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PACKING OF NON-BLOCKING FOUR-DIMENSIONAL CUBES INTO THE UNIT CUBE

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Abstract. Any collection of non-blocking four-dimensional cubes, whose total volume does not exceed 17/81, can be packed into the unit four-dimensional cube. This bound is tight for the parallel packing.

1. Introduction

Let C_n be a d-dimensional cube, for $n = 1, 2, \dots$. Moreover, let I^d be a d-dimensional cube of edges of length 1. We say that $C_1, C_2, \dots$ can be *packed* into I^d if it is possible to apply translations and rotations to the sets C_n so that the resulting translated and rotated cubes are contained in I^d and have mutually disjoint interiors. The packing is called *parallel* if each edge of any packed cube is parallel to an edge of I^d.

Meir and Moser in their seminal paper [6] showed that any family of d-dimensional cubes can be parallel packed into the unit d-dimensional cube I^d, provided that the total volume of the cubes is not greater than 2^{1-d}. Moreover, it is known that any family of d-dimensional cubes of total volume not greater than 2^{1-d} can be packed into I^d so that the uncovered part of I^d contains a cube of edge length $1 - \sqrt[d]{2}/2$ (see [5]). It is very likely that also any family of d-dimensional boxes with edge lengths not greater than 1 of total volume not greater than 2^{1-d} can be packed into I^d, but this conjecture has been confirmed only for $d = 2$ (see [1]).

Obviously, any two cubes whose sum of edge lengths is greater than 1 (and, consequently, whose total volume is greater than 2^{1-d}) cannot be parallel packed into I^d; after packing one of the cubes, there is no enough space in I^d to pack the other cube. Denote by a_n the edge length of C_n, for $n = 1, 2, \dots$. We say that the cubes $C_1, C_2, \dots$ are *non-blocking*, if $a_i + a_j \le 1$ for any $i \ne j$ (compare [2]). It is known that any collection of non-blocking squares, whose total area does not exceed 5/9, can be packed in I^2 (see [4]). Furthermore, in [3] it is shown that any collection of non-blocking three-dimensional cubes, whose total volume does not exceed 1/3, can be packed in I^3.

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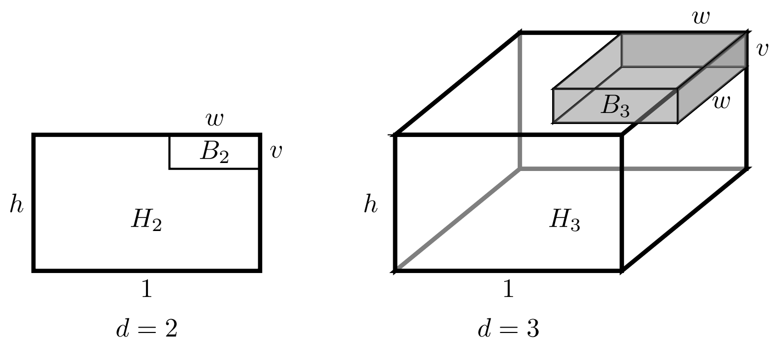

FIG. 1: H_d and B_d for $d = 2$ and $d = 3$

Conjecture 1. Any collection of non-blocking d-dimensional cubes can be parallel packed into I^d, provided that the sum of volumes of the cubes is not greater than $(2^d + 1)/3^d$.

Clearly, the upper bound $(2^d + 1)/3^d$ cannot be increased here: $2^d + 1$ cubes of edge lengths greater than $1/3$ cannot be parallel packed into I^d.

The aim of this note is to confirm this conjecture in dimension $d = 4$.

2. M_d^+-METHOD

We will use the packing method based on the methods described in [4], [3], [6] and [7]. Let $I^d = [0, 1]^d$, $H_d = [0, 1]^{d-1} \times [0, h]$ and $B_d = [1 - w, 1]^{d-1} \times [h - v, h]$, where $0 \le w < 1$ and $0 \le v < h$ (see Fig. 1). Moreover, let $\mathcal{C}$ be a collection of d-dimensional cubes $C_1, C_2, \ldots$. Assume that $a_n \ge a_{n+1}$ for $n = 1, 2, \ldots$, that $a_1 + w \le 1$ and that $a_1 + a_2 \le 1$, where a_n denotes the length of the edge of C_n. By $\mathrm{Int}B$ denote the interior of B.

