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BOUNDARY VALUE PROBLEMS FOR CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Abstract. In this article, we study the existence of solutions in a Banach space of boundary value problems for Caputo-Hadamard fractional differential inclusions of order $r \in (0,1]$.

1. Introduction

This article deals the existence of solutions for boundary value problems for fractional order differential inclusions. We consider the boundary-value problem

$$ {}^{c}_{H}D^{r} y(t) \in F(t, y(t))\,, \text{ for a.e. } \ t \in J = [1,T],\ \ 0 < r \le 1\,, \tag{1}$$

$$ ay(1) + by(T) = c\,, \tag{2}$$

where $T > 1$, ${}^{c}_{H}D^{r}$ is the Caputo-Hadamard fractional derivative of order $0 < r \le 1$, $F\colon [1,T] \times E \to \mathcal{P}(E)$ is a multivalued map, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, E is a Banach space, and a, b and c are real constants such that $a + b \neq 0$.

For boundary value problems for differential inclusions with nolocal boundary conditions and comments on their importance, we refer the reader to the papers by Gallardo [24], Karakostas and Tsamatos [31], Lomtatidze and Malaguti [37] and the references therein. Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors, for instance, Brykalov [17], Denche and Marhoune [22] and Krall [36]. Recently Ahmad, Khan and Sivasundaram [3, 32] have applied the generalized method of quasilinearization to a class of second order boundary value problem with integral boundary conditions. Some results on the existence of solutions for a class of boundary value problems for fractional order differential inclusions with integral conditions have been obtained by Benchohra et al. [9, 10, 11].

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Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, and so on (see [23, 26, 27, 29, 38, 39, 41]). However, the literature on Hadamard-type fractional differential equations has not undergone as much development; see [4, 43]. Hadamard's fractional derivative [28] of 1892 differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of the Hadamard derivative contains a logarithmic function of arbitrary exponent. The works in [4, 18, 19, 20, 33, 34, 35, 43] are major developments in the fundamental theory of Hadamard fractional calculus. A Caputo-type modification of the Hadamard fractional derivative, which is called the Caputo-Hadamard fractional derivative, was given in [30], and its fundamental theorems were proved in [1, 25].

In this paper, we present existence results for the problems (1)–(2) in the case where the right hand side is convex-valued. This result relies on the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness. Recently, this has proved to be a valuable tool in studying fractional differential equations and inclusions in Banach spaces; for additional details, see the papers of Agarwal et al. [2] and Benchohra et al. [12, 13, 14]. Our results here extend to the multivalued case some previous results in the literature and constitutes what we hope is an interesting contribution to this emerging field. We include an example to illustrate our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that are used in the remainder of this paper.

Let $C(J,E)$ be the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 1 \leq t \leq T\}\,,$$

let $L^1(J,E)$ denote the Banach space of functions $y\colon J \to E$ which are Bochner integrable with norm

$$\|y\|_{L^1} = \int_1^T |y(t)|\,dt\,.$$

$AC(J,E)$ is the space of functions $y\colon J \to E$, which are absolutely continuous whose first derivative, y', is continuous.

Let $(X, |\cdot|)$ be a Banach space. Let $P_{\mathrm{cl}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{\mathrm{cp}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$ and $P_{\mathrm{cp},c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multivalued map $G\colon X \to \mathcal{P}(X)$ is convex (closed) valued if $G(X)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.)

on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denote by $\operatorname{Fix} G$. A multivalued map $G\colon J \to P_{\mathrm{cl}}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \to d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable. Let X and Y be two sets, and $N\colon X \to p(Y)$ be a set-valued map. We define the graph of N, as

$$\operatorname{graph}(N) = \{(x, y) : x \in X, y \in N(x)\}\,.$$

For more details on multivalued maps see the books of Deimling ([21]), and Aubin et al. ([6, 7]).
Let $R > 0$, and

$$B = \{x \in E : |x| \le R\},\ \ U = \{x \in C(J, E) : \|x\| \le R\}\,,$$

clearly U is a closed subset of $C(J, B)$.

For convenience, we first recall the definition of the Kuratowski measure of noncompactness, and summarize the main properties of this measure.

Definition 2.1 ([5, 8])**.** Let E be a Banach space and let Ω_E be the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha\colon \Omega_E \to [0, \infty)$ defined by

$$\alpha(B) = \inf\{\epsilon > 0 : B \subset \bigcup_{j=1}^{m} B_j \text{ and } \operatorname{diam}(B_j) \le \epsilon\},\ \text{for } B \in \Omega_E\,.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for details, see [5], [8]).

(1) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(2) $\alpha(B) = \alpha(\overline{B})$.

(3) $A \subset B \Rightarrow \alpha(A) \le \alpha(B)$.

(4) $\alpha(A + B) \le \alpha(A) + \alpha(B)$.

