \overline{0,3}$, exist on $[t_0; T_2)$ and are non negative. Then since $q_0(t)$, $q_1(t)$, $q_2(t)$ and $q_3(t)$ are solutions of the equations (3.11), (3.12), (3.13) and (3.14) on $[t_0; T_2)$ and $q_n(t_0) > x_n(t_0)$, $n = \overline{0,3}$, the last functions (i.e. $q_n(t)$, $n = \overline{0,3}$) are also non negative on $[t_0; T_2)$, which contradicts (3.9). The obtained contradiction proves (3.8). By virtue of Lemma 2.2 from (3.8) and from the conditions of the theorem it follows that on $[t_0; T)$ the inequalities (3.10) are fulfilled. Hence the solutions $x_n(t_0)$ ($n = \overline{0,3}$) exist on $[t_0; T)$ and are non negative. Obviously $q_0(t)$, $q_1(t)$, $q_2(t)$ and $q_3(t)$ are solutions of the equations (3.11), (3.12), (3.13) and (3.14)) respectively on $[t_0; T)$ and $q_n(t_0) > x_n(t_0)$, $n = \overline{0,3}$. Therefore $q_n(t) > 0$, $t \in [t_0; T)$, $n = \overline{0,3}$. Further, the proof of the theorem is carried out similar to the proof of Theorem 3.1.

The theorem is proved. □

Theorem 3.3. *Let $a_0(t) \geq 0$, $a_n(t) \equiv 0$, $n = \overline{1,3}$, $t \geq t_0$, and*

$$\int\limits_{t_m}^{t} \exp\Big\{\int\limits_{t_m}^{t} \big[b_0(s) + c_0(s) - I_{b_0+c_0,D_0}(t_m, s)\big]ds\Big\} D_0(\tau)d\tau \leq 0\,,$$

$$t \in [t_m; t_{m+1})\,, \ m = 0, 1, \ldots$$

Then for every $\gamma_0 \geq 0$, $\gamma_n \in (-\infty; +\infty)$, $n = \overline{1,3}$, Eq. (1.2) has a solution $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ with $q_n(t_n) = \gamma_n$, $n = \overline{0,3}$ on $[t_0; +\infty)$ and

$$q_0(t) \geq 0\,, \qquad t \geq t_0\,. \tag{3.15}$$

Proof. Let $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ be the solution of Eq. (1.2) with $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$, and let $[t_0; T)$ be the maximum existence interval for $q(t)$. Show that

$$T = +\infty \,. \tag{3.16}$$

Consider the Riccati equation

$$y' + a_0(t)y^2 + [b_0(t) + c_0(t)]y + D_0(t) = 0\,, \qquad t \geq t_0\,. \tag{3.17}$$

By Theorem 2.2 from the conditions of the theorem it follows that for every $\gamma_0 \geq 0$ this equation has a solution $y_0(t)$ on $[t_0; +\infty)$ and $y_0(t) \geq 0$, $t \geq t_0$. Then using Theorem 2.1 to Eq. (3.11) and Eq. (3.17) and taking into account the fact that $q_0(t)$ is a solution to Eq. (3.11) we conclude that

$$q_0(t) \geq y_0(t) \geq 0\,, \qquad t \geq t_0\,. \tag{3.18}$$

Suppose $T < +\infty$. Then from (3.18) it follows that

$$\mathrm{tr}[A(t)Y(t)] = \int\limits_{t_0}^{t} a_0(s)q_0(s)ds \geq 0\,, \qquad t \in [t_0; T)\,.$$

By virtue of Lemma 2.1 from here it follows that $[t_0; T)$ is not the maximum existence interval for $q(t)$ which contradicts our supposition. The obtained contradiction proves (3.16). From (3.16) and (3.18) it follows (3.15). The theorem is proved. $\square$

Remark 3.2. Unlike of the conditions of Theorem 3.1 and Theorem 3.2 the conditions of Theorem 3.3 allow $D_0(t)$ to change sign in every $[t_m; t_{m+1})$, $m = 0, 1, \ldots$.

By use of Theorem 2.3 and Theorem 2.4 analogically can be proved the following two theorems respectively.

