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# AN UPPER BOUND OF A GENERALIZED UPPER HAMILTONIAN NUMBER OF A GRAPH

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Abstract. In this article we study graphs with ordering of vertices, we define a generalization called a pseudoordering, and for a graph $H$ we define the $H$-Hamiltonian number of a graph $G$. We will show that this concept is a generalization of both the Hamiltonian number and the traceable number. We will prove equivalent characteristics of an isomorphism of graphs $G$ and $H$ using $H$-Hamiltonian number of $G$. Furthermore, we will show that for a fixed number of vertices, each path has a maximal upper $H$-Hamiltonian number, which is a generalization of the same claim for upper Hamiltonian numbers and upper traceable numbers. Finally we will show that for every connected graph $H$ only paths have maximal $H$-Hamiltonian number.

## 1. Introduction

In this article we study a part of graph theory based on an ordering of vertices. We define a generalization called a pseudoordering of a graph. We will show how to generalize a Hamiltonian number, for a graph $H$ we define the $H$-Hamiltonian number of a graph $G$ and we will show that this concept is a generalization of both the Hamiltonian number and the traceable number. We get them by a special choice of graph $H$. Furthermore, we will study a maximalization of upper $H$-Hamiltonian number for a fixed number of vertices. We will show that, for a fixed number of vertices, each path has a maximal upper $H$-Hamiltonian number. From the definition it will be obvious that a lower bound of the $H$-Hamiltonian number is the number of edges $|E(H)|$ and the graph $G$ has a minimal lower $H$-Hamiltonian number if and only if $H$ is a subgraph of $G$. Now we can say that $G$ having a maximal upper $H$-Hamiltonian number is dual to $H$ being a subgraph of $G$. Furthermore, by above for every two finite graphs $G$ and $H$ such that $G$ is connected satisfying $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$, we get that $G \cong H$ if and only if the lower $H$-Hamiltonian number of $G$ is $|E(H)|$.

In [2] it is proved that $G$ has a maximal upper traceable number if and only if $G$ is a path. The same is proved for Hamiltonian number. We will show that for

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$H$ connected $G$ has a maximal $H$-Hamiltonian number if and only if $G$ is a path. This shows that this generalization of ordering of vertices is natural.

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In this article we will study a generalization of Hamiltonian spectra of undirected finite graphs. Recall that, a graph $G$ is a pair

$$G = \big(V(G), E(G)\big)\,,$$

where $V(G)$ is a finite set of vertices of $G$ and $E(G) \subseteq V(G) \times V(G)$, a symmetric Antireflexive relation, is a set of edges. We will denote an edge between $v$ and $u$ by $\{v, u\}$.

Recall that, an *ordering* on the graph $G$ is a bijection

$$f\colon \{1, 2, \dots, |V(G)|\} \to V(G)\,,$$

we denote

$$s(f, G) = \sum_{i=1}^{|V(G)|} \rho_G\big(f(i), f(i+1)\big)\,,$$

$$\bar{s}(f, G) = \sum_{i=1}^{|V(G)|-1} \rho_G\big(f(i), f(i+1)\big)\,,$$

where $\rho_G(x, y)$ is the distance of $x, y$ in the graph $G$ and $f(|V(G)| + 1) := f(1)$, for better notation. We will write only $s(f)$, $\bar{s}(f)$ if the graph is clear from context. Then

$$\{s(f, G) \mid f \ \text{ ordering on } \ G\}$$

$$\{\bar{s}(f, G) \mid f \ \text{ ordering on } \ G\}$$

are the *Hamiltonian spectrum* of the graph $G$ and the *traceable spectrum* of the graph $G$, respectively.

We want to generalize the notion of an ordering of a graph.

**Definition 1.1.** Let $G$, $H$ be graphs such that $|V(G)| = |V(H)|$ and $f\colon V(H) \to V(G)$ is a bijection, then we call $f$ a *pseudoordering* on the graph $G$ (by $H$), denote

$$s_H(f, G) = \sum_{\{x,y\} \in E(H)} \rho_G\big(f(x), f(y)\big)\,,$$

where $\rho_G(x, y)$ is the distance of $x$, $y$ in the graph $G$. We will call $s_H(f, G)$ the sum of the pseudoordering $f$. Then

$$\{s_H(f, G) \mid f \ \text{ pseudoordering on } \ G \text{ by } H\}$$

is the *$H$-Hamiltonian spectrum* of the graph $G$.

The minimum and the maximum of a Hamiltonian spectrum and of a traceable spectrum are called the (*lower*) *Hamiltonian number* and the *upper Hamiltonian number*, respectively. Furthermore, the (lower) traceable number and the upper traceable number of a graph $G$ are denoted by

$$h(G) = \min\{s(f,G) \mid f \text{ ordering on } G\}\,,$$
$$h^+(G) = \max\{s(f,G) \mid f \text{ ordering on } G\}\,,$$
$$t(G) = \min\{\bar{s}(f,G) \mid f \text{ ordering on } G\}\,,$$
$$t^+(G) = \max\{\bar{s}(f,G) \mid f \text{ ordering on } G\}\,.$$

Now we define generalized versions.