(5) $\alpha(cB) = |c|\alpha(B), c \in \mathbb{R}$.

(6) $\alpha(\operatorname{con} B) = \alpha(B)$.

Here $\overline{B}$ and $\operatorname{con} B$ denote the closure and the convex hull of the bounded set B, respectively.

For a given set V of functions $u\colon J \to E$, we set

$$V(t) = \{u(t) : u \in V\}\,, \quad t \in J\,,$$

and

$$V(J) = \{u(t) : u \in V(t)\,,\ t \in J\}\,.$$

Theorem 2.2 ([40]). *Let E be a Banach space and $C \subset L^1(J,E)$ be countable with $|u(t)| \le h(t)$ for a.e. $t \in J$ and every $u \in C$, where $h \in L^1(J,\mathbb{R}_+)$. Then the function $\phi(t) = \alpha(C(t))$ belong to $L^1(J,\mathbb{R}_+)$ and satisfies*

$$\alpha\Big(\Big\{\int_0^T u(s)\,ds, u \in C\Big\}\Big) \le 2\int_0^T \alpha\big(C(s)\big)\,ds\,.$$

Let us now recall the set-valued analog of Mönch's fixed point theorem.

Theorem 2.3 ([42]). *Let K be a closed, convex subset of a Banach space E, U a relatively open subset of K, and $N\colon \overline{U} \mapsto \mathcal{P}(K)$. Assume* graph N *is closed, N maps compact sets into relatively compact sets, and for some $x_0 \in U$, the following two conditions are satisfied:*

- *Let y belongs to $AC^n_\delta([a,b],E)$ or*

$$\begin{array}{ll} M \subset \overline{U},\ M \subset \operatorname{conv}(x_0 \cup N(M))\ \overline{M} = \overline{U} & \\ \quad \textit{with } C \textit{ a countable subset of } M \textit{ implies } \overline{M} \textit{ is compact,} \end{array} \tag{3}$$

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$$x \notin (1-\lambda)x_0 + \lambda N(x) \quad \textit{for all} \quad x \in \overline{U}\,U,\ \ \lambda \in (0,1)\,. \tag{4}$$

Then there exists $x \in \overline{U}$ with $x \in N(x)$.

Definition 2.4. A multivalued map $F\colon J \times E \to \mathcal{P}(E)$ is said to be Carathéodory if

(1) $t \to F(t,u)$ is measurable for each $u \in E$;
(2) $u \to F(t,u)$ is upper semicontinuous for almost all $t \in J$.

For each $y \in C(J,E)$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1(J,E) : v(t) \in F(t,y(t)) \ \text{ a.e. } t \in J\}\,.$$

Definition 2.5 ([34]). The Hadamard fractional integral of order $\alpha > 0$ for a function $h\colon [a,b] \to \mathbb{R}$, where a, $b \ge 0$, is defined by

$${}_HI^\alpha_a h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{h(s)}{s}\,ds\,,$$

provided the integral exists.

Definition 2.6 ([30]). Let $AC^n_\delta[a,b] = \{g : [a,b] \to \mathbb{C},\ \delta^{n-1}g \in AC[a,b]\}$ where $\delta = t\dfrac{d}{dt}$, $0 < a < b < \infty$ and let $\alpha \in \mathbb{C}$, such that $\operatorname{Re}(\alpha) \ge 0$. For a function $g \in AC^n_\delta[a,b]$ the Caputo-Hadamard derivative of fractional order α is defined as follows:

(i): If $\alpha \notin \mathbb{N}$, and $n-1 < \alpha < n$ such that $n = [\mathrm{Re}(\alpha)] + 1$, then

$$({}_H^c D_a^\alpha g)(t) = \frac{1}{\Gamma(n-\alpha)} \Big(t\frac{d}{dt}\Big)^n \int_a^t \Big(\log\frac{t}{s}\Big)^{n-\alpha-1} \delta^n g(s) \frac{ds}{s}\,,$$

(ii): If $\alpha = n \in \mathbb{N}$, then $({}_H^c D_a^\alpha g)(t) = \delta^n g(t)$,

where in both cases, $[\mathrm{Re}(\alpha)]$ denotes the integer part of the real number $\mathrm{Re}(\alpha)$ and $\log(\cdot) = \log_e(\cdot)$.

Lemma 2.7. *Let $y \in AC_\delta^n[a,b]$ or $C_\delta^n[a,b]$ and $\alpha \in \mathbb{C}$. Then*

$$I_a^\alpha ({}_H^c D_a^\alpha y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \Big(\log\frac{t}{a}\Big)^k . \tag{5}$$

3. Main results

Let us start by defining what we mean by a solution of the problem (1)–(2).

Definition 3.1. A function $y \in AC_\delta(J,E)$ is said to be a solution of (1)–(2), if there exist a function $v \in L^1(J,E)$ with $v(t) \in F(t, y(t))$ for a.e. $t \in J$ such that ${}_H^c D^r y(t) = v(t)$ on J, and the function y satisfies condition (2).