Theorem 3.4. *Let $\alpha(t)$ and $\beta(t)$ be continuously differentiable on $[t_0; +\infty)$ functions and $\alpha(t) > 0$, $\beta(t) > 0$, $t \geq t_0$,*

(A_1) $0 \leq a_0(t) \leq \alpha(t)$, $D_0(t) \leq \beta(t)$, $a_n(t) \equiv 0$, $n = \overline{1,3}$, $t \geq t_0$;

(B_1) $b_0(t) + c_0(t) \geq \frac{1}{2}\Big[\frac{\alpha'(t)}{\alpha(t)} - \frac{\beta'(t)}{\beta(t)}\Big] + \sqrt{\alpha(t)\beta(t)}$, $t \geq t_0$.

Then for every $\gamma_0 \geq -\sqrt{\frac{\beta(t_0)}{\alpha(t_0)}}$, $\gamma_n \in (-\infty; +\infty)$, $n = \overline{1,3}$, Eq. (1.2) has a solution $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ on $[t_0; +\infty)$ with $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$, and

$$q_0(t) \geq -\sqrt{\frac{\beta(t)}{\alpha(t)}}\,, \qquad t \geq t_0\,.$$

$\square$

Theorem 3.5. *Let $\alpha(t)$ and $\beta(t)$ be the same as in Theorem 3.4. If assumption (A_1) of Theorem 3.4 and the inequality*

(C_1) $b_0(t) + c_0(t) \leq \frac{1}{2}\Big[\frac{\alpha'(t)}{\alpha(t)} - \frac{\beta'(t)}{\beta(t)}\Big] - \sqrt{\alpha(t)\beta(t)}$, $t \geq t_0$, *are valid. Then for every* $\gamma_0 \ge \sqrt{\frac{\beta(t_0)}{\alpha(t_0)}}$, $\gamma_n \in (-\infty;+\infty)$, $n=\overline{1,3}$, *Eq.* (1.2) *has a solution* $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ *on* $[t_0;+\infty)$ *with* $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$, *and*

$$q_0(t) \ge \sqrt{\frac{\beta(t)}{\alpha(t)}}, \qquad t \ge t_0 .$$

□

4. The case when $a_0(t)$ may change sign

In this section we consider the case when $a_0(t)$ my change sign. Set:

$$\left[\frac{\sqrt{\sum_{n=1}^3 (b_n(t)+c_n(t))^2}}{a_0(t)}\right]_0 \equiv \begin{cases} \frac{\sqrt{\sum_{n=1}^3 (b_n(t)+c_n(t))^2}}{a_0(t)}, & \text{if} \quad a_0(t) \ne 0; \\ 0\,, & \text{if} \quad a_0(t) = 0\,, \end{cases}$$

$$\mathfrak{M}(t) \equiv \int\limits_{t_0}^{t} \|\big(d_1(\tau), d_2(\tau), d_3(\tau)\big)\| d\tau + \frac{1}{2} \sup_{\tau\in[t_0;t]} \left[\frac{\sqrt{\sum_{n=1}^3 (b_n(\tau)+c_n(\tau))^2}}{a_0(\tau)}\right]_0 ,$$

$$R_\Gamma(t) \equiv |a_0(t)|(\Gamma + \mathfrak{M}(t))^2 + \sum_{n=1}^{3} |b_n(t)+c_n(t)|(\Gamma+\mathfrak{M}(t))\,, \qquad t \ge t_0\,,$$

where $\Gamma > 0$ is a parameter. For any quaternion $q \equiv q_0 + iq_1 + jq_2 + kq_3$ $(q_n \in \mathbb{R}, n = \overline{0.3})$, set $[q]_v \equiv (q_1, q_2, q_3)$.