**Definition 1.2.**

$$h_H(G) = \min\{s_H(f,G) \mid f \text{ pseudoordering on } G\},$$
$$h_H^+(G) = \max\{s_H(f,G) \mid f \text{ pseudoordering on } G\}\,.$$

We will call them the *lower $H$-Hamiltonian number* and the *upper $H$-Hamiltonian number* of a graph $G$, respectively.

Now take $H = C_{|V(G)|}$, where $C_n$ is the cycle with $n$ vertices. When we denote the vertices of $C_{|V(G)|}$ by $\{1,2,\dots,|V(G)|\}$ we can see that

$$s(f,G) = s_{C_{|V(G)|}}(f,G)\,.$$

Analogously for $H = P_{|V(G)|-1}$, where $P_{n-1}$ is the path of length $n-1$, we get that

$$\bar{s}(f,G) = s_{P_{|V(G)|-1}}(f,G)\,.$$

**Remark 1.3.** The $C_{|V(G)|}$-Hamiltonian spectrum of a graph $G$ is equal to the Hamiltonian spectrum of $G$ for $|V(G)| \geq 3$, and the $P_{|V(G)|-1}$-Hamiltonian spectrum of $G$ is equal to the traceable spectrum of $G$ for $|V(G)| \geq 2$.

**Lemma 1.4.** *Let $G$ be a connected finite graph and $H$ be a graph such that $|V(G)| = |V(H)|$, then $h_H(G) = |E(H)|$ if and only if $H$ is isomorphic to some subgraph of $G$.*

**Proof.** Let $f\colon V(H) \to V(G)$ be a pseudoordering satisfying $s(f,G) = |E(H)|$, then $f$ is an injective graph homomorphism. The opposite implication is obvious. □

**Lemma 1.5.** *Let $G$ be a connected finite graph and $H$ be a graph such that $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$, then $h_H(G) = |E(H)|$ if and only if $H$ is isomorphic to the graph $G$.*

**Proof.** The graph $H$ is isomorphic to a subgraph of $G$ and furthermore $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, hence $H \cong G$. The opposite implication is obvious. □

## 2. Maximalization of the upper $H$-Hamiltonian number of a graph $G$

In this section we will prove that for every pair of connected graphs $H, G$ and each pseudoordering $f$ there exists a pseudoordering

$$g\colon V(H) \to \{1,2,\dots,|V(G)|\}$$

such that

$$s_H(f,G) \leq s_H(g, P_{|V(G)|-1})\,.$$

At first, let $G$ be a tree. We will only work with graphs which have at least 2 vertices.

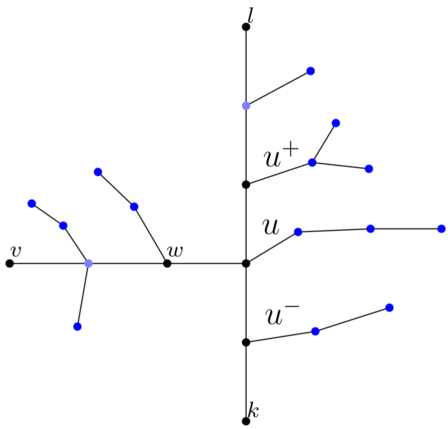
**Definition 2.1.** Let $G$ and $H$ be graphs such that $G$ is connected, $|V(G)| = |V(H)|$ and $f\colon V(H) \to V(G)$ is a pseudoordering. Furthermore, let $a, b \in V(G)$, we define $a \sim_{H,f} b$ if and only if $\{f^{-1}(a), f^{-1}(b)\} \in E(H)$.

**Definition 2.2.** Let $G$ be a tree such that $G$ is not a path. Denote three pairwise distinct leaves by $l$, $k$, $v \in V(G)$. Because $G$ is not a path then $G$ has at least 3 leaves, connect $l$, $k$ with a path $l = x_1, x_2, \dots, x_m = k$. Connect $v$, $l$ with a path $l$ $v = y_1, y_2, \dots, y_s = l$ and take the minimum of a set

$$i_m = \min\{i \mid \exists j \in \{1, \dots, m\}, y_i = x_j\}\,.$$

Take $j_m$ such that $y_{i_m} = x_{j_m}$. Now we define $u = y_{i_m}$, $w = y_{i_m - 1}$, $u^+ = x_{j_m - 1}$, $u^- = x_{j_m + 1}$.

**Example.**




**Remark 2.3.** $l \neq u \neq k$.

**Definition 2.4.** Define a set $K(v, G) \subseteq V(G)$ as a set of vertices $z \in V(G)$ such the path between $z$ and $l$ uses the edge $\{w, u\}$.

