Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
20 (2004), 53–61
www.emis.de/journals

STABILITY OF NOOR ITERATIONS WITH ERRORS FOR GENERALIZED NONLINEAR COMPLEMENTARITY PROBLEMS

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Abstract. In this paper, we introduce and study a class of generalized nonlinear complementarity problems and construct an iterative algorithm, called the Noor iterations with errors, by using the change of variables technique. We establish the existence and uniqueness of solution of the generalized nonlinear complementarity problem and the convergence and stability of iterative sequence generated by the algorithm.

1. Introduction

It is well known that the complementarity theory has a lot of applications in diverse fields of mathematical, regional, physical, and engineering sciences ([1], [3], [4], [6]-[12], [14] -[16]). In 1980, Van Bokhoven used first the change of variables technique to study a class of linear complementarity problems in R^n. Afterwards, Noor [9] and Noor and Zarae [15] modified the change of variables technique to suggest some iterative methods for solving some classes of nonlinear complementarity problems in R^n. Recently, Ahmad, Kazmi and Rehman [1] and Noor and Al-Said [14] have extended the results of Noor [9], Noor and Zarae [15] and Van Bokhoven [16] to the implicit complementarity problem in the infinite-dimensional spaces and the generalized strongly nonlinear complementarity problem in Hilbert spaces, respectively. On the other hand, Noor [13] introduced and studied a class of three-step approximation schemes for general variational inequalities.

Inspired and motivated by the research work in [1], [9], [13]-[16], in this paper, we introduce and study a new class of generalized nonlinear complementarity problems in Hilbert spaces. Using the change of variables technique, we obtain that the generalized nonlinear complementarity problem and the fixed point problem are equivalent. Using this equivalence, we suggest and analysis a new unified and general algorithm, which is called the Noor iteration with errors, for computing the approximate solution of the generalized nonlinear complementarity problem. Under certain conditions, we establish the existence and uniqueness of solution of the generalized nonlinear complementarity problem, and the convergence and stability of iterative sequence of generated by the algorithm. Our results are an extension and improvements of previously known results.

2. Preliminaries

Let H be a real Hilbert space on which the inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a nonempty closed convex cone of H, P_K

2000 *Mathematics Subject Classification.* 47J20, 49J40.
Key words and phrases. Generalized nonlinear complementarity problem, change of variables, Noor iteration with errors, stability.
This work was supported by Korea Research Foundation Grant (KRF-2003-015-C00039).

denote the projection of H onto K and A, B and $N\colon H \times H \to H$ be nonlinear mappings. We now consider the following problem.

The generalized nonlinear complementarity problem consists in finding $u \in H$ such that:

$$u \in K, N(Au, Bu) \in K^* \text{ and } \langle N(Au, Bu), u\rangle = 0, \tag{2.1}$$

where $K^* = \{y \in H : \langle y, x\rangle \geq 0,\ \forall x \in K\}$ is a convex polar cone of K in H.

In case $N(u, v) = u + v$ for all $u, v \in H$, then problem (2.1) is equivalent to finding $u \in H$ such that

$$u \in K, Au + Bu \in K^* \text{ and } \langle Au + Bu, u\rangle = 0, \tag{2.2}$$

which is called the strongly nonlinear complementarity problem, or the generalized strongly nonlinear complementarity problem or the generalized mildly nonlinear complementarity problem, introduced and studied by Noor [8]-[10] and Noor and Al-Said [14]. In [9], Noor proved that a wide class of problems arising in fluid flow through porous media, lubrication problems, contact problems elasticity, economics, and structural analysis can be studied by the generalized strongly nonlinear complementarity problem.

If $N(u, v) = u$ for all $u, v \in H$, then problem (2.1) is equivalent to finding $u \in H$ such that

$$u \in K, Au \in K^* \text{ and } \langle Au, u\rangle = 0, \tag{2.3}$$

which is called the generalized complementarity problem. For the applications, numerical methods and formulations, see [4], [15] and the references therein.