Theorem 4.1. *Let* $\alpha_m(t)$ *and* $\beta_m(t)$, $m = 1,2$ *be the same as in Theorem 2.5. If:*

1) $a_n(t) \equiv 0$, $n = \overline{1,3}$;
2) $\alpha_1(t) \le a_0(t) \le \alpha_2(t)$, $\beta_1(t) \le R_\Gamma(t) + d_0(t) \le \beta_2(t)$, $t \in [t_0;\tau_0)$;
3) $b_0(t) + c_0(t) \ge \frac{1}{2}\big(\frac{\alpha_m'(t)}{\alpha_m(t)} - \frac{\beta_m'(t)}{\beta_m(t)}\big) + 2(-1)^m\sqrt{\alpha_m(t)\beta_m(t)}$, $t \in [t_0;\tau_0)$, $m = 1,2$;
4) $b_0(t) + c_0(t) \ge 2|a_0(t)|R_\Gamma(t)$, $t \in [t_0;\tau_0)$;
5) $\operatorname{supp}(b_n(t)+c_n(t)) \subset \operatorname{supp} a_0(t)$, $n = \overline{1,3}$, *the function* $\left[\frac{\sqrt{\sum_{n=1}^3 (b_n(t)+c_n(t))^2}}{a_0(t)}\right]_0$ *is bounded on* $[t_0;\tau_0)$ *if* $\tau_0 < +\infty$ *and is locally bounded on* $[t_0;\tau_0)$ *if* $\tau_0 = +\infty$,

then for every $\gamma_0 \in \Big[-\sqrt{\frac{\beta_2(t_0)}{\alpha_2(t_0)}};\sqrt{\frac{\beta_1(t_0)}{\alpha_1(t_0)}}\Big]$, $\gamma_n \in \mathbb{R}$, $n = \overline{1,3}$, *with* $\|(\gamma_1,\gamma_2,\gamma_3)\| \le \Gamma$ *Eq.* (1.1) *has a solution* $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ *on* $[t_0;\tau_0)$ *satisfying* *the initial conditions* $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$, *and*

$$-\sqrt{\frac{\beta_2(t)}{\alpha_2(t)}} \le q_0(t) \le \sqrt{\frac{\beta_1(t)}{\alpha_1(t)}}\,, \qquad t \in [t_0; \tau_0)\,; \tag{4.1}$$

$$\|[q(t)]_v\| \le \|[q(t_0)]_v| + \mathfrak{M}(t)\,, \quad t \in [t_0; \tau_0)\,. \tag{4.2}$$

If $\tau_0 < +\infty$ *then* $q(t)$ *is continuable on* $[t_0; \tau_0]$.

Proof. Let $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ be the solution of Eq. (1.1) with $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$, and let $[t_0; T)$ be the maximum existence interval for $q(t)$. We must show that

$$T \ge \tau_0\,. \tag{4.3}$$

Under the restriction 1) the system (2.1) takes the form

$$\begin{cases} q_0' + a_0(t)q_0^2 + \{b_0(t) + c_0(t)\}q_0 - P(t, q_1, q_2, q_3) = 0; \\ \widetilde{q}\,' + \mathcal{L}_{q_0}(t)\widetilde{q} - f_{q_0}(t) = 0\,, \qquad t \ge t_0\,, \end{cases} \tag{4.4}$$

where

$$f_{q_0}(t) \equiv \big(\big(b_1(t)+c_1(t)\big)q_0+d_1(t), \big(b_2(t)+c_2(t)\big)q_0+d_2(t), \big(b_3(t)+c_3(t)\big)q_0+d_3(t)\big)\,,$$

$$\mathcal{L}_{q_0}(t) \equiv$$
$$\begin{pmatrix} b_0(t) + c_0(t) + 2a_0(t)q_0 & c_3(t) - b_3(t) & b_2(t) - c_2(t) \\ b_3(t) - c_3(t) & b_0(t) + c_0(t) + 2a_0(t)q_0 & c_1(t) - b_1(t) \\ c_2(t) - b_2(t) & b_1(t) - c_1(t) & b_0(t) + c_0(t) + 2a_0(t)q_0 \end{pmatrix},$$

$t \ge t_0$, $\widetilde{q} \equiv (q_1, q_2, q_3)$. Since the hermitian part $\mathcal{L}^H_{q_0(t)}(t)$ of the matrix $\mathcal{L}_{q_0(t)}(t)$ is $\mathcal{L}^H_{q_0(t)}(t) = \operatorname{diag}\big\{b_0(t) + c_0(t) + 2a_0(t)q_0(t), b_0(t) + c_0(t) + 2a_0(t)q_0(t), b_0(t) + c_0(t) + 2a_0(t)q_0(t)\big\}$, by the second equation of the system (4.4) $\|[q(t)]_v\|$ we have the estimate (see [8, p. 56, Lemma 4.2]):