**Remark 2.5.** $K(v, G)$ is the connected component of $\big(V(G), E(G) \setminus \{w, u\}\big)$, $G$ without edge $\{w, u\}$, which contains $v$.

**Lemma 2.6.** (i) *Paths between vertices from $K(v, G)$ don't use the edge $\{w, u\}$.*

(ii) *Paths between vertices from $V(G) \setminus K(v, G)$ don't use the edge $\{w, u\}$.*

(iii) *Paths joining a vertex from $V(G) \setminus K(v, G)$ to a vertex from $K(v, G)$ use the edge $\{w, u\}$.*

**Proof.** Because $G$ is a tree, there is a unique path between each pair of vertices, then it is obvious by remark 2.5. □

**Definition 2.7.** Define graphs

$$\bar{G} = \big(V(G), E(G) \setminus \{\{w,u\}\} \cup \{\{w,l\}\}\big)\,,$$
$$\tilde{G} = \big(V(G), E(G) \setminus \{\{w,u\}\} \cup \{\{w,k\}\}\big)\,.$$

**Lemma 2.8.** *$\bar{G}$ and $\tilde{G}$ are trees.*

**Proof.** At first we show connectivity, let $a$, $b \in V(G)$, connect them with a path. If both are in $K(v,G)$ or in $V(G) \setminus K(v,G)$, then by Lemma 2.6, the path in $G$ uses only edges which are also in $\bar{G}, \tilde{G}$. Hence it is path also there.

Let $a \in K(v,G)$ and $b \in V(G)\setminus K(v,G)$. We can see $w \in K(v,G)$, by Lemma 2.6 a path between $a$ and $w$, $a = a_1, a_2, \dots, a_p = w$, doesn't use $\{w,u\}$ and all vertices of this path are in $K(v,G)$. If not, there is a path between vertices from $K(v,G)$ and $V(G) \setminus K(v,G)$ which doesn't use $\{w,u\}$, that is a contradiction with Lemma 2.6. Connect $l$ and $b$ with a path, $l = b_1, b_2, \dots, b_q = b$. It doesn't use $\{w,u\}$ and all vertices are in $V(G) \setminus K(v,G)$. Then $a = a_1, a_2, \dots, a_p = w,\ l = b_1, b_2, \dots, b_q = b$ is a path between $a$, $b$ in the graph $\bar{G}$, analogously for $\tilde{G}$.

Now we show that they don't contain a cycle, for contradiction suppose that $\bar{G}$ contains a cycle $C \subseteq \bar{G}$. If $C$ doesn't use the edge $\{w,l\}$, then $C \subseteq G$, but $G$ is a tree, this is a contradiction. If $C$ uses $\{w,l\}$, then there exists a path in $G$ between $w,l$, which doesn't use the edge $\{w,l\}$. Then there exists a path in $G$ between $w,l$, which doesn't use the edge $\{w,u\}$, but $w \in K(v,G)$ and $l \in V(G) \setminus K(v,G)$, that is contradiction with Lemma 2.6. Analogously for $\tilde{G}$. □

We want to show that

$$s_H(G,f) \le s_H(\bar{G},f)$$

or

$$s_H(G,f) \le s_H(\tilde{G},f)\,.$$

**Lemma 2.9.**

$$a,b \in K(v,G), \quad \text{then} \quad \rho_G(a,b) = \rho_{\bar{G}}(a,b) = \rho_{\tilde{G}}(a,b)\,,$$
$$a,b \in V(G) \setminus K(v,G), \quad \text{then} \quad \rho_G(a,b) = \rho_{\bar{G}}(a,b) = \rho_{\tilde{G}}(a,b)\,.$$

**Proof.** A path in $G$ between $a,b$, by Lemma 2.6, doesn't use $\{u,w\}$ , hence it is a path in $\bar{G}$ and $\tilde{G}$ too, then the distance of $a,b$ is the same in $G$, $\bar{G}$ and $\tilde{G}$. □

**Definition 2.10.** Define subsets

$$F^+,\ F^-,\ F^0 \subseteq K(v,G) \times \big(V(G) \setminus K(v,G)\big)$$

such that $(a,b) \in F^+$ if a path between $a$, $b$ uses the edge $\{u,u^+\}$. $(a,b) \in F^-$ if a path between $a,b$ uses the edge $\{u,u^-\}$ and $(a,b) \in F^0$ if a path between $a$, $b$ doesn't use neither $\{u,u^-\}$ nor $\{u,u^+\}$.