It is worth mentioning that problem (2.1) can be written as

$$u \in K, v = N(Au, Bu) \in K^* \text{ and } \langle v, u\rangle = 0. \tag{2.4}$$

Let us recall the following concepts. For each $u \in H$, we define the absolute value of u as follows:

$$|u| = u^+ + u^-,\ u^+ = \sup\{0, u\} \text{ and } u^- = -\inf\{0, u\}.$$

It is known that for any arbitrary element $u \in H$, we get that $u = u^+ - u^-$ and $\langle u^+, u^-\rangle = 0$. Using the idea and technique of Noor [10], Noor and Al-Said [14] and Noor and Zarae [15], and for all $z \in H$, we consider the following change of variables:

$$u = \frac{|z| + z}{2} = z^+ = P_K(z),\ v = \frac{|z| - z}{\rho} = \frac{2z^{-1}}{\rho} = \frac{2}{\rho}(P_K(z) - z),$$

where $\rho > 0$ is a constant. It is easy to verify that the generalized nonlinear complementarity problem (2.1) has a solution $u \in H$ if and only if the mapping $G\colon H \to H$ defined by

$$G(z) = (1 - t)z^+ + t\Big(z^+ - \frac{\rho}{2}N(Az^+, Bz^+)\Big) \text{ for all } z \in H \tag{2.5}$$

has a fixed point $z \in H$, where t is a constant in $(0, 1]$ and

$$u = z^+ = P_K(z). \tag{2.6}$$

Invoking the method of Noor [13], by (2.5) and (2.6) we suggest the following algorithms for the generalized nonlinear complementarity problem (2.1):

Algorithm 2.1 (Noor iteration with errors). Given $z_0 \in H$, compute the sequence $\{z_n\}_{n\geq 0}$ by the iterative schemes

$$\begin{aligned} x_n &= (1-\gamma_n)z_n^+ + \gamma_n\Big(z_n^+ - \frac{\rho}{2}N(Az_n^+, Bz_n^+)\Big) + s_n,\ z_n^+ = P_K(z_n), \\ y_n &= (1-\beta_n)z_n^+ + \beta_n\Big(x_n^+ - \frac{\rho}{2}N(Ax_n^+, Bx_n^+)\Big) + q_n,\ x_n^+ = P_K(x_n), \\ z_{n+1} &= (1-\alpha_n)z_n^+ + \alpha_n\Big(y_n^+ - \frac{\rho}{2}N(Ay_n^+, By_n^+)\Big) + p_n,\ y_n^+ = P_K(y_n) \end{aligned} \tag{2.7}$$

for all $n \geq 0$, where $\{s_n\}_{n\geq 0}$, $\{q_n\}_{n\geq 0}$ and $\{p_n\}_{n\geq 0}$ are the sequences of the elements of H introduced to take into account possible inexact computations, and the sequences $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ satisfy

$$0 \leq \alpha_n, \beta_n, \gamma_n \leq 1 \text{ for all } n \geq 0 \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty. \tag{2.8}$$

As special cases of the Noor iteration with errors, we have

Algorithm 2.2 (Ishikawa iteration with errors). Given $z_0 \in H$, compute the sequence $\{z_n\}_{n\geq 0}$ by the iterative schemes

$$\begin{aligned} x_n =&\ (1-\gamma_n)z_n^+ + \gamma_n\Big(z_n^+ - \tfrac{\rho}{2}N(Az_n^+, Bz_n^+)\Big) + s_n,\ z_n^+ = P_K(z_n), \\ z_{n+1} =&\ (1-\beta_n)z_n^+ + \beta_n\Big(x_n^+ - \tfrac{\rho}{2}N(Ax_n^+, Bx_n^+)\Big) + q_n,\ x_n^+ = P_K(x_n) \end{aligned}$$

for all $n \geq 0$, where $\{s_n\}_{n\geq 0}$ and $\{q_n\}_{n\geq 0}$ are the sequences of the elements of H introduced to take into account possible inexact computations, and the sequences $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ satisfy

$$0 \leq \beta_n, \gamma_n \leq 1 \text{ for all } n \geq 0 \text{ and } \sum_{n=0}^{\infty} \beta_n = \infty.$$