To prove the existence of a solution to (1)–(2), we need the following auxiliary lemma

Lemma 3.2. *Let $h\colon J \to E$ be a continuous function. A function y is a solution of the fractional integral equation*

$$\begin{aligned} y(t) = {} & \frac{1}{\Gamma(r)} \int_1^t \Big(\log\frac{t}{s}\Big)^{r-1} h(s)\frac{ds}{s} \\ & - \frac{1}{a+b}\Big[\frac{b}{\Gamma(r)} \int_1^T \Big(\log\frac{T}{s}\Big)^{r-1} h(s)\frac{ds}{s} - c\Big]\,, \end{aligned} \tag{6}$$

if and only if y is a solution of the fractional boundary value problem,

$${}_H^c D^r y(t) = h(t)\,, \quad 0 < r \le 1\,, \tag{7}$$

$$a y(1) + b y(T) = c\,. \tag{8}$$

Proof. Assume y satisfies (7). Then Lemma 2.7 implies that

$$y(t) = {}_H I^r h(t) + y(1)\,.$$

The boundary condition (8) implies that

$$a y(1) + b y(T) = {}_H I^r h(t) + (a+b) y(1) = c\,,$$

and

$$y(1) = \frac{c}{a+b} - b \quad \frac{{}_H I^r h(t)}{a+b}\,.$$

Finally, we obtain the solution (6)

$$y(t) =_H I^r h(t) - \frac{b}{a+b} \ {}_H I^r h(t) + \frac{c}{a+b}\,.$$

Conversely, it is clear that if y satisfies equation (6), then equations (7)–(8) hold. □

Theorem 3.3. *Assume the following hypotheses hold:*

(H1) $F\colon J \times E \to P_{\mathrm{cp,c}}(E)$ *is a Carathéodory multi-valued map.*

(H2) *There exists a function* $p \in C(J,E)$ *such that*

$$\|F(t,u)\|_p := \sup\{|v| : v(t) \in F(t,y)\} \le p(t)\,,$$

for each $(t,y) \in J \times E$.

(H3) *There exists* $l > 0$ *such that*

$$H_d(F(t,x), F(t,\bar{x})) \le l|x - \bar{x}| \quad \text{for every} \quad x, \bar{x} \in E\,.$$

(H4) *For each bounded set* $B \subset C(J,E)$ *and for each* $t \in J$, *we have*

$$\alpha(F(t,B,) \le p(t)\alpha(B)\,,$$

where α *is a measure of noncompactness on* E.

(H5) *The function* $\phi = 0$ *is the unique solution in* $C(J,E)$ *satisfying*

$$\begin{aligned}\phi(t) \le 2\Big\{&\frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} \varphi(s,\phi(s))\frac{ds}{s} \\ &- \frac{1}{a+b}\Big[\frac{b}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} \varphi(s,\phi(s))\frac{ds}{s} - c\Big]\Big\}, \quad \text{for } t \in J\,.\end{aligned} \tag{9}$$

Then the BVP (1)–(2) *has at least one solution in* J.

Proof. First we transform problem (1)–(2) into a fixed point problem. Consider the multivalued operator

$$N(y) = \left\{ h \in AC_\delta(J,E) : \begin{array}{rcl} (Ny)(t) & = & \dfrac{1}{\Gamma(r)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} - \dfrac{1}{a+b} \\ & \times & \left[\dfrac{b}{\Gamma(r)}\displaystyle\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s} - c\right], \\ & & v \in S_{F,y} \end{array} \right\}.$$

Clearly, from Lemma 3.2, the fixed points of N are solutions to (1)–(2). We shall show that N satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in C(J,E)$.*

Indeed, if h_1, h_2 belong to $N(y)$, then there exist v_1, $v_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$\begin{aligned} h_i(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_i(s) \frac{ds}{s} \\ - \frac{1}{a+b}\Big[\frac{b}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_i(s) \frac{ds}{s} - c\Big], \quad i = 1, 2\,. \end{aligned}$$

Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned} \big(dh_1 + (1-d)h_2\big)(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} [dv_1(s) + (1-d)v_2(s)] \frac{ds}{s} \\ - \frac{1}{a+b}\Big[\frac{b}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} [dv_1(s) + (1-d)v_2(s)] \frac{ds}{s} - c\Big]\,. \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y)\,.$$

Step 2: *$N(M)$ is relatively compact for each compact $M \subset \overline{U}$.*

