A note on centralizers in q-deformed Heisenberg algebras

Volodymyr Mazorchuk

Max-Planck-Institut für Mathematik, Bonn, Germany[*] [†]

Abstract

We reprove and generalize several results (including the main one) from the recent monograph [3] using the technique of generalized Weyl algebras.

1 Introduction

Let $\mathfrak{k}$ be a field, J a set and $\mathbf{q} = (q_i)_{i\in J} \in \mathfrak{k}^J$. The authors of [3] define the *q-deformed Heisenberg algebra* as an associative unital $\mathfrak{k}$-algebra, $\mathcal{H}(\mathbf{q}, J)$, generated over $\mathfrak{k}$ by $\{X_i, Y_i | i \in J\}$ subject to the following relations: $[A_i, A_j] = [B_i, B_j] = [A_i, B_j] = 0$, $i \neq j$; $A_iB_i - q_iB_iA_i = 1$, $i \in J$. Approx. 400 papers, where $\mathcal{H}(\mathbf{q}, J)$, its properties, generalizations and several physical applications were studied, are cited in [3] and we refer the reader to [3] for these details.

One of the principal theorems in [3] states that, for $|J| = 1$ and q not a root of unity, any two commuting elements in $\mathcal{H}(q) = \mathcal{H}(\mathbf{q}, J)$ are algebraically dependent ([3, Theorem 7.4]).

The aim of this note is to show (in Section 2) how one can quickly obtain this result and even generalize it to a wider class of algebras, if one realizes that q-deformed Heisenberg algebras belong to the class of generalized Weyl algebras (GWAs), introduced by V.Bavula in the late 80's. There are several advantages of this approach. First of all, this drastically simplifies the proof and avoids lengthy calculations. Then, next to a generalization of [3, Theorem 7.4] we get a generalization of another central result [3, Theorem 6.6], where the centralizer of an element in $\mathcal{H}(q)$ is described. We also get some additional information, e.g. the commutativity of the centralizer, which appears to be new. In Section 3 we consider the root of unity case, in which GWAs have large centers. In this case we obtain a generalization of [3, Theorem 7.5] and [3, Corollary 6.12]. Finally, in Section 4 we use highest weight modules over GWAs to construct their realizations by difference operators acting on a polynomial ring. This generalizes results from [3, Chapter 8].

Our arguments in the proof of Theorem 1 are very close to those of [2], but, formally, V.Bavula considers a slightly different class of algebras (e.g. $\mathfrak{k}$ is supposed to be of characteristic zero) and one has to insert a small preliminary step to be able to transfer his proof to the case we consider here.

Generalized Weyl algebras are associated with a ring, R, central elements $0 \neq t_i$, $i \in J$, and pairwise commuting automorphisms σ_i, $i \in J$, of R such that $\sigma_i(t_j) = t_j$, $i \neq j$.

[*]on leave from: Chalmers Tekniska Högskola och Göteborgs Universitet, Göteborg, Sweden, e-mail: volody@math.chalmers.se

[†]Junior Associate of the ICTP

The corresponding *generalized Weyl algebra* $A(R, \{t_i\}, \{\sigma_i\})$ is defined as a ring, obtained by adjoining to R symbols $\{X_i, Y_i | i \in J\}$ which satisfy the following relations: $Y_i X_i = t_i$, $X_i Y_i = \sigma_i(t_i)$, $i \in J$; $X_i a = \sigma_i(a) X_i$, $a Y_i = Y_i \sigma_i(a)$, $i \in J$, $a \in R$; $[X_i, Y_j] = [X_i, X_j] = [Y_i, Y_j] = 0$, $i \neq j$. This algebra possesses a natural $\mathbb{Z}^J$-gradation. To get $\mathcal{H}(\mathbf{q}, J)$ one should take $R = \mathfrak{k}[h_i, i \in J]$, $t_i = h_i$, $i \in J$, and σ_i defined by $\sigma_i(h_j) = h_j$, $i \neq j$ and $\sigma_i(h_i) = q_i h_i + 1$ (this formally works only for $q_i \neq 0$ as σ_i would not be an automorphism otherwise, however, the definition of GWA can be extended to the case when σ_i are endomorphisms, moreover, this does not affect our applications to results in [3], because $q_i \neq 0$ is assumed there).