**Lemma 2.11.** *$F^+$, $F^-$, $F^0$ are pairwise disjoint and*

$$F^+ \cup F^- \cup F^0 = K(v,G) \times \big(V(G) \setminus K(v,G)\big)\,.$$

**Proof.** From the definition of $F^+$, $F^-$, $F^0$ we have $F^-$ and $F^0$, $F^+$ and $F^0$ are disjoint. Let $(a,b) \in F^+ \cap F^-$, then the path between $a$, $b$ uses edges $\{u, u^-\}, \{u, u^+\}$ and by lemma 2.6, it also uses the edge $\{w, u\}$. Hence it is a path which has a vertex of degree 3 and that is contradiction. $\square$

**Lemma 2.12.** *Let $x$, $\bar{x} \in K(v,G)$ and $y$, $\bar{y} \in V(G) \setminus K(v,G)$ such that $(x,y) \in F^+$ and $(\bar{x},\bar{y}) \in F^-$. Then*

$$\rho_{\bar{G}}(x,y) + \rho_{\bar{G}}(\bar{x},\bar{y}) \geq \rho_G(x,y) + \rho_G(\bar{x},\bar{y})\,,$$

$$\rho_{\tilde{G}}(x,y) + \rho_{\tilde{G}}(\bar{x},\bar{y}) \geq \rho_G(x,y) + \rho_G(\bar{x},\bar{y})\,.$$

*Moreover, both sides are equal, in the first inequality, if and only if $y = l$ and, in the second inequality, if and only if $\bar{y} = k$.*

**Proof.** Let $z$ denote the first common vertex of paths $Q\colon l = y_1, y_2, \dots, y_s = k$ and $P\colon y = x_1, x_2, \dots, x_m = x$. Consider

$$i_m = \min\{i \mid \exists j \in \{1, \dots, m\}, y_i = x_j\}$$

and therefore $z = y_{i_m}$, let $T$ be the path from $z$ to $l$, we will show that $z$ is the only one common vertex of $T$ and $P$, vertices from $P$ split into the 4 subpaths, $P_1$ from $y$ to $z$, $P_2$ from $z$ to $u$, edge $\{u,w\}$ and $P_3$ from $w$ to $x$. Vertices from $P_1$ are not in $Q$ (except for $z$) from the definition of $z$. Vertices from $P_2$ are not in $T$ (except for $z$) from the uniqueness of paths in trees and vertices from $P_3$ belong to $K(v,G)$ and every vertex of $T$ belongs to $V(G) \setminus K(v,G)$. By composition of paths $P_1$, $T, \{l,w\}$, $P_3$, we get a path from $y$ to $x$ in the graph $\bar{G}$.

Let $\bar{P}$ denote the path from $\bar{y}$ to $\bar{x}$, analogously define $\bar{z}$ as the first common vertex of paths $\bar{P}$ and $Q$ (first in the direction from $\bar{y}$ to $\bar{x}$ ). We split $\bar{P}$ into the subpaths $\bar{P}_1$ from $\bar{y}$ to $\bar{z}$ , $\bar{P}_2$ from $\bar{z}$ to $u$, edge $\{u,w\}$ and $\bar{P}_3$ from $u$ to $\bar{x}$. Let $\bar{T}$ be the path from $u$ to $l$, analogously we get that $u$ is the only one common vertex of $\bar{P}$ and $\bar{T}$. Hence $\bar{P}_1, \bar{P}_2, \bar{T}, \{l,w\}, \bar{P}_3$ is a path between $\bar{y}, \bar{x}$ in the graph $\bar{G}$.

And for paths from $u$ to $z$ and from $u$ to $\bar{z}$, $u$ is the only one common vertex, by uniqueness of path in trees.

Now we can calculate

$$\rho_G(x,y) = \rho_G(x,w) + 1 + \rho_G(u,z) + \rho_G(z,y)\,,$$

$$\rho_G(\bar{x},\bar{y}) = \rho_G(\bar{x},w) + 1 + \rho_G(u,\bar{z}) + \rho_G(\bar{z},\bar{y})\,,$$

$$\rho_{\bar{G}}(x,y) = \rho_G(x,w) + 1 + \rho_G(l,z) + \rho_G(z,y)\,,$$

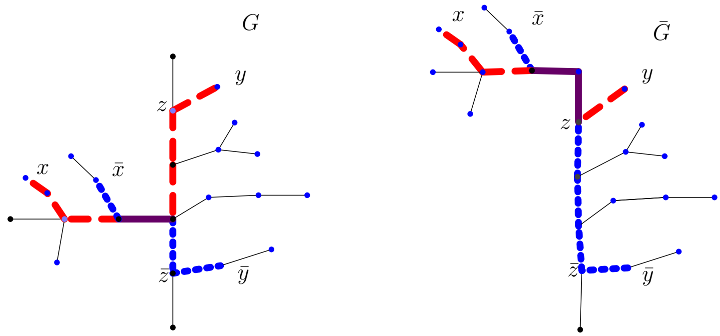
$$\rho_{\bar{G}}(\bar{x},\bar{y}) = \rho_G(\bar{x},w) + 1 + \rho_G(l,z) + \rho_G(z,u) + \rho_G(u,\bar{z}) + \rho_G(\bar{z},\bar{y})\,,$$

hence

$$\rho_{\bar{G}}(\bar{x},\bar{y}) + \rho_{\bar{G}}(x,y) = \rho_G(\bar{x},\bar{y}) + \rho_G(x,y) + 2\rho_G(l,z)\,.$$