Let $M \subset \overline{U}$ be a compact set and let $\{h_n\}$ by any sequence of elements of $N(M)$. We show that $\{h_n\}$ has a convergent subsequence by using the Arzela-Ascoli criterion of compactness in $C(J, B)$. Since $\{h_n\} \in N(M)$, there exist $y_n \in M$ and $v_n \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s) \frac{ds}{s} \\ - \frac{1}{a+b}\Big[\frac{b}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_n(s) \frac{ds}{s} - c\Big], \end{aligned}$$

for $n \ge 1$. Using Theorem 2.2 and the properties of the Kuratowski measure of noncompactness, we have

$$\begin{aligned} \alpha(\{h_n(t)\}) \le 2\Bigg\{ \frac{1}{\Gamma(r)} \int_1^t \alpha\Big(\Big(\log \frac{t}{s}\Big)^{r-1} \frac{v_n(s)}{s} : n \ge 1\Big)\, ds \\ - \frac{1}{a+b}\Big[\frac{b}{\Gamma(r)} \int_1^T \alpha\Big(\Big(\log \frac{T}{s}\Big)^{r-1} \frac{v_n(s)}{s} : n \ge 1\Big)\, ds - c\Big]\Bigg\}. \end{aligned} \tag{10}$$

On the other hand, since $M(s)$ is compact in E, the set $\{v_n(s) : n \ge 1\}$ is compact. Consequently, $\alpha(\{v_n(s) : n \ge 1\}) = 0$ for a.e. $s \in J$.

Furthermore,

$$\alpha\left(\left\{\left(\log \frac{t}{s}\right)^{r-1} \frac{v_n(s)}{s}\right\}\right) = \left(\log \frac{t}{s}\right)^{r-1} \frac{1}{s} \alpha(\{v_n(s) : n \ge 1\}) = 0\,,$$

and

$$\alpha\left(\left\{\left(\log\frac{T}{s}\right)^{r-1}\frac{v_n(s)}{s}\right\}\right)=\left(\log\frac{T}{s}\right)^{r-1}\frac{1}{s}\alpha(\{v_n(s):n\geq 1\})=0\,,$$

for a.e. t, $s\in J$. Hence, from this and (10), $\{h_n(t):n\geq 1\}$ is relatively compact with respect to α for each $t\in J$. In addition, for each t_1, $t_2\in J$ with $t_1<t_2$, we have

$$\begin{aligned}|h_n(t_2)-h_n(t_1)| &= \left|\frac{1}{\Gamma(r)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{r-1}-\left(\log\frac{t_1}{s}\right)^{r-1}\right]v_n(s)\frac{ds}{s}\right.\\ &\qquad\left.+\frac{1}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\right|\\ &\leq \frac{p(t)}{\Gamma(r)}\int_1^{t_1}\left[\left(\log\frac{t_1}{s}\right)^{r-1}-\left(\log\frac{t_2}{s}\right)^{r-1}\right]\frac{ds}{s}\\ &\qquad+\frac{p(t)}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-1}\frac{ds}{s}\,.\end{aligned}$$

As $t_1\to t_2$, the right hand side of the above inequality tends to zero. This shows that $\{h_n : n\geq 1\}$ is equicontinuous. Consequently, $\{h_n : n\geq 1\}$ is relatively compact in $C(J,B)$.

Step 3: *The graph of N is closed.*

Let $y_n\to y_*$, $h_n\in N(y_n)$, and $h_n\to h_*$. We need to show that $h_*\in N(y_*)$. Now $h_n\in N(y_n)$ means that there exists $v_n\in S_{F,y_n}$ such that, for each $t\in J$,

$$\begin{aligned}h_n(t) &= \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\\ &\qquad-\frac{1}{a+b}\left[\frac{b}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1}v_n(s)\frac{ds}{s}-c\right],\quad v_n\in S^1_{F,y_n}\,.\end{aligned}$$

We must show that there exists $v_*\in S_{F,y_*}$ such that for each $t\in J$

$$\begin{aligned}h_*(t) &= \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}v_*(s)\frac{ds}{s}\\ &\qquad-\frac{1}{a+b}\left[\frac{b}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1}v_*(s)\frac{ds}{s}-c\right],\quad v_*\in S^1_{F,y_*}\,.\end{aligned}$$

Since $F(t,\cdot,\cdot)$ is upper semicontinuous, for every $\epsilon>0$, there exists $n_0(x)$ such that for every $n\geq n_0$, we have $v_n\in F(t,y(t),x(t))\subset F(t,y_*(t),x_*(t))+\epsilon B(0,1)$ a.e. $t\in J$. And since F has compact values, there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(.)\to v_*\quad\text{as}\quad m\to\infty\,,$$

$$v_* \in F\big(t, y_*(t)\big) \quad \text{as} \quad t \in J\,.$$

For every $w(t) \in F(t, y_*(t))$, we have

$$|v_{n_m} - v_*| \le |v_{n_m} - w(t)| + |w(t) - v_*|$$

and so

$$|v_{n_m} - v_*| \le d\big(v_{n_m}(t), F(t, y_*(t))\big)\,.$$

By an analogous relation obtained by interchanging the roles of v_{n_m} and v_*, it follows that

$$\begin{aligned} |v_{n_m} - v_*| &\le H_d(F(t, y_{n_m}(t), F(t, y_*(t))) \\ &\le l|y_{n_m} - y_*|\,. \end{aligned}$$