2 Main result

Consider the GWA $A = A(R, t, \sigma)$, where $R = \mathfrak{k}[H]$, $t \in R \setminus \mathfrak{k}$ and $\sigma(H) = qH + 1$ for $q \neq 0$ not a root of unity from $\mathfrak{k}$ (in particular, in this case $\mathfrak{k}$ is infinite). As $|J| = 1$ we will write simply X and Y for X_i, Y_i. A is an integral domain and $\mathbb{Z}$-graded with $A_0 = R$, $A_i = RX^i$ and $A_{-i} = RY^i$, $i \in \mathbb{N}$. We set $A_\pm = \oplus_{i \in \mathbb{N}} A_{\pm i}$.

Theorem 1. *The centralizer $C(f)$ of any non-scalar element $f \in A$ is a commutative algebra and a free $\mathfrak{k}[f]$-module of finite rank r. Moreover, r divides both the maximal degree $\pi_+(f)$ and the minimal degree $\pi_-(f)$ in the graded decomposition of f.*

This is a generalization of [2, Theorem 7] and Amitsur's theorem on centralizers in Weyl algebra ([1]). Our proof follows closely [2, Chapter 7] (where the case $q = 1$ and $\mathrm{char}(\mathfrak{k}) = 0$ was considered) with some differences on the first stage caused by a different choice of σ. But before presenting it we give two immediate corollaries of Theorem 1:

Corollary 1. *Two commuting elements of A, in particular, of $\mathcal{H}(q)$, are algebraically dependent.*

Corollary 2. *If $f \in A$ is such that $\pi_+(f)$ and $\pi_-(f)$ are relatively prime then $C(f) = \mathfrak{k}[f]$.*

Proof of Theorem 1. **Step 1.** Let $\mathbb{Z}$ act on $\mathfrak{k}$ via $1(x) = qx + 1$. We claim that the only finite orbit of this action is $\{(1-q)^{-1}\}$.

Indeed, $n(x) = q^n x + (q^n - 1)/(q-1)$ and $n(x) = x$ implies $x = (q^n - 1)/((q-1)(1-q^n)) = (1-q)^{-1}$.

Step 2. σ extends to an automorphism of $\mathfrak{k}(H)$. By abuse of notation we will denote this extension by σ as well. We claim that $\sigma^n(p) = p$ for some $p \in \mathfrak{k}(H)$ and some $n \in \mathbb{N}$ implies $p \in \mathfrak{k}$.

Indeed, let $p = q_1(H)/q_2(H)$. Adding to $\mathfrak{k}$ all roots of q_1 and q_2 if necessary, we may assume that $p = \alpha \dfrac{(H - \alpha_1) \dots (H - \alpha_i)}{(H - \beta_1) \dots (H - \beta_j)}$. As $\mathfrak{k}$ is infinite, $\sigma(p) = p$ implies that the multisets $\{\alpha_1, \dots, \alpha_i\}$ and $\{\beta_1, \dots, \beta_j\}$ are stable under the $\mathbb{Z}$-action from Step 1. As they are finite we get that the only possibility is $\alpha_s = \beta_s = c = (1-q)^{-1}$ and hence $p = \alpha(H - c)^l$, $l \in \mathbb{Z}$. Now, as q is not a root of unity, we compare the leading coefficients in p and $\sigma(p)$ and conclude that $l = 0$.

The referee has pointed out that there is a shorter way to get the above result using the division algorithm and comparing the leading coefficients. However, we decided to keep the above proof as we will use the description of finite $\mathbb{Z}$-orbits later in the proof of Theorem 3.

From now on, Bavula's proof formally works without any change, but we repeat it for convenience of the reader.