The description is inductive with respect to d.

- For $d = 2$, the method M_2^+ (presented in [4]) is as follows. Squares $C_1, C_2, \ldots$ are packed into $H_2 \setminus \mathrm{Int}B_2$ in layers $L_1, L_2, \ldots$. The first layer is either the rectangle $[0, 1] \times [0, a_1]$ if $\big([0, 1] \times [0, a_1]\big) \cap \mathrm{Int}B_2 = \emptyset$ or the rectangle $[0, 1 - w] \times [0, a_1]$, otherwise (see Fig. 2). The squares $C_1, C_2, \ldots$ are packed into H_2 along the base of the first layer L_1 from left to right. If C_{n_1} is the first square that cannot be packed in that way, then the new layer L_2, of height a_{n_1}, is created directly above L_1. The base of L_2 is either equal to 1 if $\big([0, 1] \times [a_1, a_1 + a_{n_1}]\big) \cap \mathrm{Int}B_2 = \emptyset$ or equal to $1 - w$, otherwise. The squares $C_{n_1}, C_{n_1+1}, \ldots$ are packed into H_2 along the base of the second layer from left to right. If C_{n_2} is the first square that cannot be packed in that way in the second layer, then the new layer L_3, of height a_{n_2}, is created directly above the second layer. The base of L_3 is either equal to 1 if $\big([0, 1] \times [a_1 + a_{n_1}, a_1 + a_{n_1} + a_{n_2}]\big) \cap \mathrm{Int}B_2 = \emptyset$ or equal to $1 - w$, otherwise, etc.

- Assume that $d \geq 3$ and that the methods M_j^+ are described for $j = 2, 3, \ldots, d-1$. Cubes from $\mathcal{C}$ are packed into H_d in layers $L_1, L_2, \ldots$ similarly as in the method of Meir and Moser [6]. The base of each layer is a unit (d-1)-dimensional cube. The first layer is the box $[0,1]^{d-1} \times [0, a_1]$. The cubes are packed in L_1 so that the $(d-1)$-dimensional bottoms of the cubes are packed into the $(d-1)$-dimensional bottom of the layer according to the method M_{d-1}^+ (where $h = 1$, i.e., where $H_{d-1} = I^{d-1}$). If $\big([0,1]^{d-1} \times [0, a_1]\big) \cap \mathrm{Int}B_d = \emptyset$, then $B_{d-1} = \emptyset$ in the M_{d-1}^+-method, otherwise $B_{d-1} = [1-w, 1]^{d-1}$. If C_{n_1} is the first cube that cannot be packed in L_1, then the new layer L_2, of height a_{n_1}, is created directly above L_1. The cubes $C_{n_1}, C_{n_1+1}, \ldots$ are packed into L_2 so that the $(d-1)$-dimensional bottoms of the cubes are packed into the $(d-1)$-dimensional bottom of the layer according to the M_{d-1}^+-method. If $\big([0,1]^{d-1} \times [a_1, a_1 + a_{n_1}]\big) \cap \mathrm{Int}B_d = \emptyset$, then $B_{d-1} = \emptyset$ in the M_{d-1}^+-method, otherwise $B_{d-1} = [1-w, 1]^{d-1}$. If C_{n_2} is the first cube that cannot be packed in that way in the second layer, then the new layer L_3, of height a_{n_2}, is created directly above the second layer, etc. If t is an integer such that $a_1 + a_{n_1} + \ldots + a_t > h$, then we stop the packing process; there is no empty space in H_d to create a new layer to pack C_{n_t} (see Fig. 2, where $n_t = z$).