Therefore,

$$\begin{aligned} |h_n(t) - h_*(t)| \le{}& \frac{1}{\Gamma(r)} \int_1^t \Big(\log\frac{t}{s}\Big)^{r-1} l|v_{n_m} - v_*|\,ds \\ &+ \frac{1}{a+b}\Big[\frac{b}{\Gamma(r)} \int_1^t \Big(\log\frac{t}{s}\Big)^{r-1} l|v_{n_m} - v_*|\,ds\Big] \\ \le{}& \frac{\Big(1 + \frac{1}{a+b}\Big) l(\log T)^r}{\Gamma(r+1)} \|y_{n_m} - y_*\|_{L^1}\,. \end{aligned}$$

Hence

$$\|h_n(t) - h_*(t)\|_\infty \to 0 \quad \text{as} \quad m \to \infty\,.$$

Step 4: *M is relatively compact in $C(J, B)$.*

Suppose $M \subset \overline{U}$, $M \subset \text{conv}(0 \cup N(M))$, and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. Using an argument similar to the one used in Step 2 shows that $N(M)$ is equicontinuous. Then, since $M \subset \text{conv}(0 \cup N(M))$, we see that M is equicontinuous as well.

To apply the Arzela-Ascoli theorem, it remains to show that $M(t)$ is relatively compact in E for each $t \in J$. Since $C \subset M \subset \text{conv}(0 \cup N(M))$ and C is countable, we can find a countable set $H = \{h_n : n \ge 1\} \subset N(M)$ with $C \subset \text{conv}(0 \cup H)$.Then, there exist $y_n \in M$ and $v_n \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) ={}& \frac{1}{\Gamma(r)} \int_1^t \Big(\log\frac{t}{s}\Big)^{r-1} v_n(s)\frac{ds}{s} \\ &- \frac{1}{a+b}\Big[\frac{b}{\Gamma(r)} \int_1^T \Big(\log\frac{T}{s}\Big)^{r-1} v_n(s)\frac{ds}{s} - c\Big]\,. \end{aligned}$$

From $M \subset \overline{C} \subset \overline{\text{conv}}(0 \cup (H))$, and according, to Theorem 2.2, we have $\alpha(M(t)) \le \alpha(\overline{C}(t)) \le \alpha(H(t)) = \alpha(\{h_n(t) : n \ge 1\})$.

Using (10) and the fact that $v_n(s) \in M(s)$, we obtain

$$\alpha(M(t)) \leq 2\Bigg(\frac{1}{\Gamma(r)}\int_1^t \alpha\Big(\Big\{\Big(\log\frac{t}{s}\Big)^{r-1} v_n(s)\Big\}\Big)\frac{ds}{s}$$
$$-\frac{1}{a+b}\Big[\frac{b}{\Gamma(r)}\int_1^T \alpha\Big(\Big\{\Big(\log\frac{T}{s}\Big)^{r-1} v_n(s)\frac{ds}{s}\Big\}\Big) - c\Big]\Bigg).$$

Now, since $v_n(s) \in M(s)$, we have

$$\alpha(M(t)) \leq 2\Bigg(\frac{1}{\Gamma(r)}\int_1^t \alpha\Big(\Big\{\Big(\log\frac{t}{s}\Big)^{r-1}\frac{v_n(s)}{s} : n \geq 1\Big\}\Big)\, ds$$
$$-\frac{1}{a+b}\Big[\frac{b}{\Gamma(r)}\int_1^T \alpha\Big(\Big\{\Big(\log\frac{T}{s}\Big)^{r-1}\frac{v_n(s)}{s} : n \geq 1\Big\}\Big)\, ds - c\Big]\Bigg).$$

Also, since $v_n(s) \in M(s)$, we have

$$\alpha\Big(\Big\{\Big(\log\frac{t}{s}\Big)^{r-1}\frac{v_n(s)}{s}; n \geq 1\Big\}\Big) = \Big(\log\frac{t}{s}\Big)^{r-1}\frac{1}{s}\alpha(M(s)),$$

and

$$\alpha\Big(\Big\{\Big(\log\frac{T}{s}\Big)^{r-1}\frac{v_n(s)}{s}; n \geq 1\Big\}\Big) = \Big(\log\frac{T}{s}\Big)^{r-1}\frac{1}{s}\alpha(M(s)),$$

and it follows that

$$\alpha(M(t)) \leq 2\Bigg(\frac{1}{\Gamma(r)}\int_1^t \Big(\log\frac{t}{s}\Big)^{r-1}\alpha(M(s))\frac{ds}{s}$$
$$-\frac{1}{a+b}\Big[\frac{b}{\Gamma(r)}\int_1^T \Big(\log\frac{T}{s}\Big)^{r-1}\alpha(M(s))\frac{ds}{s}) - c\Big]\Bigg)$$
$$\leq 2\Bigg(\frac{1}{\Gamma(r)}\int_1^t \Big(\log\frac{t}{s}\Big)^{r-1}\psi(s, \alpha(M(s)))\frac{ds}{s}$$
$$-\frac{1}{a+b}\Big[\frac{b}{\Gamma(r)}\int_1^T \Big(\log\frac{T}{s}\Big)^{r-1}\psi(s, \alpha(M(s)))\frac{ds}{s} - c\Big]\Bigg).$$