Step 3. Let $g \in A$, $\pi_+(g) = n > 0$ and $g_1, g_2 \in C(g)$, $m = \pi_+(g_1) = \pi_+(g_2) \geq 0$, such that m-th graded terms of g_1 and g_2 are equal $b_1 X^m$ and $b_2 X^m$, $b_1, b_2 \in R$, respectively. Then b_1 and b_2 are linearly dependent.

Let bX^n be the highest term of g, $0 \neq b \in R$. From $[g, g_i] = 0$ we get $b\sigma^n(b_i) = \sigma^m(b)b_i$, $i = 1, 2$. Hence $\sigma^n(b_1/b_2) = b_1/b_2$ and the statement follows from Step 2.

Step 4. Denote $m = \pi_-(f)$ and $n = \pi_+(f)$. If $m = n = 0$ then $f \in R$ and $C(f) = R$ follows from the graded decomposition of A. Clearly, R is a free $\mathfrak{k}[f]$-module of rank $\deg_R(f)$ – the degree of the polynomial f.

Assume $n > 0$ (the case $m < 0$ is analogous). Then $C(f) \cap A_- = \varnothing$. Otherwise there exists $g \in C(f) \cap A_-$ of largest possible degree $\pi_+(g) = -k < 0$. Then $\pi_+(g^{ni} f^{2ki}) = \pi_+(f^{ki}) = nki \geq 0$ for all $i \in \mathbb{N}$ but, as $t \notin \mathfrak{k}$, the $\mathfrak{k}[H]$-coefficients of X^{nki} in the graded decomposition of $g^{ni} f^{2ki}$ and f^{ki} have different degrees for sufficiently large i, which contradicts Step 3.

Let κ be the composition of π_+ with $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Then $G = \kappa(C(f) \setminus \{0\})$ is a cyclic group of order r, which divides n. Let $G = \{m_1 = 0, \ldots, m_r\}$. For each m_i choose $g_i \in C(f)$ such that $\kappa(g_i) = m_i$ and the number $\pi_+(g_i)$ be the smallest possible. From $C(f) \cap A_- = \varnothing$ and Step 3 it follows that such g_i do exist and their highest terms are unique up to non-zero scalars. In particular, we can set $g_1 = 1$. Assume $\sum_i g_i \varphi_i = 0$ for some $\varphi_i \in \mathfrak{k}[H]$ and not all φ_i are zero. Then there should exist i, j such that $\pi_+(g_i \varphi_i) = \pi_+(g_j \varphi_j)$. But then $\kappa(g_i) = \kappa(g_j)$ and we obtain a contradiction. Thus the right $K[f]$-module M, generated by $\{g_i\}$, is free.

Step 5. Now we claim that $C(f) = M$, in fact, we need $C(f) \subset M$. If $g \in C(f)$ and $\pi_+(g) = 0$ then Step 3 and $C(f) \cap A_- = \varnothing$ imply $g \in \mathfrak{k} = g_1\mathfrak{k}$. If $\pi_+(g) = k > 0$, then there exists i such that $\kappa(g) = \kappa(g_i)$ and $\pi_+(g_i) \leq k$. Hence $k = \pi_+(g_i f^s)$ for some $s \in \mathbb{Z}_+$. Applying Step 3 one more time we get $\lambda \in \mathfrak{k}$ such that $\pi_+(g - \lambda g_i f^s) < k$ and the proof is completed by induction on k.

Step 6. Finally, we claim that $C(f)$ is commutative. Choose $g \in C(f)$ such that $\kappa(g)$ is a generator of G. Denote by $E \subset C(f)$ the commutative subalgebra, generated by f and g. By Step 3, Step 4 and the induction on the degree of elements, the $\mathfrak{k}[f]$-module $C(f)/E$ is finite-dimensional, hence for any $u \in C(f)$ there is $0 \neq P \in \mathfrak{k}[f]$ such that $Pu \in E$. Let $v \in C(f)$ be arbitrary and $Qv \in E$ for some $0 \neq Q \in \mathfrak{k}[f]$. Then $PQuv = (Pu)(Qv) = (Qv)(Pu) = PQvu$. Since A is an integral domain, $uv = vu$ and the proof is complete. □

In the same way as in [2], Theorem 1 immediately implies the following.