Lemma 1. *If C_z is the first cube from $\mathcal{C}$ that cannot be packed into $H_d \setminus \mathrm{Int}B_d$ by the M_d^+-method, then the total volume of cubes $C_1, C_2, \ldots, C_z$ plus the volume of B is greater than $a_1^d + (1-a_1)^{d-1}(h - a_1)$.*

Proof. The proof for $d = 2$ and for $d = 3$ is given in [4] and [3]. Assume that $d \geq 3$ and that the statements holds for each dimension $j = 2, 3, \ldots, d-1$.

The cubes from $\mathcal{C}$ are packed into $H_d \setminus \mathrm{Int}B_d$ by the M_d^+-method. Clearly, the volume of B_d is equal to $w^{d-1}v$ and the volume of the $(d-1)$-dimensional bottom of B equals w^{d-1}. Let $n_0 = 1$. Clearly, $z = n_t$, where t is the smallest integer such that $a_1 + a_{n_1} + \ldots + a_{n_t} > h$ (see Fig. 2).

Since C_{n_1} cannot be packed in the first layer L_1, by the inductive assumption we conclude that the sum of volumes of $(d-1)$-dimensional bottoms of cubes $C_1, C_2, \ldots, C_{n_1}$ is greater than $a_1^{d-1} + (1-a_1)^{d-1}$, provided that $L_1 \cap \mathrm{Int}B_d = \emptyset$ or greater than $a_1^{d-1} + (1-a_1)^{d-1} - w^{d-1}$, provided that $L_1 \cap \mathrm{Int}B_d \neq \emptyset$ (the volume of B_{d-1} equals w^{d-1}). Thus the sum of volumes of cubes in L_1 is greater than $a_1^d + \big[(1-a_1)^{d-1} - a_{n_1}^{d-1}\big] \cdot a_{n_1}$, provided that $L_1 \cap \mathrm{Int}B_d = \emptyset$ or greater than $a_1^d + \big[(1-a_1)^{d-1} - a_{n_1}^{d-1} - w^{d-1}\big] \cdot a_{n_1}$, provided that $L_1 \cap \mathrm{Int}B_d \neq \emptyset$.

Let k be the smallest integer such that

$$a_1 + a_{n_1} + \ldots + a_{n_{k-1}} + v > h$$

(see Fig. 2, where $k = 2$). If $j \in \{1, \ldots, k-1\}$, then the total volume of $(d-1)$-dimensional bottoms of cubes packed in L_j is greater than

$$a_{n_{j-1}}^{d-1} + (1 - a_{n_{j-1}})^{d-1} - a_{n_j}^{d-1}\,.$$

This means that the sum of volumes of cubes in L_j is greater than

$$a_{n_{j-1}}^{d} + \big[(1 - a_{n_{j-1}})^{d-1} - a_{n_j}^{d-1}\big] a_{n_j}\,.$$

If $j \in \{k, \dots, t\}$, then the total volume of $(d-1)$-dimensional bottoms of cubes packed in L_j is greater than

$$a_{n_{j-1}}^{d-1} + (1 - a_{n_{j-1}})^{d-1} - w^{d-1} - a_{n_j}^{d-1}.$$

This means that the sum of volumes of cubes in L_j is greater than

$$a_{n_{j-1}}^{d} + \left[(1 - a_{n_{j-1}})^{d-1} - w^{d-1} - a_{n_j}^{d-1}\right] a_{n_j}.$$

As a consequence, the sum of volumes of $C_1, \dots, C_z$ is greater than

$$\begin{aligned}
&a_1^d + \left[(1-a_1)^{d-1} - a_{n_1}^{d-1}\right] a_{n_1} \\
&\quad + a_{n_1}^d + \left[(1-a_{n_1})^{d-1} - a_{n_2}^{d-1}\right] a_{n_2} \\
&\quad + \ldots + a_{n_{k-2}}^d + \left[(1-a_{n_{k-2}})^{d-1} - a_{n_{k-1}}^{d-1}\right] a_{n_k-1} \\
&\quad + a_{n_{k-1}}^d + \left[(1-a_{n_{k-1}k})^{d-1} - w^{d-1} - a_{n_k}^{d-1}\right] a_{n_k} + \ldots \\
&\quad + a_{n_{t-1}}^d + \left[(1-a_{n_{t-1}})^{d-1} - w^{d-1} - a_{n_t}^{d-1}\right] a_{n_t} + a_{n_t}^d \\
&\geq a_1^d + (1-a_1)^{d-1}(a_{n_1} + \ldots + a_{n_t}) - w^{d-1}(a_{n_k} + \ldots + a_{n_{t-1}}).
\end{aligned}$$