$$\begin{aligned} \|[q(t)]_v\| \le \|[q(t_0)]_v\| \exp\Big\{-\int\limits_{t_0}^{t} \big(b_0(\tau) + c_0(\tau) + 2a_0(\tau)q_0(\tau)\big)d\tau\Big\} \\ + \int\limits_{t_0}^{t} \exp\Big\{-\int\limits_{\tau}^{t} \big(b_0(s) + c_0(s) + 2a_0(s)q_0(s)\big)\, ds\Big\}\|f_{q_0(\tau)}(\tau)\|\, d\tau\,, \\ t \in [t_0; t_1)\,. \end{aligned} \tag{4.5}$$

From the condition 4) of the theorem it follows that

$$b_0(t) + c_0(t) + 2a_0(t)q_0(t) \ge 0\,, \qquad t \in [t_0; t_1)\,, \tag{4.6}$$

for some $t_1 > t_0$. Show that

$$-\sqrt{\frac{\beta_2(t)}{\alpha_2(t)}} \le q_0(t) \le \sqrt{\frac{\beta_1(t)}{\alpha_1(t)}}\,, \qquad t \in [t_0; T_2)\,; \tag{4.7}$$

$$\|[q(t)]_v\| \le \|[q(t_0)]_v\| + \mathfrak{M}(t)\,, \qquad t \in [t_0 : T_2)\,. \tag{4.8}$$

From (4.5) and (4.6) it follows

$$\|[q(t)]_v\| \le \|[q(t_0)]_v\| + \frac{1}{2}\exp\Big\{-\int_{t_0}^{t}\big(b_0(s)+c_0(s)+2a_0(s)q_0(s)\big)\,ds\Big\}$$
$$\times \int_{t_0}^{t}\Big(\exp\Big\{\int_{t_0}^{\tau}\big(b_0(s)+c_0(s)+2a_0(s)q_0(s)\big)\,ds\Big\}\Big)'$$
$$\times \left[\frac{\sqrt{\sum_{n=1}^{3}(b_n(\tau)+c_n(\tau))^2}}{a_0(\tau)}\right]_0 d\tau + \int_{t_0}^{t}\|\big(d_1(\tau),d_2(\tau),d_3(\tau)\big)\|d\tau\,,$$

for $t \in [t_0; t_1)$. From here from (4.6) and 5) it follows (4.8). Since $\|[q(t_0)]_v| \le \Gamma$ from (4.8) we obtain

$$-R_\Gamma(t) + q_0(t) \le P\big(t, q_1(t), q_2(t), q_3(t)\big) \le R_\Gamma(t) + q_0(t)\,, \qquad t \in [t_0; t_1)\,.$$

From here and from 2) it follows

$$\beta_1(t) \le P\big(t, q_1(t), q_2(t), q_3(t)\big) \le \beta_2(t)\,, \qquad t \in [t_0; t_1)\,. \tag{4.9}$$

Consider the Riccati equation

$$r' + a_0(t)r^2 + \{b_0(t) + c_0(t)\}r - P\big(t, q_1(t), q_2(t), q_3(t)\big) = 0\,, \qquad t \in [t_0; t_1)\,. \tag{4.10}$$

Let $r(t)$ be a solution of this equation with $r(t_0) = q_0(t_0)$. Then by virtue of Theorem 2.1 from 1), 2) and (4.9) it follows that $r(t)$ exists on $[t_0; t_1)$ and

$$-\sqrt{\frac{\beta_2(t)}{\alpha_2(t)}} \le r(t) \le \sqrt{\frac{\beta_1(t)}{\alpha_1(t)}}, \qquad t \in [t_0; t_1)\,.$$