Corollary 3. *1. Any maximal commutative subalgebra of A has the form $C(f)$ for some non-scalar element $f \in A$.*

2. *If $f, g \in A$ commute then $C(f) = C(g)$.*

3. *If C is a maximal commutative subalgebra of A and $f \in A$ such that $p(f) \in C$ for some $p(f) \in \mathfrak{k}[f]$ then $f \in C$.*

4. *The intersection of two distinct maximal commutative subalgebras of A is $\mathfrak{k}$.*

5. *The center of A equals $\mathfrak{k}$.*

3 Root of unity case

Assume now that $q^l = 1$, $l \in \mathbb{N} \setminus \{1\}$, and $q^i \neq 1$, $i = 1, \ldots, l-1$ or $q = 1$ and $\mathrm{char}(\mathfrak{k}) = l > 0$. Then the center of the algebra A is quite big and can be completely described. Set $R \ni F = \prod_{i=0}^{l-1} \sigma(H)$ and $W = \langle \sigma \rangle$. The next Theorem is a generalization of [3, Corollary 6.12].

Theorem 2. *The center $Z(A)$ of A equals $B = \langle F, X^l, Y^l \rangle = \langle X^l, Y^l \rangle$ and A is a finitely-generated $Z(A)$-module.*

Proof. **Step 1.** $B = \langle F, X^l, Y^l \rangle = \langle X^l, Y^l \rangle \subset Z(A)$.

If we put $n = l$ into the formula for $n(x)$ in Step 1 of Theorem 1, we get $l(x) = x$ and hence $\sigma^l(f) = f$ for any $f \in R$. Hence $\sigma^l = 1$. From the definition of A we get $fX^l = X^l f$ and $fY^l = Y^l f$ for all $f \in R$. Moreover, $X^l Y = X^{l-1}(XY) = X^{l-1}\sigma(t) = tX^{l-1} = (YX)X^{l-1} = YX^l$. Analogously, $Y^l X = XY^l$. As $\sigma^l = 1$, we have $\sigma(F) = F$ and hence $FX = XF$ and $FY = YF$. Thus the subalgebra of A, generated by F, X^l and Y^l is contained in $Z(A)$. The equality $\langle F, X^l, Y^l \rangle = \langle X^l, Y^l \rangle$ follows from $F = X^l Y^l = Y^l X^l$.

Step 2. $R \cap Z(A) = \{f \in R | \sigma(f) = f\} = \mathfrak{k}[F]$.

The first equality is obvious and hence it is enough to prove that for $f \in R$ the equality $\sigma(f) = f$ implies $f \in \mathfrak{k}[F]$. Let $f(H) \in \mathfrak{k}[H]$ be non-constant and $\hat{\mathfrak{k}}$ be the decomposition field of f. Then $f = \beta \prod_{j=1}^{k} (H - \alpha_j)$ and it is enough to consider the case $f = \prod_{w \in W} w(H - \alpha)$. If $q = 1$, then $\sigma^i(H - s) = H + i - s = H - s$ if and only if $l | s$ and hence the orbit of $(H - s)$ under the W action contains precisely l elements. If $q^l = 1$ then $\sigma^i(H - s) = q^i H + (q^i - 1)/(q - 1) - s$ and again we get that each orbit contains precisely l elements. In particular, $\deg(f) \geq l$. So, it is enough to prove the statement for $f = \prod_{i=0}^{l-1} \sigma^i(H - s)$. But $F = \prod_{i=0}^{l-1} \sigma^i(H)$ and $\deg(f - F) < l$, hence $f - F$ is constant.

Step 3. $Z(A)$ is a graded subalgebra of A, $Z(A)_i = Z(A) \cap A_i \neq 0$ if and only if $l | i$ and $Z(A)_i = X^i \mathfrak{k}[F]$. In particular, $Z(A) = B$.