Also, the function φ given by $\varphi(t) = \rho(M(t))$ belongs to $C(J, E)$. Consequently by $(H3)$, $\varphi = 0$; that is, $\rho(M(t)) = 0$ for all $t \in J$. Now, by the Arzela-Ascoli theorem, M is relatively compact in $C(J, E)$.

Step 5: *The apriori estimate.*

Let $h \in C(J, E)$ such that $y \in \lambda N(y)$ for some $0 < \lambda < 1$. Then

$$h(t) = \frac{1}{\Gamma(r)}\int_1^t \Big(\log\frac{t}{s}\Big)^{r-1} v(s)\frac{ds}{s}$$

$$- \frac{1}{a+b}\Big[\frac{b}{\Gamma(r)}\int_1^T \Big(\log\frac{T}{s}\Big)^{r-1} v(s)\frac{ds}{s} - c,\Big] \quad v \in S_{F,y}\,.$$

For each $t \in J$, we have

$$\begin{aligned}
\|N(y)\| &\le \frac{1}{\Gamma(r)}\int_1^t \Big(\log\frac{t}{s}\Big)^{r-1}|v(s)|\frac{ds}{s} \\
&\quad - \frac{1}{a+b}\Big[\frac{b}{\Gamma(r)}\int_1^T \Big(\log\frac{T}{s}\Big)^{r-1}|v(s)|\frac{ds}{s} - c\Big] \\
&\le \frac{(\log t)^r}{\Gamma(r+1)}\int_1^t p(s)\,ds \\
&\quad - \frac{1}{a+b}\Big[\frac{b(\log T)^r}{\Gamma(r+1)}\int_1^T p(s)ds - c\Big] \\
&\le \Big(1-\frac{b}{a+b}\Big)\frac{(\log T)^r}{\Gamma(r+1)}\int_1^T p(s)\,ds + \frac{c}{a+b}\,,
\end{aligned}$$

where

$$\|p\|_\infty \sup\big\{|p(t)| : t \in J\big\}\,.$$

Then

$$\|y\|\Bigg(\Big(1-\frac{b}{a+b}\Big)\frac{(\log T)^r}{\Gamma(r+1)}\int_1^T p(s)\,ds + \frac{c}{a+b}\Bigg) := R\,.$$

Hence the condition (4) is satisfied. As a consequence of Steps 1–5 and Theorem 2.3, we conclude that N has a fixed point $x \in C(J,E)$ which is a solution of problem 1–(2). This concludes the proof. □

3.1. **An example.** We conclude this paper with an example to illustrate our main result. Let

$$E = l^1 = \Big\{(y_1, y_2, \dots, y_n, \dots),\ \sum_1^\infty |y_n| < \infty\Big\}\,,$$

be our Banach space with norm

$$\|y\|_E = \sum_1^\infty |y_n|\,.$$

We apply Theorem 3.3 to the the following fractional differential inclusion,

$$ {}^c_H D^r y(t) \in F_n(t, y(t))\,, \quad \text{for a.e.} \quad t \in J = [1,e],\ \ 0 < r \le 1\,, \tag{11}$$

$$ ay(1) + by(e) = c\,, \tag{12}$$

where

$$F_n\big(t, y(t)\big) = \big\{v \in E : f_n(t, y(t)) \le v \le g_n\big(t, y(t)\big)\big\}\,,$$

and where f_n, $g_n\colon J \times E \times E \mapsto E$. We assume that for each $t \in [1,e]$, $f_n(t,\cdot,\cdot)$ is lower semi-continuous (i.e., the set $\{y \in E : f_n(t, y(t)) > \mu_1\}$ is open for each $\mu_1 \in$

$\mathbb{R}$), and assume that for each $t \in [1, e]$, $g_n(t, \cdot, \cdot)$ is upper semi-continuous (i.e., the set $\{y \in E : g_n(t, y(t)) < \mu_2\}$ is open for each $\mu_2 \in \mathbb{R}$), with $y = (y_1, y_2, \ldots, y_n, \ldots)$.