Obviously,

$$a_1 + a_{n_1} + \ldots + a_{n_t} > h$$

as well as

$$a_{n_k} + \ldots + a_{n_{t-1}} < v$$

(see Fig. 2, where $k = 2$ and $t = 5$).

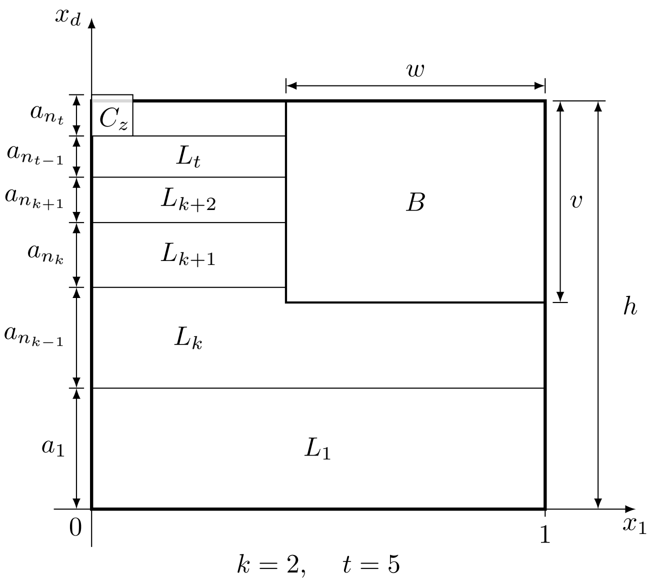

FIG. 2: Projection of H_d onto $x_1 x_d$ plane.

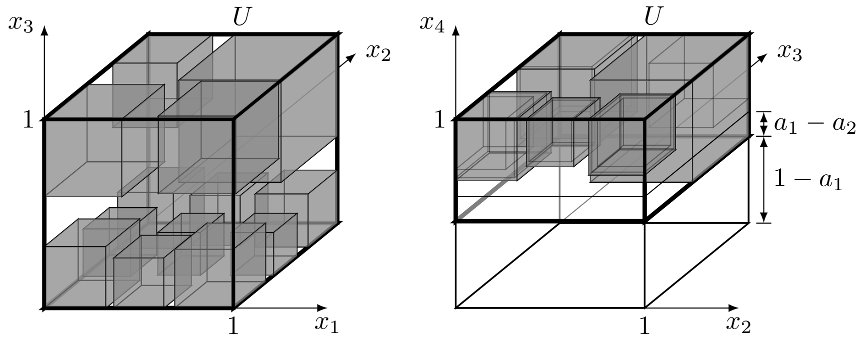

FIG. 3: Packing cubes $C_1, C_2, \ldots, C_{13}$ into U when $a_5 \leq 1/3$.

Thus the sum of volumes of $C_1, \ldots, C_z$ is greater than

$$a_1^d + (1 - a_1)^{d-1}(h - a_1) - w^{d-1}v.$$

□

3. Packing of non-blocking four-dimensional cubes into I^4

Let $\mathcal{C}$ be a collection of cubes $C_1, C_2, \ldots$. Assume that $a_1 + a_2 \leq 1$ and that $a_n \geq a_{n+1}$, where a_n denotes the edge length of C_n for $n = 1, 2 \ldots$.

Let

$$\begin{aligned} U &= [0,1] \times [0,1] \times [0,1] \times [1 - a_1, 1]\,, \\ H_4 &= [0,1] \times [0,1] \times [0,1] \times [0, 1 - a_2]\,, \\ B_4 &= [1 - a_1, 1] \times [1 - a_1, 1] \times [1 - a_1, 1] \times [1 - a_1, 1 - a_2]\,. \end{aligned}$$

Clearly, if $a_1 \neq a_2$, then B_4 is a box of size $a_1 \times a_1 \times a_1 \times (a_1 - a_2)$.