Algorithm 2.3 (Mann iteration with errors). Given $z_0 \in H$, compute the sequence $\{z_n\}_{n\geq 0}$ by the iterative schemes

$$z_{n+1} = (1-\gamma_n)z_n^+ + \gamma_n\Big(z_n^+ - \frac{\rho}{2}N(Az_n^+, Bz_n^+)\Big) + s_n,\ z_n^+ = P_K(z_n)$$

for all $n \geq 0$, where $\{s_n\}_{n\geq 0}$ is the sequence of the elements of H introduced to take into account possible inexact computations, and the sequence $\{\gamma_n\}_{n\geq 0}$ satisfies

$$0 \leq \gamma_n \leq 1 \text{ for all } n \geq 0 \text{ and } \sum_{n=0}^{\infty} \gamma_n = \infty.$$

Remark 2.1. If $\gamma_n = \beta_n = 0$, $\alpha_n = 1$, $s_n = q_n = p_n = 0$ for all $n \geq 0$ and $N(u,v) = u + v$ for all $u, v \in H$, then Algorithm 2.1 reduces to Algorithm 3.1 in [14]. On the other hand, Algorithms 3.3 and 3.5 in [10] are special cases of Algorithm 2.1.

Definition 2.1. Let $A\colon H \to H$ and $N\colon H \times H \to H$ be mappings.
(i) N is said to be γ-strongly monotone with respect to A in the first argument if there exists a constant $\gamma > 0$ such that

$$\langle N(Ax,z) - N(Ay,z), x - y\rangle \geq \gamma\|x-y\|^2 \text{ for all } x, y, z \in H;$$

(ii) N is said to be α-Lipschitz continuous in the first argument if there exists a constant $\alpha > 0$ such that

$$\|N(x,z) - N(y,z)\| \leq \alpha\|x - y\| \text{ for all } x, y \in H.$$

(iii) A is said to be δ-Lipschitz continuous if there exists a constant $\delta > 0$ such that

$$\|Ax - Ay\| \leq \delta\|x - y\| \text{ for all } x, y \in H.$$

It follows from (i), (ii) and (iii) that $\alpha\delta \geq \gamma$. Similarly, we can define the Lipschitz continuity of N in the second argument.

Definition 2.2 ([2])**.** Let $T\colon H \to H$ be a mapping and $x_0 \in H$. Assume that $x_{n+1} = f(T, x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n\geq 0} \subset H$. Suppose that $F(T) = \{x \in H : x = Tx\} \neq \emptyset$ and $\{x_n\}_{n\geq 0}$ converges to some $u \in F(T)$. Let $\{z_n\}_{n\geq 0}$ be an arbitrary sequence in H and $\varepsilon_n = \|z_{n+1} - f(T, z_n)\|$ for all $n \geq 0$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} z_n = u$, then the iteration procedure defined by $x_{n+1} = f(T, x_n)$ is said to be T-stable or stable with respect to T.

Harder and Hicks [2] proved how such a sequence $\{z_n\}_{n\geq 0}$ could arise in practice and demonstrated the importance of investigating the stability of various iterative schemes for various classes of nonlinear mappings.

Lemma 2.1 ([5])**.** *Let $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ be nonnegative sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + t_n b_n + c_n, \ \forall n \geq 0,$$

where $\{t_n\}_{n\geq 0} \subset [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

We now study the existence and uniqueness of solution of the generalized nonlinear complementarity problem (2.1) and establish the convergence and stability of iterative sequence of generated by Algorithm 2.1.

Theorem 3.1. *Let $A, B\colon H \to H$ be δ-Lipschitz continuous and η-Lipschitz continuous, respectively. Let $N\colon H \times H \to H$ be α-Lipschitz continuous in the first argument and β-Lipschitz continuous in the second argument, and γ-strongly monotone with respect to A in the first argument. Assume that*

$$\lim_{n\to\infty} \beta_n \|s_n\| = \lim_{n\to\infty} \|q_n\| = 0, \tag{3.1}$$

and one of the conditions (3.2) *and* (3.3) *holds:*

$$\sum_{n=0}^{\infty} \|p_n\| < \infty; \tag{3.2}$$

there exists a nonnegative sequence $\{d_n\}_{n\geq 0}$ such that

$$\|p_n\| = d_n \alpha_n \text{ for all } n \geq 0 \quad \text{and} \quad \lim_{n\to\infty} d_n = 0. \tag{3.3}$$