Obviously $q_0(t)$ is a solution of Eq. (4.7) on $[t_0; t_1)$. Hence by the uniqueness theorem $q_0(t)$ coincides with $r(t)$ on $[t_0; t_1)$, and therefore (4.7) is valid. Let T_1 be the upper bound of all $t_1 \in [t_0; T)$ for which (4.7)–(4.9) are satisfied. We assert that

$$T_1 = T\,. \tag{4.11}$$

Indeed otherwise from (4.7) it follows that

$$q_0(t) \ge -\sqrt{\frac{\beta_2(T_1)}{\alpha_2(T_1)}}\,.$$

From here and from 4) it follows that $b_0(t) + c_0(t) + 2a_0(t)q_0(t) \ge 0,\ \ t \in [T_1; T_2)$ for some $T_2 > T_1$. Hence

$$b_0(t) + c_0(t) + 2a_0(t)q_0(t) \ge 0\,,\ \ t \in [t_0; T_2)\,. \tag{4.12}$$

Then repeating the arguments of the proof of (4.7) and (4.8) we conclude that

$$-\sqrt{\frac{\beta_2(t)}{\alpha_2(t)}} \le q_0(t) \le \sqrt{\frac{\beta_1(t)}{\alpha_1(t)}}\,, \qquad t \in [t_0; T_2)\,;$$

$$\|[q(t)]_v\| \le \|[q(t_0)]_v\| + \mathfrak{M}(t)\,, \qquad t \in [t_0 : T_2)\,,$$

which with (4.12) contradicts the definition of T_1. The obtained contradiction proves (4.11). Thus

$$-\sqrt{\frac{\beta_2(t)}{\alpha_2(t)}} \le q_0(t) \le \sqrt{\frac{\beta_1(t)}{\alpha_1(t)}}\,, \qquad t \in [t_0; T)\,;$$

$$\|[q(t)]_v\| \le \|[q(t_0)]_v\| + \mathfrak{M}(t)\,, \qquad t \in [t_0 : T)\,.$$

By virtue of Lemma 2.1 from here it follows (4.3) and fulfillment of (4.1) and (4.2). If $\tau_0 < +\infty$ then by Lemma 2.1 from (4.1) and (4.2) it follows that $q(t)$ is continuable on $[t_0; \tau_0]$.
The theorem is proved. $\square$

Let $\tau_0 < +\infty$. Set:

$$\mathfrak{M}^*(t) \equiv \int\limits_t^{\tau_0} \|(d_1(\tau), d_2(\tau), d_3(\tau))\|\, d\tau + \frac{1}{2} \sup_{\tau \in [t;\tau_0]} \left[\frac{\sqrt{\sum_{n=1}^3 (b_n(\tau) + c_n(\tau))^2}}{a_0(\tau)}\right]_0,$$

$$R^*_\Gamma(t) \equiv |a_0(t)|(\Gamma + \mathfrak{M}^*(t))^2 + \sum_{n=1}^3 |b_n(t) + c_n(t)|(\Gamma + \mathfrak{M}^*(t))\,, \qquad t \in [t_0; \tau_0]\,.$$

Corollary 4.1. *Let $\alpha_m(t)$ and $\beta_m(t)$, $m = 1, 2$, be continuously differentiable on $[t_0; \tau_0]$ functions such that $(-1)^m \alpha_m(t) > 0$, $(-1)^m \beta(t) > 0$, $t \in [t_0; \tau_0]$, $m = 1, 2$. If:*

1) $a_n(t) \equiv 0,\ \ n = \overline{1, 3}$;

1*) $\alpha_1(t) \le a_0(t) \le \alpha_2(t)$;

2*) $b_0(t) + c_0(t) \le -\frac{1}{2}\big(\frac{\alpha'_m(t)}{\alpha_m(t)} - \frac{\beta'_m(t)}{\beta_m(t)}\big) + 2(-1)^m \sqrt{\alpha_m(t)\beta_m(t)},\ \ t \in [t_0; \tau_0]$, $m = 1, 2$;

3*) $b_0(t) + c_0(t) \le -2|a_0(t)| R^*_\Gamma(t),\ \ t \in [t_0; \tau_0]$;

4*) $\operatorname{supp}(b_n(t) + c_n(t)) \subset \operatorname{supp} a_0(t),\ \ n = \overline{1, 3}$, *the function* $\left[\frac{\sqrt{\sum_{n=1}^3 (b_n(t)+c_n(t))^2}}{a_0(t)}\right]_0$ *is bounded on* $[t_0; \tau_0]$, *then for every* $\gamma_0 \in \Big[-\sqrt{\frac{\beta_1(\tau_0)}{\alpha_1(\tau_0)}}; \sqrt{\frac{\beta_2(\tau_0)}{\alpha_2(\tau_0)}}\Big]$, $\gamma_n \in \mathbb{R}$, $n = \overline{1,3}$, *with* $\|(\gamma_1, \gamma_2, \gamma_3)\| \le \Gamma$ *Eq.* (1.1) *has a solution* $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ *on* $[t_0; \tau_0]$ *satisfying the initial conditions* $q_n(\tau_0) = \gamma_n$, $n = \overline{0,3}$, *and*