- The first cube C_1 is packed into U at the vertex $(1, 1, 1, 1)$.
- The cubes C_2, C_3, C_4 are packed into U at the vertices of $(0, 0, 1, 1)$, $(1, 0, 1, 1)$ and $(0, 1, 1, 1)$, respectively.
- The cubes $C_5, C_6, C_, C_8$ are packed into U at the vertices of $(0, 0, 0, 1)$, $(1, 0, 0, 1)$, $(0, 1, 0, 1)$ and $(1, 1, 0, 1)$, respectively.
- If $a_5 > 1/3$, then no more cube will be packed into $U \setminus H_4$.
- If $a_5 \leq 1/3$, then $C_9, \ldots, C_{13}$ are packed into $U \setminus \mathrm{Int} H_4$ (see Fig. 3, a detailed description can be found in Appendix A).
- The remaining cubes are packed into $H_4 \setminus \mathrm{Int} B_4$ in corresponding layers L_i $(i = 1, 2, \ldots)$ by the M_4^+-method (see Fig. 2 and Fig. 4).

Theorem 2. *Any collection of non-blocking four-dimensional cubes with total volume not greater than* $17/81$ *can be packed into* I^4.

Proof. Denote by $C_1, C_2, \ldots$ the cubes in the collection. Without loss of generality we can assume that $a_1 \geq a_2 \geq \ldots$, where a_n is the edge length of C_n, for

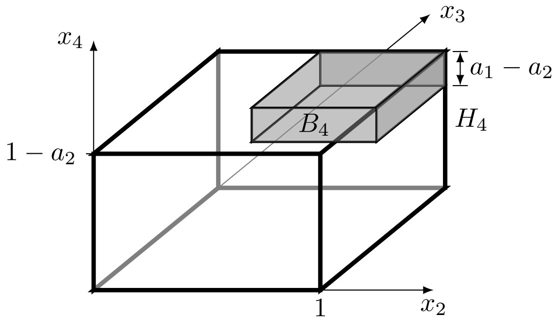

FIG. 4: Three-dimensional top of H_4.

$n = 1, 2, \ldots$. We will show that if the cubes cannot be packed into I^4, then $a_1^4 + a_2^4 + \ldots > 17/81$, which is a contradiction.

Consider three cases.

Case 1: $a_9 > 1/3$.

Eight first cubes were packed at the top of I^4 and the remaining cubes $C_9, C_{10}, \ldots$ were packed into H_4. The volume of B_4 is equal to $a_1^3(a_1 - a_2)$. If the cubes cannot be packed into I^4, then, by Lemma 1 (for $d = 4$ and $h = 1 - a_2$), the sum of volumes of the cubes is greater than

$$\begin{aligned}&(a_1^4 + a_2^4 + \cdots + a_8^4) + a_9^4 + (1 - a_9)^3(1 - a_2 - a_9) - a_1^3(a_1 - a_2)\\ &= a_1^3 a_2 + a_2^4 + \cdots + a_9^4 + (1 - a_9)^3(1 - a_2 - a_9)\\ &\geq 2a_2^4 + 7a_9^4 + (1 - a_2 - a_9)(1 - a_9)^3.\end{aligned}$$

Consider the function

$$\varphi(a_2, a_9) = 2a_2^4 + 7a_9^4 + (1 - a_2 - a_9)(1 - a_9)^3$$

in the domain given by inequalities $1/3 < a_9 \leq a_2 \leq 1/2$.