Set $F = (F_1, F_2, \ldots, F_n, \ldots)$, $f = (f_1, f_2, \ldots, f_n, \ldots)$, $g = (g_1, g_2, \ldots, g_n, \ldots)$. Assume that there exists $p \in C([1, e], \mathbb{R}^+)$ such that,

$$\begin{aligned} \|F(t,u)\|_{\mathcal{P}} &= \sup\big\{|v|, v(t) \in F\big(t, y(t)\big)\big\} \\ &= \max\big(|f_n(t, y(t))|, |g_n(t, y(t))|\big) \\ &\leq p(t)\,, \quad \text{for each } \ t \in [1, e], \ y \in E. \end{aligned}$$

It is clear that F is compact and convex-valued, and it is upper semi-continuous, and furthermore, we assume that for $(t, y) \in J \times E$. We also assume that for each bounded set $B \subset C(J, E)$ and for each $t \in J$, we have

$$\alpha(F(t, B) \leq p(t)\alpha(B)\,,$$

where α is a measure of noncompactness on E, and the function $\phi = 0$ is the unique solution in $C(J, E)$ of

$$\begin{aligned} \phi(t) \leq 2\Bigg\{ &\frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \varphi(s, \phi(s)) \frac{ds}{s} \\ &- \frac{1}{a+b} \left[\frac{b}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s}\right)^{r-1} \varphi(s, \phi(s)) \frac{ds}{s} - c\right] \Bigg\}, \ \text{ for } \ t \in J\,. \end{aligned}$$

Since all the conditions of Theorem 3.3 are satisfied, the problem (11)–(12) has at least one solution y on $[1, e]$.

References

[1] Adjabi, Y., Jarad, F., Baleanu, D., Abdeljawad, T., *On Cauchy problems with Caputo Hadamard fractional derivatives*, J. Comput. Anal. Appl. **21** (4) (2016), 661–681.

[2] Agarwal, R.P., Benchohra, M., Seba, D., *On the application of measure of noncom-pactness to the existence of solutions for fractional differential equations*, Results Math. **55** (2009), 221–230.

[3] Ahmad, B., Khan, R.A., Sivasundaram, S., *Generalized quasilinearization method for a first order differential equation with integral boundary condition*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **12** (2) (2005), 289–296.

[4] Ahmad, B., Ntouyas, S.K., *Initial value problems for hybrid Hadamard fractional equations*, Electron. J. Differential Equ. **2014** (161) (2014), 8 pp.

[5] Akhmerov, R.R., Kamenski, M.I., Patapov, A.S., Rodkina, A.E., Sadovski, B.N., *Measures of noncompactness and condensing operators* (translated from the 1986 russian original by a. iacop), operator theory: advances and applications, vol. 55, Birkhäuser Verlag, Basel, 1992.

[6] Aubin, J.P., Cellina, A., *Differential inclusions*, Springer-Verlag, Berlin-Heidelberg, New York, 1984.

[7] Aubin, J.P., Frankowska, H., *Set-valued analysis*, Birkhäuser, Boston, 1990.

[8] Banas, J., Goebel, K., *Measure of noncompactness in Banach spaces*, Lecture Notes in Pure and Appl. Math., Marcel Dekker, New York, 1980.

[9] Belarbi, A., Benchohra, M., *Existence results for nonlinear boundary-value problems with integral boundary conditions*, Electron. J. Differential Equ. **6** (2005), 1–10.

[10] Benchohra, M., Hamani, S., *Boundary value problems for differential inclusions with fractional order*, Discuss. Math. Differ. Incl. Control Optim. **28** (2008), 147–164.

[11] Benchohra, M., Hamani, S., *Nonlinear boundary value problems for differential inclusions with Caputo fractional derivative*, Topol. Methods Nonlinear Anal. **32** (1) (2008), 115–130.

[12] Benchohra, M., Henderson, J., Seba, D., *Measure of noncompactness and fractional differential equations in Banach spaces*, Commun. Appl. Anal. **12** (2008), 419–428.

[13] Benchohra, M., Henderson, J., Seba, D., *Measure of noncompactness and fractional and-hyperbolic partial fractional differential equations in Banach space*, PanAmer. Math. J. **20** (2010), 27–37.

[14] Benchohra, M., Henderson, J., Seba, D., *Boundary value problems for fractional differential inclusions in Banach space*, Fract. Differ. Calc. **2** (2012), 99–108.

[15] Benhamida, W., Hamani, S., *Measure of noncompactness and Caputo-Hadamard fractional differantial equations in Banach spaces*, Eur. Bull. Math. **1** (3) (2018), 98–103.

[16] Benhamida, W., Hamani, S., Henderson, J., *Boundary value problems for Caputo-Hadamard fractional differential equations*, Adv. Theor. Nonlinear Anal. Appl. **2** (3) (2018), 138–145.

[17] Brykalov, S.A., *A second order nonlinear problem with two-point and integral boundary conditions*, Georgian Math. J. **1** (1994), 243–249.