Now we get our inequality and we see that both are equal if and only if $l = z$. But $l$ is a leaf, hence $z$ is a leaf, then $y = z = l$. For $\tilde{G}$ analogously. $\square$

**Example.** Paths between $x$, $y$ and $\bar{x}$, $\bar{y}$ in graphs $G$ and $\bar{G}$.




**Lemma 2.13.** *Let* $(x, y) \in F^0$ *then*

$$\rho_{\bar{G}}(x, y) > \rho_G(x, y) ,$$

$$\rho_{\tilde{G}}(x, y) > \rho_G(x, y) .$$

**Proof.** Let $P$ be a path from $x$ to $y$ and $Q$ be a path from $l$ to $k$ in $G$, for $P$ and $Q$, $u$ is the only one common vertex because $(x, y) \in F^0$. Hence $x \to w - l \to u \to y$ is a path in $\bar{G}$, where paths of type $a \to b$ are subpaths of $P$ and $Q$ and $-$ denotes an edge. Now we can calculate the following

$$\rho_{\bar{G}}(x, y) = \rho_G(x, u) + 1 + \rho_G(l, u) + \rho_G(u, y) = \rho_G(x, y) + \rho_G(l, u)$$

and from $l \neq u$ we have inequality.

For $\tilde{G}$ analogously. □

**Lemma 2.14.**

$$\rho_{\bar{G}}(x, y) > \rho_G(x, y) \quad \textit{for} \quad (x, y) \in F^- ,$$

$$\rho_{\tilde{G}}(x, y) > \rho_G(x, y) \quad \textit{for} \quad (x, y) \in F^+ .$$

**Proof.** We will prove the first inequality. As well as in lemma 2.12 denote $z$ the first common vertex of paths from $y$ to $x$ and from $k$ to $l$, formally we can define it as well as in lemma 2.12. Now we consider a path
$x \to w - l \to u \to z \to y$. Hence

$$\begin{aligned}\rho_{\bar{G}}(x, y) &= \rho_G(x, w) + 1 + \rho_G(l, u) + \rho_G(u, z) + \rho_G(z, y) \\ &= \rho_G(x, y) + \rho_G(l, u)\end{aligned}$$

and from $l \neq u$ we have inequality.

For second inequality analogously. □

**Definition 2.15.** Let $G$ be a tree and $H$ be a graph such that

$$|V(G)| \;=\; |V(H)|$$

and

$$f \colon V(H) \to V(G)$$

is a pseudoordering, we define a set

$$L = \{(x,y) \in K(v,G) \times \big(V(G) \setminus K(v,G)\big) \mid x \sim_{H,f} y\},$$

where $K(v,G)$ is the set from Definition 2.4.

**Lemma 2.16.** *Let $G$ be a tree and $H$ be a graph such that, $|V(G)| = |V(H)|$ and*

$$f\colon V(H) \to V(G)$$

*is a pseudoordering. Then*

$$s_H(f,\bar{G}) \ge s_H(f,G)$$

*or*

$$s_H(f,\tilde{G}) \ge s_H(f,G)\,,$$

*the first case occurs when*

$$|L \cap F^+| \le |L \cap F^-|\,,$$

*the second case occurs when*

$$|L \cap F^+| \ge |L \cap F^-|\,.$$

**Proof.** Denote $n^+ = |L \cap F^+|$, $n^- = |L \cap F^-|$, $m = |L \cap F^0|$,

$$\bar{m} = \frac{\big|\{(x,y) \in \big(K(v,G)^2\big) \cup \big(\big(V(G) \setminus K(v,G)\big)^2\big) \mid x \sim_{H,f} y\}\big|}{2}\,,$$

where square $K(v,G)^2$ means $K(v,G) \times K(v,G)$. $\bar{m}$ is number of edges $\{x,y\} \in E(H)$, which satisfy that $f(x)$ and $f(y)$ lie in the same component of

$$(V(G), E(G) \setminus \{w,u\})\,.$$

Let $n^+ \ge n^-$, the second case is analogous, we rearrange the sum $s_H(f,G)$ in this way

$$s_H(f,G) = \sum_{i=1}^{n^-} \big(\rho_G(x_i,y_i) + \rho_G(\bar{x}_i,\bar{y}_i)\big) + \sum_{i=n^-+1}^{n^+} \rho_G(x_i,y_i)$$

$$+ \sum_{i=1}^{m} \rho_G(a_i,b_i) + \sum_{i=1}^{\bar{m}} \rho_G(c_i,d_i)\,,$$

where

$$(x_i,y_i) \in F^+, \quad (\bar{x}_i,\bar{y}_i) \in F^-, \quad (a_i,b_i) \in F^0,$$

$$(c_i,d_i) \in \{(x,y) \in \big(K(v,G)^2\big) \cup \big(\big(V(G) \setminus K(v,G)\big)^2\big) \mid x \sim_{H,f} y\}\,.$$