Since $\varphi'_{a_2}(a_2, a_9) = 8a_2^3 - (1 - a_9)^3 \geq 8a_9^3 - (1 - a_9)^3 > 0$ for $1/3 < a_9 \leq a_2$, we get

$$\varphi(a_2, a_9) \geq \varphi(a_9, a_9) = 9a_9^4 + (1 - 2a_9)(1 - a_9)^3\,.$$

Let $\varphi_1(a_9) = 9a_9^4 + (1 - 2a_9)(1 - a_9)^3$. Since

$$\varphi_1'(a_9) = 44a_9^3 - 21a_9^2 + 18a_9 - 5$$

and

$$\varphi_1''(a_9) = 132a_9^2 - 42a_9 + 18 > 0$$

it follows that

$$\varphi_1'(a_9) > \varphi_1'(1/3) > 0$$

for $a_9 > 1/3$ and that

$$\varphi(a_2, a_9) > \varphi_1(1/3) = 17/81\,.$$

Case 2: $a_9 \leq 1/3$ *and* $a_5 > 1/3$.

Eight first cubes were packed at the top of I^4 and the remaining cubes $C_9, C_{10}, \dots$ were packed into H_4. If the cubes cannot be packed into I^4, then, by Lemma 1, the sum of volumes of the cubes is greater than

$$\begin{aligned}
&a_1^4 + a_2^4 + \cdots + a_8^4 + a_9^4 + (1-a_9)^3(1-a_2-a_9) - a_1^3(a_1-a_2) \\
&= a_1^3 a_2 + a_2^4 + (a_3^4 + a_4^4 + a_5^4) + (a_6^4 + a_7^4 + a_8^4 + a_9^4) + (1-a_9)^3(1-a_2-a_9) \\
&\geq 2a_2^4 + 3 \cdot \big(\frac{1}{3}\big)^4 + 4a_9^4 + (1-a_2-a_9)(1-a_9)^3 .
\end{aligned}$$

The function

$$2x^4 + 3 \cdot \left(\frac{1}{3}\right)^4 + 4y^4 + (1-x-y)(1-y)^3$$

reaches values not smaller than 17/81 in the domain D given by the inequalities $0 < y \leq 1/3 < x \leq 1/2$ (see Appendix B)

Case 3: $a_5 \leq 1/3$.

Thirteen cubes were packed at the top of I^4.

If the cubes cannot be packed into I^4, then, by Lemma 1, the sum of volumes of the cubes is greater than

$$\begin{aligned}
&(a_1^4 + a_2^4 + \cdots + a_{13}^4) + a_{14}^4 + (1-a_{14})^3(1-a_2-a_{14}) - a_1^3(a_1-a_2) \\
&= a_1^3 a_2 + a_2^4 + \cdots + a_{14}^4 + (1-a_{14})^3(1-a_2-a_{14}) \\
&\geq 2a_2^4 + 12a_{14}^4 + (1-a_2-a_{14})(1-a_{14})^3 .
\end{aligned}$$

The function $2x^4 + 12y^4 + (1-x-y)(1-y)^3$ reaches values not smaller than 17/81 for $0 < y < 1/3$ and $y \leq x \leq 1/2$ (see Appendix C). □

Appendix A.

The first eight cubes are packed in the following places:

$[1-a_1, 1] \times [1-a_1, 1] \times [1-a_1, 1] \times [1-a_1, 1]$,
$[0, a_2] \times [0, a_2] \times [1-a_2, 1] \times [1-a_2, 1]$,
$[1-a_3, 1] \times [0, a_3] \times [1-a_3, 1] \times [1-a_3, 1]$,
$[0, a_4] \times [1-a_4, 1] \times [1-a_4, 1] \times [1-a_4, 1]$,
$[0, a_5] \times [0, a_5] \times [0, a_5] \times [1-a_5, 1]$,
$[1-a_6, 1] \times [0, a_6] \times [0, a_6] \times [1-a_6, 1]$,
$[0, a_7] \times [1-a_7, 1] \times [0, a_7] \times [1-a_7, 1]$,
$[1-a_8, 1] \times [1-a_8, 1] \times [0, a_8] \times [1-a_8, 1]$.