Suppose that there exists a positive constant ρ satisfying

$$\rho\beta\eta < 2 \tag{3.4}$$

and one of the following conditions

$$\beta\eta < \gamma, \ \rho < 4\frac{\gamma - \beta\eta}{\alpha^2\delta^2 - \beta^2\eta^2}, \tag{3.5}$$

$$\alpha\delta < \beta\eta, \ \rho > 4\frac{\beta\eta - \gamma}{\beta^2\eta^2 - \alpha^2\delta^2}. \tag{3.6}$$

Then the mapping G defined by (2.5) *and* (2.6) *satisfies that*

$$\|G(x) - G(y)\| \leq (1 - t(1 - \theta))\|x - y\| \text{ for all } x, y \in H,$$

where

$$\theta = \frac{1}{2}\rho\beta\eta + \sqrt{1 - \rho\gamma + \frac{1}{4}\rho^2\alpha^2\delta^2}. \tag{3.7}$$

Moreover, G has a unique fixed point $z \in H$ and the sequence $\{z_n\}_{n\geq 0}$ generated by Algorithm 2.1 *converges strongly to z.*

Proof. Let x, y be arbitrary elements in H. Since N is α-Lipschitz continuous, β-Lipschitz continuous in the second argument, and γ-strongly monotone with respect to A in the first argument, A and B are δ-Lipschitz continuous and η-Lipschitz continuous, respectively, it follows that

$$\begin{aligned}
&\|x^+ - y^+ - \frac{1}{2}\rho(N(Ax^+, Bx^+) - N(Ay^+, Bx^+))\|^2 \\
&= \|x^+ - y^+\|^2 - \rho\langle N(Ax^+, Bx^+) - N(Ay^+, Bx^+), x^+ - y^+\rangle \\
&\quad + \frac{1}{4}\rho^2\|N(Ax^+, Bx^+) - N(Ay^+, Bx^+)\|^2 \\
&\leq (1 - \rho\gamma + \frac{1}{4}\rho^2\alpha^2\delta^2)\|x^+ - y^+\|^2
\end{aligned} \tag{3.8}$$

and

$$\|N(Ay^+, Bx^+) - N(Ay^+, By^+)\| \leq \beta\eta\|x^+ - y^+\|. \tag{3.9}$$

In view of (2.5), (2.6), (3.7)-(3.9) and the nonexpansivity of P_K, we deduce that

$$\begin{aligned}
&\|G(x) - G(y)\| \\
&= \|(1-t)(x^+ - y^+) + t[x^+ - y^+ - \frac{1}{2}\rho(N(Ax^+, Bx^+) - N(Ay^+, By^+))]\| \\
&\leq (1-t)\|x^+ - y^+\| + t\|x^+ - y^+ - \frac{1}{2}\rho(N(Ax^+, Bx^+) - N(Ay^+, Bx^+))\| \\
&\quad + \frac{1}{2}t\rho\|N(Ay^+, Bx^+) - N(Ay^+, By^+)\| \\
&\leq (1 - t + t\sqrt{1 - \rho\gamma + \frac{1}{4}\rho^2\alpha^2\delta^2} + \frac{1}{2}t\rho\beta\eta)\|x^+ - y^+\| \\
&= (1 - t(1-\theta))\|P_K(x) - P_K(y)\| \\
&\leq (1 - t(1-\theta))\|x - y\|.
\end{aligned}$$

Notice that (3.4) and (3.7) mean that

$$\begin{aligned}
\theta < 1 &\Leftrightarrow 1 - \rho\gamma + \frac{1}{4}\rho^2\alpha^2\delta^2 < 1 - \rho\beta\eta + \frac{1}{4}\rho^2\beta^2\eta^2 \\
&\Leftrightarrow \frac{1}{4}\rho^2(\alpha^2\delta^2 - \beta^2\eta^2) < \rho(\gamma - \beta\eta).
\end{aligned} \tag{3.10}$$

Now we consider the following three cases:
Case 1. Suppose that $\beta\eta < \gamma$. Note that $\alpha\delta \geq \gamma$. Then (3.10) implies that

$$\theta < 1 \Leftrightarrow \rho < 4\frac{\gamma - \beta\eta}{\alpha^2\delta^2 - \beta^2\eta^2}.$$