Now, by Lemma 2.12

$$\rho_G(x_i,y_i) + \rho_G(\bar{x}_i,\bar{y}_i) \le \rho_{\tilde{G}}(x_i,y_i) + \rho_{\tilde{G}}(\bar{x}_i,\bar{y}_i)\,,$$

by Lemma 2.14

$$\rho_G(x_i,y_i) \le \rho_{\tilde{G}}(x_i,y_i)\,,$$

by Lemma 2.13
$$\rho_G(a_i, b_i) \le \rho_{\tilde{G}}(a_i, b_i)$$
and by Lemma 2.9
$$\rho_G(c_i, d_i) = \rho_{\tilde{G}}(c_i, d_i)\,.$$
Hence
$$\begin{aligned} s_H(f,G) &\le \sum_{i=1}^{n^-}(\rho_{\tilde{G}}(x_i,y_i) + \rho_{\tilde{G}}(\bar{x}_i,\bar{y}_i)) \\ &\quad + \sum_{i=n^-+1}^{n^+} \rho_{\tilde{G}}(x_i,y_i) + \sum_{i=1}^{m}\rho_{\tilde{G}}(a_i,b_i) + \sum_{i=1}^{\bar{m}}\rho_{\tilde{G}}(c_i,d_i) \\ &= s_H(f,\tilde{G})\,. \end{aligned}$$
□

**Lemma 2.17.** *Let $G$ be a tree and $H$ be a graph such that, $|V(G)| = |V(H)|$ and $f\colon V(H) \to V(G)$ is a pseudoordering. Then there exists a pseudoordering*
$$g\colon V(H) \to \{x_1, x_2, \dots, x_{|V(G)|}\} = V(P_{|V(G)|-1}) \quad \textit{such that}$$
$$s_H(f,G) \le s_H(g, P_{|V(G)|-1})\,.$$

**Proof.** We denote
$$\alpha(G) = \sum_{\substack{v\in V(G)\\ \deg_G v \ge 3}} \deg_G v,$$
from the definition of $u$, $l$ and $k$ we know that $\deg_G u \ge 3$ and $\deg_G l = \deg_G k = 1$. From the construction of $\bar{G}$ and $\tilde{G}$ we have $\deg_{\bar{G}} u = \deg_{\tilde{G}} u \le \deg_G u$, $\deg_{\bar{G}} l = \deg_{\tilde{G}} k = 2$ and all other vertices have the same degree as before. Hence
$$\alpha(\bar{G}) < \alpha(G)\,,$$
$$\alpha(\tilde{G}) < \alpha(G)\,.$$
Let $S$ be a tree, which is not a path, we choose any three pairwise distinct leaves in $V(S)$ and define $S^*$ as one of graphs $\bar{S}, \tilde{S}$, which satisfy $s_H(f, S^*) \ge s_H(f, S)$. Denote $G_0 = G$ and for $i \ge 0$ denote $G_{i+1} = G_i^*$ if $G_i$ is not a path, otherwise define $G_{i+1} = G_i$. For contradiction we assume that the tree $G_i$ is not a path for every $i \in \mathbb{N}_0$. We know $\alpha(G_i) \in \mathbb{N}_0$ for every $i$ and
$$\alpha(G_{i+1}) \le \alpha(G_i) - 1\,,$$
hence
$$\alpha(G_{\alpha(G_0)+1}) \le \alpha(G_0) - \alpha(G_0) - 1 = -1$$
and this is contradiction. Therefore there exists some $j$ such that $G_j$ is a path, from Lemma 2.16 we get
$$s_H(f, G_{i+1}) \ge s_H(f, G_i)$$

and hence

$$s_H(f, G_j) \geq s_H(f, G)\,.$$

□

**Theorem 2.18.** *Let $G$ and $H$ be graphs such that $G$ is connected, $|V(G)| = |V(H)|$ and $f\colon V(H) \to V(G)$ is a pseudoordering, then there exists a pseudordering*

$$g\colon V(H) \to \{x_1, x_2, \dots, x_{|V(G)|}\} = V(P_{|V(G)|-1}) \quad \textit{such that}$$

$$s_H(f, G) \leq s_H(g, P_{|V(G)|-1})\,.$$

**Proof.** Let $K$ be any spanning tree of $G$, $x$, $y \in V(G)$, we connect $x$ and $y$ with a path in graph $K$, this path is also a path in $G$. Hence

$$\rho_G(x, y) \leq \rho_K(x, y)$$

for every $x, y$, hence

$$s_H(f, G) \leq s_H(f, K),$$

by Lemma 2.17 there exists a pseudoordering

$$g\colon V(H) \to \{x_1, x_2, \dots, x_{|V(G)|}\} = V(P_{|V(G)|-1}) \quad \text{such that}$$

$$s_H(f, G) \leq s_H(f, K) \leq s_H(g, P_{|V(G)|-1})\,.$$

□

**Corollary 2.19.** *Let $G$ and $H$ be graphs such that $G$ is connected, $|V(G)| = |V(H)|$, then*

$$h^+_H(G) \leq h^+_H(P_{|V(G)|-1})\,.$$

## 3. Graphs with a maximal upper H-Hamiltonian number

In this section we will prove that if in Corollary 2.19 the graph $H$ is connected, then in the inequality in Corollary 2.19 both sides are equal.