If $a_5 \leq 1/3$, then there is enough empty space between C_5, C_6, C_7 and C_8 to pack the next five cubes, for example in the following places:

$[\frac{1}{2} - \frac{1}{2}a_9, \frac{1}{2} + \frac{1}{2}a_9] \times [0, a_9] \times [0, a_9] \times [1-a_9, 1]$,
$[0, a_{10}] \times [\frac{1}{2} - \frac{1}{2}a_{10}, \frac{1}{2} + \frac{1}{2}a_{10}] \times [0, a_{10}] \times [1-a_{10}, 1]$,
$[\frac{1}{2} - \frac{1}{2}a_{11}, \frac{1}{2} + \frac{1}{2}a_{11}] \times [1-a_{11}, 1] \times [0, a_{11}] \times [1-a_{11}, 1]$,
$[1-a_{12}, 1] \times [\frac{1}{2} - \frac{1}{2}a_{12}, \frac{1}{2} + \frac{1}{2}a_{12}] \times [0, a_{12}] \times [1-a_{12}, 1]$,
$[\frac{1}{2} - \frac{1}{2}a_{13}, \frac{1}{2} + \frac{1}{2}a_{13}] \times [\frac{1}{2} - \frac{1}{2}a_{13}, \frac{1}{2} + \frac{1}{2}a_{13}] \times [0, a_{13}] \times [1-a_{13}, 1]$.

Appendix B.

We find the global minimum of the function

$$f(x,y) = 2x^4 + 3\cdot\left(\frac{1}{3}\right)^4 + 2y^4 + (1-x-y)(1-y)^3$$

in the domain D_f given by the following inequalities:

$$\begin{cases} \frac{1}{3} \le x \le \frac{1}{2} \\ 0 \le y \le \frac{1}{3} \end{cases}.$$

Since $f'_x(x,y) = 8x^3 - (1-y)^3$, the equation $f'_x(x,y) = 0$ implies that $x = \frac{1}{2} - \frac{1}{2}y$. Moreover $f'_y(x,y) = 16y^3 - (1-y)^3 - 3(1-y)^2(1-x-y)$. Hence

$$f'_y\left(\frac{1}{2} - \frac{1}{2}y, y\right) = 16y^3 - (1-y)^3 - 3(1-y)^2(1 - \frac{1}{2} + \frac{1}{2}y - y) = 16y^3 - \frac{5}{2}(1-y)^3 = 0$$

at $y_0 = \frac{5}{2\sqrt[3]{100}+5} > \frac{1}{3}$. Thus there is no stationary point in D_f.

The boundary of the rectangle D_f consists of four segments.

- The segment $y = 1/3$ with $1/3 \le x \le 1/2$. The function
$$f_1(x) = f\left(x, \frac{1}{3}\right) = 2x^4 - \frac{8}{27}x + \frac{23}{81}$$
for $x \in [1/3, 1/2]$ reaches its lowest value $17/81$ at $x = 1/3$.
- The segment $x = 1/3$ with $0 \le y \le 1/3$. Consider the function
$$f_2(y) = f\left(\frac{1}{3}, y\right) = 5y^4 - \frac{11}{3}y^3 + 5y^2 - 3y + \frac{59}{81}$$
for $y \in [0, 1/3]$. Since
$$f'_2(y) = 20y^3 - 11y^2 + 10y - 3$$
and
$$f''_2(y) = 60y^2 - 22y + 10 > 0\,,$$
it follows that
$$f'_2(y) \le f'_2\left(\frac{1}{3}\right) = -\frac{4}{27} < 0$$
for $y \le 1/3$, i.e., the function f_2 is decreasing in interval $[0, 1/3]$. Thus $f_2(y) \ge f_2(1/3) = 17/81$ for $0 \le y \le 1/3$.
- The segment $x = 1/2$ with $0 \le y \le 1/3$. Consider the function
$$f_3(y) = f\left(\frac{1}{2}, y\right) = 5y^4 - \frac{9}{2}y^3 + \frac{9}{2}y^2 - \frac{5}{2}y + \frac{143}{216}\,.$$
for $y \in [0, 1/3]$. Since
$$f'_3(y) = 20y^3 - \frac{27}{2}y^2 + 9y - \frac{5}{2}$$
and
$$f''_2(y) = 60y^2 - 27y + 9 > 0$$
it follows that
$$f'_3(y) \le f'_3(1/3) = -\frac{7}{27} < 0$$

for $y \le 1/3$, i.e., the function f_3 is decreasing for $y \in [0, 1/3]$. Thus $f_3(y) \ge f_3(1/3) = 145/648 > 17/81$ for $0 \le y \le 1/3$.