If $z \in Z(A)$ and $z = \sum_{i \in \mathbb{Z}} z_i$ is a graded decomposition of z, from $zX = Xz$, $zY = Yz$, $zH = Hz$ we get $z_i X = X z_i$, $z_i Y = Y z_i$ and $z_i H = H z_i$ and hence all $z_i \in Z(A)$. Therefore $Z(A)$ is also graded. If l does not divide i, then $\sigma^i(H) \neq H$ and we have $X^i H \neq H X^i$ and $Y^i H \neq H Y^i$. Hence $Z(A)_i = 0$. If $l | i$ then for $f \in R$ from $(X^i f)X = X(X^i f)$ it follows that $\sigma(f) = f$ and hence $f \in \mathfrak{k}[F]$ by Step 2.

Step 4. A is a finitely generated B-module.

As a system of $2l^2$ generators of A over B one cane take, e.g. $Y^i H^j, X^i H^j$, $0 \leq i, j \leq l-1$. Theorem is proved. □

From Theorem 2 we get the following generalization of [3, Theorem 7.5].

Corollary 4. *If $f, g \in A$ such that $fg = gf$ then there is $P(x, y) \in Z(A)(x, y)$ such that $P(f, g) = 0$.*

I would like to finish this section with a counterexample to the conjecture on [3, page 126], where the authors ask if two commuting elements $\alpha, \beta \in \mathcal{H}(q)$, whose degrees are relatively prime with l, will be algebraically dependent over $\mathfrak{k}$. Take X and XY^l, the elements of degrees 1 and $l - 1$ respectively, both relatively prime with l. For $0 \neq p(x, y) \in \mathfrak{k}[x, y]$ each summand of $p(X, XY^l)$ is homogeneous in A and $p(X, XY^l) = 0$ should be checked on all homogeneous components. As A is an integral domain and X is itself homogeneous, we can assume that $\deg_A(p(X, XY^l)) = 0$. Then $p(X, XY^l) = \sum_i a_i X^{il} Y^{il}$ with $a_i \in \mathfrak{k}$. Set $f_i = X^{il} Y^{il}$. As $t \notin \mathfrak{k}$, $\deg f_i = \deg(t)^{il} > 0$ and hence $\deg(f_i) \neq \deg(f_j)$, $i \neq j$. From this we get that $p(X, XY^l)$ is non-zero as a sum of polynomials with increasing degrees.

4 Realization by q-difference operators

Here we assume R to be commutative. Let $A = A(R, t_i, \sigma_i)$ be a GWA and $\mathfrak{n}$ be a maximal ideal of R containing t_i for all $i \in J$. Set $X = (X_i)$, $Y = (Y_i)$, and for $l = (l_i) \in \mathbb{Z}_+^J$ put $X^l = \prod_i X_i^{l_i}$. Let $R_\mathfrak{n} = R/\mathfrak{n}$ and $\varphi : R \to R/\mathfrak{n}$ be the canonical projection. Let $I_\mathfrak{n}$ denote the left ideal in A, generated by $\mathfrak{n}$ and all X_i. From $t_i \in \mathfrak{n}$ it follows that $I_\mathfrak{n} \cap A_0 = \mathfrak{n}$ and thus for $l \in \mathbb{Z}_+^J$ there holds $I_\mathfrak{n} \cap A_{-l} = Y^l \mathfrak{n}$. Denote by $M(\mathfrak{n})$ the left module $A/I_\mathfrak{n}$, which is non-zero since $(A/I_\mathfrak{n})_0 \neq 0$. As $I_\mathfrak{n}$ is a $\mathbb{Z}^J$-graded ideal, the module $M(\mathfrak{n})$ is also $\mathbb{Z}^J$-graded and is naturally identified with the polynomial ring $R_\mathfrak{n}[Y_i]$ (with right $R_\mathfrak{n}$-coefficients). If $Y^l \in R_\mathfrak{n}[Y_i]$ is a monomial, the action of A on Y^l is defined by $Y_j(Y^l) = Y_j Y^l$, $r(Y^l) = Y^l \varphi((\prod_i \sigma_i^{l_i})(r))$ and $X_j(Y^l) = (1 - \delta_{l_j,0}) \prod_i Y^{l_i - \delta_{i,j}} \varphi(\sigma_j^{l_j}(t_j))$.