**Remark 3.1.** For easier writing, we will denote vertices of $H$ the same as vertices of $G$, we will rename them in this way $v \in H \mapsto f(v)$. We can naturally see it as graph with two sets of edges.

In inequalities in Lemma 2.16 both sides are equal under specific conditions, if $L \cap F^0 \neq \emptyset$, then in Lemma 2.13 there is a strict inequality and then also the same happens in Theorem 2.18.

If $(L \setminus K(v, G) \times \{l\}) \cap F^+ \neq \emptyset$, then in Lemma 2.12 there is a strict inequality and then also the same happens in Theorem 2.18. Analogously if

$$(L \setminus K(v, G) \times \{k\}) \cap F^- \neq \emptyset\,.$$

Overall we get that the only nontrivial case is

$$L \subseteq K(v, G) \times \{k, l\}\,. \tag{1}$$

**Remark 3.2.** Remark 3.1 holds for every triple of distinct leaves $k$, $l$, $v$ in $G$.

**Lemma 3.3.** *Let $G$ be a tree, $H$ connected graph such that $|V(G)| = |V(H)|$ and $f\colon V(H) \to V(G)$ is a pseudoordering, which satisfy*

$$s_H(f, G) = h_H^+(P_{|V(G)|-1}),$$

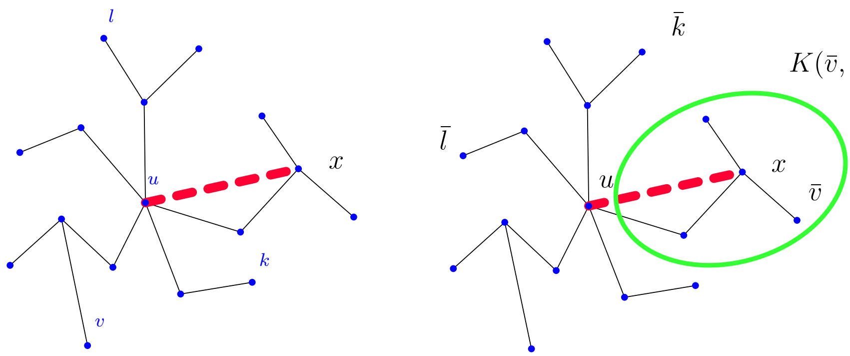
*then $G$ is path.*

**Proof.** For contradiction suppose that $G$ is not a path, then there exist three pairwise distinct leaves $k, l, v$, we denote in the same way as before, vertex $u$ and set of vertices $K(v, G)$. Because graph $H$ is connected there exists a vertex $x$ such that $\{u, x\} \in E(H)$. Let $X \subseteq V(G)$ be a set of vertices of components of graph $G \setminus u$, containing $x$. $G \setminus u$ has, by definition of $u$, at least 3 components. Let now $\bar{v}$ be an arbitrary leaf (leaf in $G$) in $X$. Choose $\bar{k}$, $\bar{l}$ as arbitrary leaves in pairwise distinct components of $G \setminus u$ and different from $X$.

Now $(x, u) \in \bar{L}$, where $\bar{L}$ is alternative of $L$ for $\bar{k}$, $\bar{l}$, $\bar{v}$ and by Remark 3.1 for $\bar{k}$, $\bar{l}$, $\bar{v}$ and by $k \neq u \neq l$ we get contradiction.

□

**Example.** We show the idea of the last proof in the following picture.




**Remark 3.4.** Let $G$ be a graph with a maximal $H$-Hamiltonian number, then every spanning tree of $G$ has a maximal $H$-Hamiltonian number, therefore every spanning tree is a path. We will show that the only graphs with this property are cycles and paths.

**Lemma 3.5.** *Let $G$ be a connected graph such that $|V(G)| \geq 2$, then there is a vertex, which is not an articulation point.*

**Proof.** Consider a block-cut tree of $G$ and a block $B$, which is a leaf of the block-cut tree or if this tree has only one vertex, then $B = G$. $B$ is, by definition of a block, 2-connected. Because $B$ is leaf we get that in $B$ there is only one articulation and in $B$ there are at least 2 vertices. Hence in $B$ there is at least one vertex, which is not an articulation point. □

**Lemma 3.6.** *Let $G$ be a finite connected graph such that $|V(G)| \geq 2$ and every spanning tree of $G$ is a path, then $G$ is a path or a cycle.*