- The segment $y = 0$ with $1/3 \le x \le 1/2$. The function

$$f_4(x) = f(x, 0) = 2x^4 - x + 1 + \frac{1}{27} > 2\Big(\frac{1}{3}\Big)^4 - \frac{1}{2} + \frac{28}{27} = \frac{91}{162} > \frac{17}{81}\,.$$

Appendix C.

We will show that the global minimum of the function

$$g(x, y) = 2x^4 + 12y^4 + (1 - x - y)(1 - y)^3$$

in the extended (for simplicity of calculations) domain D_g given by the following inequalities:

$$\begin{cases} 0 \le x \le \frac{1}{2} \\ 0 \le y \le \frac{1}{3} \end{cases}$$

is greater than 17/81. Since $g'_x = 8x^3 - (1 - y)^3$, the equation $g'_x(x, y) = 0$ implies that $x = \frac{1}{2} - \frac{1}{2}y$. Moreover $g'_y(x, y) = 48y^3 - (1 - y)^3 - 3(1 - y)^2(1 - x - y)$. Hence

$$g'_y\Big(\frac{1}{2} - \frac{1}{2}y, y\Big) = 48y^3 - \frac{5}{2}(1 - y)^3 = 0$$

at $y_0 = \frac{5}{2\sqrt[3]{300}+5} < \frac{1}{3}$. Then $x_0 = \frac{\sqrt[3]{300}}{2\sqrt[3]{300}+5} \in [1/3, 1/2]$ and

$$g(x_0, y_0) = \frac{1500}{(2\sqrt[3]{300} + 5)^3} \approx 0.2412 > \frac{17}{81}\,.$$

The boundary of the rectangle D_g consists of four segments.

- The segment $y = 1/3$ with $0 \le x \le 1/2$. The function

$$g_1(x) = g\Big(x, \frac{1}{3}\Big) = 2x^4 - \frac{8}{27}x + \frac{28}{81}$$

for $x \in [0, 1/2]$ reaches its lowest value 22/81 at $x = 1/3$.

- The segment $x = 0$ with $0 \le y \le 1/3$. The function

$$g_2(y) = g\Big(0, y\Big) = 12y^4 + (1 - y)^4$$

for $y \in [0, 1/3]$ reaches its lowest value $12/(1 + \sqrt[3]{12})^3 \approx 0.3371 > 17/81$ at $y = 1/(1 + \sqrt[3]{12})$.

- The segment $x = 1/2$ with $0 \le y \le 1/3$. Consider the function

$$g_3(y) = g\Big(\frac{1}{2}, y\Big) = 2\Big(\frac{1}{2}\Big)^4 + 12y^4 + \Big(\frac{1}{2} - y\Big)(1 - y)^3$$

for $y \in [0, 1/3]$. Since $y \le 1/3$, we get

$$g_3(y) \ge \frac{1}{4} + 12y^4 + \Big(\frac{1}{2} - y\Big)\Big(1 - \frac{1}{3}\Big)^3 = 12y^4 - \frac{8}{27}y + \frac{59}{216}$$

Let $\psi(y) = 12y^4 - \frac{8}{27}y + \frac{59}{216}$. Since ψ for $y \in [0, 1/3]$ reaches its lowest value $(177 - 8\sqrt[3]{36})/648 \approx 0.23238 > 17/81$ at $x = 1/(3\sqrt[3]{6})$, it follows that $g_3(y) > 17/81$.

- The segment $y = 0$ with $0 \leq x \leq 1/2$. The function
$$g_4(x) = g(x, 0) = 2x^4 - x + 1 > -\frac{1}{2} + 1 = \frac{1}{2} > \frac{17}{81}\,.$$

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