[18] Butzer, P.L., Kilbas, A.A., Trujillo, J.J., *Composition of Hadamard-type fractional integration operators and the semigroup property*, J. Math. Anal. Appl. **269** (2002), 387–400.

[19] Butzer, P.L., Kilbas, A.A., Trujillo, J.J., *Fractional calculus in the Mellin setting and Hadamard-type fractional integrals*, J. Math. Anal. Appl. **269** (2002), 1–27.

[20] Butzer, P.L., Kilbas, A.A., Trujillo, J.J., *Mellin transform analysis and integration by parts for Hadamard-type fractional integrals*, J. Math. Anal. Appl. **270** (2002), 1–15.

[21] Deimling, .K., *Multivalued differential equations*, Walter De Gruyter, Berlin-New York, 1992.

[22] Denche, M., Marhoune, A.L., *High order mixed-type differential equations with weighted integral boundary conditions*, Electron. J. Differential Equ. **2000** (60) (2000), 1–10.

[23] Diethelm, K., Freed, A.D., *On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity*, Scientifice Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties (Keil, F., Mackens, W., Voss, H., Werther, J., eds.), Springer-Verlag, Heidelberg, 1999, pp. 217–224.

[24] Gallardo, J.M., *Second order differential operators with integral boundary conditions and generation of semigroups*, Rocky Mountain J. Math. **30** (2000), 1265–1292.

[25] Gambo, Y.Y., Jarad, F., Baleanu, D., Abdeljawad, T., *On Caputo modification of the Hadamard fractional derivatives*, Adv. Difference Equ. **2014** (10) (2014), 12 pp.

[26] Gaul, L., Klein, P., Kempfle, S., *Damping description involving fractional operators*, Mech. Systems Signal Processing **5** (1991), 81–88.

[27] Glockle, W.G., Nonnenmacher, T.F., *A fractional calculus approach of self-similar protein dynamics*, Biophys. J. **68** (1995), 46–53.

[28] Hadamard, J., *Essai sur l'étude des fonctions donnees par leur development de Taylor*, J. Math. Pure Appl. **8** (1892), 101–186.

[29] Hilfer, R., *Applications of fractional calculus in physics*, World Scientific, Singapore, 2000.

[30] Jarad, F., Abdeljawad, T., Baleanu, D., *Caputo-type modification of the Hadamard fractional derivatives*, Adv. Difference Equ. **2012** (142) (2012), 8 pp.

[31] Karakostas, G.L., Tsamatos, P.Ch., *Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary value problems*, Electron. J. Differential Equ. **2002** (30) (2002), 1–17.

[32] Khan, R. A., *The generalized method of quasilinearization and nonlinear boundary value problems with integral boundary conditions*, Electron. J. Qual. Theory Differ. Equ. **2003 (10)** (2003), 1–15.

[33] Kilbas, A.A., *Hadamard-type fractional calculus*, J. Korean Math. Soc. **38** (6) (2001), 1191–1204.

[34] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and applications of fractional differential equations*, North-Holland Math. Studies, Elsevier Science B.V., Amsterdam, 2006.

[35] Klimek, M., *Sequential fractional differential equations with Hadamard derivative*, Commun. Nonlinear Sci. Numer. Simul. **16** (12) (2011), 4689–4697.

[36] Krall, A.M., *The adjoint of a differential operator with integral boundary conditions*, Proc. Amer. Math. Soc. **16** (1965), 738–742.

[37] Lomtatidze, A., Malaguti, L., *On a nonlocal boundary value problems for second order nonlinear singular differential equations*, Georgian Math. J. **7** (2000), 133–154.

[38] Mainardi, F., *Fractional calculus: Some basic problems in continuum and statistical mechanics*, Fractals and Fractional Calculus in Continuum Mechanics (Carpinteri, A., Mainardi, F., eds.), Springer-Verlag, Wien, 1997, pp. 291–348.

[39] Metzler, F., Schick, W., Kilian, H.G., Nonnenmacher, T.F., *Relaxation in filled polymers: A fractional calculus approach*, J. Chem. Phys. **103** (1995), 7180–7186.

[40] Miller, K.S., Ross, B., *An introduction to the fractional calculus and differential equations*, John Wiley, New York, 1993.

[41] Oldham, K.B., Spanier, J., *The fractional calculus*, Academic Press, New York, London, 1974.

[42] O'Regan, D., Precup, R., *Fixed point theorems for set-valued maps and existence prin-ciples for integral inclusions*, J. Math. Anal. Appl. **245** (2000), 594–612.

[43] Thiramanus, P., Ntouyas, S.K., Tariboon, J., *xistence and uniqueness results for Hadamard-type fractional differential equations with nonlocal fractional integral boundary conditions*, Abstr. Appl. Anal. **2014** (2014), 9 pp., Art. ID 902054.

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