$$-\sqrt{\frac{\beta_1(t)}{\alpha_1(t)}} \le q_0(t) \le \sqrt{\frac{\beta_2(t)}{\alpha_2(t)}}, \qquad t \in [t_0; \tau_0]; \tag{4.13}$$

$$\|[q(t)]_v\| \le \|[q(\tau_0)]_v\| + \mathfrak{M}^*(t), \qquad t \in [t_0; \tau_0]. \tag{4.14}$$

Proof. Set: $\lambda_0 \equiv t_0 = \tau_0$, $\widetilde{a}(t) \equiv -a(\lambda_0 - t)$, $\widetilde{b}(t) \equiv -b(\lambda_0 - t)$, $\widetilde{c}(t) \equiv -c(\lambda_0 - t), \widetilde{d}(t) \equiv -d(\lambda_0 - t)$, $\widetilde{a}_0(t) \equiv -a_0(\lambda_0 - t)$, $\widetilde{b}_n(t) \equiv -b_n(\lambda_0 - t)$, $\widetilde{c}_n(t) \equiv -c_n(\lambda_0 - t)$, $\widetilde{d}_n(t) \equiv -d_n(\lambda_0 - t)$,

$$\widetilde{\mathfrak{M}}(t) \equiv \int\limits_{t_0}^{t} \|(\widetilde{d}_1(\tau), \widetilde{d}_2(\tau), \widetilde{d}_3(\tau))\| d\tau + \frac{1}{2}\sup_{\tau\in[t_0;t]} \left[\frac{\sqrt{\sum_{n=1}^{3}(\widetilde{b}_n(\tau) + \widetilde{c}_n(\tau))^2}}{\widetilde{a}_0(\tau)}\right]_0,$$

$$\widetilde{R}_\Gamma(t) \equiv |\widetilde{a}_0(t)|(\Gamma + \widetilde{\mathfrak{M}}(t))^2 + \sum_{n=1}^{3} |\widetilde{b}_n(t) + \widetilde{c}_n(t)|(\Gamma + \widetilde{\mathfrak{M}}(t)), \qquad t \in [t_0; \tau_0],$$

where

$$\left[\frac{\sqrt{\sum_{n=1}^{3}(\widetilde{b}_n(t) + \widetilde{c}_n(t))^2}}{\widetilde{a}_0(t)}\right]_0 \equiv \begin{cases} \frac{\sqrt{\sum_{n=1}^{3}(\widetilde{b}_n(t)+\widetilde{c}_n(t))^2}}{\widetilde{a}_0(t)}, & \text{if } \widetilde{a}_0(t) \ne 0; \\ 0, & \text{if } \widetilde{a}_0(t) = 0. \end{cases}$$

In Eq. (1.1) make the substitution

$$q(t) = u(\lambda_0 - t), \qquad t \in [t_0; \tau_0].$$

we obtain

$$u' + u\widetilde{a}(t)u + \widetilde{b}(t)u + u\widetilde{c}(t) + \widetilde{d}(t) = 0, \qquad t \in [t_0; \tau_0]. \tag{4.15}$$

It is not difficult to verify that

$$\widetilde{\mathfrak{M}}(\lambda_0 - t) = \mathfrak{M}^*(t), \qquad \widetilde{R}_\Gamma(\lambda_0 - t) = R^*_\Gamma(t), \qquad t \in [t_0; \tau_0.$$