Theorem 3. *Let $A = A(R, t_i, \sigma_i)$, where $R = \mathfrak{k}[H_i]_{i \in J}$ and σ_i are defined as follows: $\sigma_i(H_j) = H_j$, $j \neq i$, $\sigma_i(H_i) = q_i H_i + 1$. Assume that $h_i - (1 - q_i)^{-1} \notin \mathfrak{n}$, $q_i \neq 1$, and for all i the parameter q_i is either not a root of unity or possibly equals 1 if $\mathrm{char}(\mathfrak{k}) = 0$. Then the annihilator $Ann_A(M(\mathfrak{n}))$ is zero. In all other cases it is non-zero.*

Proof. Clearly, $Ann_A(M(\mathfrak{n}))$ is a $\mathbb{Z}^J$-graded ideal of A and we need $Ann_A(M(\mathfrak{n})) \cap A_l = 0$, $l \in \mathbb{Z}^j$, only. Then $Ann_A(1) \cap R = \mathfrak{n}$ and hence $Ann_A(Y^l) \cap R = (\prod_i \sigma_i^{-l_i})(\mathfrak{n})$. By Step 1 and Step 2 of Theorem 1, the condition $h_i - (1 - q_i)^{-1} \notin \mathfrak{n}$ guarantees that the orbit of $\mathfrak{n}$ under $W = \langle \sigma_i \rangle$ is infinite and hence $\cap_{w \in W} w(\mathfrak{n}) = 0$. This implies, in particular, $R \cap Ann_A(M(\mathfrak{n})) = 0$. As $Ann_A(M(\mathfrak{n}))$ is a $\mathbb{Z}^J$-graded ideal and A is an integral domain, this automatically implies $Ann_A(M(\mathfrak{n})) = 0$.

If $h_i - (1 - q_i)^{-1} \in \mathfrak{n}$, we have $(h_i - (1 - q_i)^{-1}) \in \mathfrak{n}$ and $w((h_i - (1 - q_i)^{-1})) = (h_i - (1 - q_i)^{-1})$ for any $w \in W$. Hence $(h_i - (1 - q_i)^{-1}) \subset Ann_A(M(\mathfrak{n}))$. If $q_i^l = 1$ or $q_i = 1$ and $\mathrm{char}(\mathfrak{k}) = l$, then $X_i^l \in Ann_A(M(\mathfrak{n}))$ by Theorem 2. This completes the proof. □

Now, if we write $\mathcal{H}(\mathbf{q}, J)$ as the GWA from Section 1 and set $\mathfrak{n} = (h_i)$, the formula $X_j(Y^l) = (1 - \delta_{l_j,0}) \prod_i Y^{l_i - \delta_{i,j}} \varphi(\sigma_j^{l_j}(t_j))$ will read $X_j(Y^l) = (1 - \delta_{l_j,0})(\sum_{s=0}^{l_j - 1} q_j^s) \prod_i Y^{l_i - \delta_{i,j}}$, which is precisely the q_j-difference operator on $\mathfrak{k}[Y]$ and we get the following refinement of [3, Theorem 8.1, Theorem 8.3]:

Corollary 5. *Let $\mathfrak{n} = (h_i)$. Then $Ann_A(M(\mathfrak{n})) = 0$ if and only if all q_i are either non-roots of unity or some $q_i = 1$ and $\mathrm{char}(\mathfrak{k}) = 0$. In particular, in these cases $\mathcal{H}(\mathbf{q}, J)$ can be realized via q-difference operators acting on $\mathfrak{k}[Y]$.*

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