**Proof.** We will prove it by induction with respect to the number of vertices. Let $n$ be the number of vertices, for $n = 2$ and $n = 3$ it is obviously true. Let it be true for $n \geq 3$, let $G$ be a graph with $n + 1$ vertices such that every spanning tree of $G$ is a path. Let $v \in V(G)$ be a vertex, which is not an articulation point, by lemma 3.5 it exists. We denote $G'$ the subgraph induced by the set of vertices $V(G) \setminus \{v\}$. $G'$ is connected, we will show that every spanning tree of $G'$ is a path. Let there exist a spanning tree which is not a path, let $u \in V(G)$ be a vertex such that $\{v, u\} \in E(G)$. Now when we add this edge to the spanning tree, we get a spanning tree of $G$, which is not a path and it is a contradiction. By induction hypothesis $G'$ is a path or a cycle, we denote $A = \{u \in V(G) | \{v, u\} \in E(G)\}$. For contradiction we assume $G'$ is a cycle and let $u \in A$, in $G'$ be an edge $e$ such that $u$ is not incident to $e$. Consider the subgraph $B$ of $G$, $B = (V(G), E(G') \setminus e \cup \{v, u\})$, and this is a spanning tree of $G$ which is not path, contradiction.

Therefore $G'$ is a path, let $x$, $y$ be endpoints of this path, for contradiction we assume that there exists some another vertex $u \in A$. Hence $G'$ together with $\{u, v\}$ form a spanning tree which is not a path. Hence $A \subseteq \{x, y\}$, because $G$ is connected we get also $A \neq \emptyset$. Finally there are the two cases for $G$, if $|A| = 1$, then $G$ will be a path and if $|A| = 2$, then $G$ will be a cycle. □

**Theorem 3.7.** *Let $G$ and $H$ be connected finite graphs such that $|V(G)| = |V(H)|$, then*

$$h_H^+(G) \leq h_H^+(P_{|V(G)|-1}),$$

*moreover, both sides are equal if and only if $G$ is a path.*

**Proof.** The first part follows from Theorem 2.18, let $G$ be a graph, $f$ be a pseudoordering such that

$$s_H(f, G) = h_H^+(G) = h_H^+(P_{|V(G)|-1}).$$

From the proof of Theorem 2.18 we know that every spanning tree also satisfies the equation above. Hence, by Lemma 3.3, every spanning tree of $G$ is a path. By Lemma 3.6 $G$ is a path or a cycle, for contradiction we assume, that it is a cycle. We denote $n = |V(G)|$, we will show that there are two vertices $v, u \in V(G)$ such that $v \sim_{H,f} u$ and $\rho_G(u, v) < \frac{n}{2}$.

Because $G$ is cycle, $|V(H)| = n \geq 3$ and $H$ is connected we see that there is a vertex of degree at least 2. Let $v$ be a vertex such that $deg_H(v) \geq 2$, there exists at least two vertices $u$ such that $v \sim_{H,f} u$. There exists at most one vertex such that $\rho_G(u, v) \geq \frac{n}{2}$, hence at least one of them satisfies $\rho_G(u, v) < \frac{n}{2}$.

Now we connect $v$ and $u$ with a shorter path in $G$. Let $e$ be some edge on this path, we define a graph $\bar{G} = \big(V(G), E(G) \setminus e\big)$, it is a path, where every distance is greater or equal as in $G$. But $\rho_G(u, v) < \rho_{\bar{G}}(u, v)$ and then

$$s_H(f, \bar{G}) = s_H(f, \bar{G}) > h_H^+(P_{|V(G)|-1}),$$

and this is contradiction with Theorem 2.18. □

## 4. Conclusion

When we use following equations which can be found for example in [2, Theorem 1.3] and [2, Corollary 2.2]

$$h^+(P_{|V(G)|-1}) = \left\lfloor \frac{|V(G)|^2}{2} \right\rfloor , \quad t^+(P_{|V(G)|-1}) = \left\lfloor \frac{|V(G)|^2}{2} \right\rfloor - 1\,.$$

This result is also calculated in [1] and when we use Theorem 3.7 for $H = P_{|V(G)|-1}$ and for $H = C_{|V(G)|}$ we get the following theorem.

**Theorem 4.1** ([2]).

$$h^+(G) \le \left\lfloor \frac{|V(G)|^2}{2} \right\rfloor , \quad t^+(G) \le \left\lfloor \frac{|V(G)|^2}{2} \right\rfloor - 1\,.$$

*Moreover, both sides are equal if and only if $G$ is a path.*

First part is [2, Corollary 2.2] and second part is [2, Theorem 4.2]. Now we can see, that Theorem 3.7 is generalization of Theorem 4.1 which is from article [2].

## References

[1] Dzúrik, M., *Metrické vlastnosti grafů*, bachelor thesis (2018).

[2] Okamoto, F., Zhang, P., *On upper traceable numbers of graphs*, Math. Bohem. **133** (2008), 389–405.

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