From here and from the conditions 1), 1*)–4*) of the corollary we get:

$$\widetilde{\alpha}_1(t) \le \widetilde{a}_0(t) \le \widetilde{\alpha}_2(t), \qquad \widetilde{\beta}_1(t) \le \widetilde{R}_\Gamma(t) + \widetilde{d}_0(t) \le \widetilde{\beta}_2(t),$$
$$\widetilde{b}_0(t) + \widetilde{c}_0(t) \ge 2|\widetilde{a}_0(t)|\widetilde{R}_\Gamma(t),$$
$$\widetilde{b}_0(t) + \widetilde{c}_0(t) \ge \frac{1}{2}\left(\frac{\widetilde{\alpha}'_m(t)}{\widetilde{\alpha}_m(t)} - \frac{\widetilde{\beta}'_m(t)}{\widetilde{\beta}_m(t)}\right) + 2(-1)^m\sqrt{\widetilde{\alpha}_m(t)\widetilde{\beta}_m(t)}, \qquad t \in [t_0; \tau_0],$$

where $\widetilde{\alpha}_m(t) \equiv -\alpha_{3-m}(\lambda_0 - t)$, $\widetilde{\beta}_m(t) \equiv -\beta_{3-m}(\lambda_0 - t)$, $m = 1, 2$, $t \in [t_0; \tau_0]$, $\operatorname{supp}(\widetilde{b}_n(t) + \widetilde{c}_n(t)) \subset \operatorname{supp} \widetilde{a}_0(t)$, $n = \overline{1,3}$, the function $\left[\frac{\sqrt{\sum_{n=1}^{3}(\widetilde{b}_n(t)+\widetilde{c}_n(t))^2}}{\widetilde{a}_0(t)}\right]_0$ is bounded on $[t_0; \tau_0]$. By Theorem 4.1 from here is seen that for every

$\gamma_0 \in \left[-\sqrt{\frac{\widetilde{\beta}_2(t_0)}{\widetilde{\alpha}_2(t_0)}}; \sqrt{\frac{\widetilde{\beta}_1(t_0)}{\widetilde{\alpha}_1(t_0)}}\right], \gamma_n \in$ R, $n = \overline{1,3}$, with $||(\gamma_1, \gamma_2, \gamma_3)|| \le \Gamma$ Eq. (4.15) has a solution $u(t) \equiv u_0(t) - iu_1(t) - ju_2(t) - ku_3(t)$ on $[t_0; \tau_0]$ and

$$-\frac{\widetilde{\beta}_2(t)}{\widetilde{\alpha}_2(t)} \le u_0(t) \le \frac{\widetilde{\beta}_1(t)}{\widetilde{\alpha}_1(t)}\,,$$

$$\|[u(t)]_v\| \le \|[u(t_0)]_v\| + \widetilde{\mathfrak{M}}(t)\,, \qquad t \in [t_0; \tau_0]\,.$$

From here it follows that Eq. (1.1) has a solution $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ on $[t_0; \tau_0]$, satisfying the initial conditions $q_n(\tau_0) = \gamma_n, \ n = \overline{0,3}$ and the estimates (4.13) and (4.14) are valid.
The corollary is proved. □

5. A completely non conjugation theorem

Consider the linear system

$$\begin{cases} \phi' = C(t)\phi + A(t)\psi\,; \\ \psi' = -D(t)\phi - B(t)\psi\,, \qquad t \ge t_0\,. \end{cases} \tag{5.1}$$

where $\phi = \phi(t)$ and $\psi = \psi(t)$ are the unknown continuously differentiable vector functions of dimension 4, $A(t), \ B(t), \ \ C(t)$ and $D(t)$ are the same matrix functions as in (2.5).

Definition 5.1. We will say that the solution $(\phi(t), \psi(t))$ of the system (5.1) satisfies the completely non conjugation condition if $\phi(t) \ne \theta, \ \ \psi(t) \ne \theta \ \ t \ge t_0$, where θ is the null vector of dimension 4.

Theorem 5.1. *Let the conditions of Theorem 3.1 (of Theorem 3.2) are satisfied. Then the solution* $(\phi(t), \psi(t))$ *of the system* (5.1) *with* $\psi(t_0) = (\gamma_0 E - \gamma_1 I - \gamma_2 J - \gamma_3 K)\phi(t_0) \ne \theta$, *where* $\gamma_n \ge 0, \ \ n \in \mathfrak{S}(\ne \emptyset), \ \ \sum\limits_{n\in\mathfrak{S}} \gamma_n \ne 0, \ \ \gamma_n \in (-\infty; +\infty)$, $n \in \mathfrak{O}$ *(where* $\gamma_n > 0$, $n = \overline{0,3}$*) satisfies of the completely non conjugation condition.*

Proof. Let the conditions of Theorem 3.1 (of Theorem 3.2) be satisfied and let $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ be the solutions of Eq. (1.2) with $q_n(t_0) = \gamma_0$, $n = \overline{0,3}$ By virtue of Theorem 3.1 (Theorem 3.2) $q(t)$ exists on $[t_0; +\infty)$. From the condition $\sum\limits_{n\in\mathfrak{S}} \gamma_n > \ 0 \ \ (\gamma_n > 0$, $n = \overline{0,3})$ it follows that

$$q(t) \ne 0\,, \qquad t \ge t_0\,. \tag{5.2}$$

By (2.4) $Y_1(t) \equiv \widehat{q(t)}$ is a solution of Eq. (2.3) on $[t_0; +\infty)$. From (5.2) it follows that

$$\det Y_1(t) \ne 0\,, \qquad t \ge t_0\,. \tag{5.3}$$

Let $\Phi_1(t)$ be the solution of the matrix equation

$$\Phi' = [A(t)Y_1(t) + C(t)]\Phi = 0\,, \qquad t \ge t_0\,,$$

satisfying the initial condition $\Phi_1(t_0) = E$. Them by the Liouville's formula we have

$$\det \Phi(t) = \exp\left\{ \int\limits_{t_0}^{t} \operatorname{tr}\left[A(\tau)Y_1(\tau) + C(\tau)\right] d\tau \right\} > 0\,, \qquad t \geq t_0\,. \tag{5.4}$$

Let $(\phi(t), \psi(t))$ be the solution of the system (5.1) satisfying the initial condition of the theorem. Then

$$\phi(t) = \Phi(t)\phi(t_0)\,, \qquad \psi(t) = Y_1(t)\Phi(t)\phi(t_0)\,.$$

From here from (5.3) and (5.4) it follows that $\phi(t) \neq \theta,\ \ \psi(t) \neq \theta,\ \ t \geq t_0$.
The theorem is proved. $\square$

Remark 5.1. Except in a special case when $A(t)$ and $D(t)$ are diagonal matrices and $C(t) = B^*(t),\ \ t \geq t_0$ (here $*$ is the transpose sign) the system (5.1) is not hamiltonian.

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References

[1] Campos, J., Mawhin, J., *Periodic solutions of quaternionic-valued ordinary differential equations*, Annali di Mathematica **185** (2006), 109–127.

[2] Christiano, V., Smarandache, F., *An Exact Mapping from Navier - Stokes Equation to Schrodinger Equation via Riccati Equation*, Progress in Phys. **1** (2008), 38–39.

[3] Gibbon, J.D., Halm, D.D., Kerr, R.M., Roulstone, I., *Quaternions and particle dynamics in the Euler fluid equations*, Nonlinearity **19** (2006), 1962–1983.

[4] Grigorian, G.A., *On two comparison tests for second-order linear ordinary differential equations*, Differ. Uravn. **47** (9) (2011), 1225–1240, Russian. Translation in Differ. Equ. 47 (2011), no. 9, 1237–1252.

[5] Grigorian, G.A., *Two comparison criteria for scalar Riccati equations with applications*, Russian Math. (Iz. VUZ) **56** (11) (2012), 17–30.

[6] Grigorian, G.A., *Global solvability of scalar Riccati equations*, Izv. Vissh. Uchebn. Zaved. Mat. **3** (2015), 35–48.

[7] Grigorian, G.A., *Global solvability criteria for scalar Riccati equations with complex coefficients*, Differ. Uravn. **53** (4) (2017), 459–464.

[8] Hartman, Ph., *Ordinary differential equations*, Classics in Applied Mathematics, vol. 38, SIAM, Philadelphia, 2002.

[9] Leschke, K., Morya, K., *Application of quaternionic holomorphic geometry to minimal surfaces*, Complex Manifolds **3** (2006), 282–300.

[10] Wilzinski, P., *Quaternionic - valued differential equations. The Riccati equation*, J. Differential Equations **247** (2009), 2163–2187.

[11] Zoladek, H., *Classifications of diffeomormhisms of S^4 induced by quaternionic Riccati equations with periodic coefficients*, Topol. methods Nonlinear Anal. **33** (2009), 205–215.

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