Case 2. Suppose that $\beta\eta = \gamma$. Since $\alpha\delta \geq \gamma$, it follows from (3.10) that

$$\theta < 1 \Leftrightarrow 0 \leq \frac{1}{4}\rho^2(\alpha^2\delta^2 - \beta^2\eta^2) < \rho(\gamma - \beta\eta) = 0,$$

which is a contradiction.
Case 3. Suppose that $\beta\eta > \gamma$. If $\alpha\delta \geq \beta\eta$, then (3.10) means that

$$\theta < 1 \Leftrightarrow 0 \leq \frac{1}{4}\rho^2(\alpha^2\delta^2 - \beta^2\eta^2) < \rho(\gamma - \beta\eta) < 0,$$

which is impossible. Hence $\alpha\delta < \beta\eta$. According to (3.10), we know that

$$\theta < 1 \Leftrightarrow \rho > 4\frac{\beta\eta - \gamma}{\beta^2\eta^2 - \alpha^2\delta^2}.$$

Thus $\theta < 1$ is equivalent to (3.5) and (3.6). It follows from (3.5), (3.6) and $t \in (0, 1]$ that $1 - t(1-\theta) < 1$. Hence G has a unique fixed point $z \in H$ and

$$
\begin{aligned}
z &= (1-\gamma_n)z^+ + \gamma_n\Big(z^+ - \frac{\rho}{2}N(Az^+, Bz^+)\Big) \\
&= (1-\beta_n)z^+ + \beta_n\Big(z^+ - \frac{\rho}{2}N(Az^+, Bz^+)\Big) \\
&= (1-\alpha_n)z^+ + \alpha_n\Big(z^+ - \frac{\rho}{2}N(Az^+, Bz^+)\Big),
\end{aligned}
\tag{3.11}
$$

where $z^+ = P_K(z)$. By virtue of (2.7), (3.8), (3.9) and (3.11), we infer that

$$
\begin{aligned}
\|x_n - z\| &= \|(1-\gamma_n)(z_n^+ - z^+) \\
&\quad + \gamma_n[z_n^+ - z^+ - \frac{1}{2}\rho(N(Az_n^+, Bz_n^+) - N(Az^+, Bz^+))] + s_n\| \\
&\le (1-\gamma_n)\|z_n^+ - z^+\| + \gamma_n\|z_n^+ - z^+ \\
&\quad - \frac{1}{2}\rho(N(Az_n^+, Bz_n^+) - N(Az^+, Bz_n^+))]\| \\
&\quad + \frac{1}{2}\gamma_n\rho\|N(Az^+, Bz_n^+) - N(Az^+, Bz^+)\| + \|s_n\| \\
&\le (1-\gamma_n)\|z_n^+ - z^+\| + \gamma_n\theta\|z_n^+ - z^+\| + \|s_n\| \\
&= (1-\gamma_n(1-\theta))\|P_K(z_n) - P_K(z)\| + \|s_n\| \\
&\le (1-\gamma_n(1-\theta))\|z_n - z\| + \|s_n\| \\
&\le \|z_n - z\| + \|s_n\|.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\|y_n - z\| &\le (1-\beta_n)\|z_n^+ - z^+\| + \beta_n\theta\|x_n^+ - z^+\| + \|q_n\| \\
&\le (1-\beta_n(1-\theta))\|z_n - z\| + \beta_n\|s_n\| + \|q_n\| \\
&\le \|z_n - z\| + \beta_n\|s_n\| + \|q_n\|
\end{aligned}
$$

and

$$
\begin{aligned}
\|z_{n+1} - z\| &\le (1-\alpha_n)\|z_n^+ - z^+\| + \alpha_n\theta\|y_n^+ - z^+\| + \|p_n\| \\
&\le (1-\alpha_n(1-\theta))\|z_n - z\| + \alpha_n(\beta_n\|s_n\| + \|q_n\|) + \|p_n\|.
\end{aligned}
\tag{3.12}
$$

Suppose that (3.2) holds. Set $a_n = \|z_n - z\|$, $b_n = (1-\theta)^{-1}(\beta_n\|s_n\| + \|q_n\|)$, $c_n = \|p_n\|$ and $t_n = (1-\theta)\alpha_n$ for all $n \ge 0$. It follows from (2.8), (3.1), (3.2) and Lemma 2.1 that $\lim_{n\to\infty} z_n = z$.

Suppose that (3.3) holds. Put $a_n = \|z_n - z\|$, $b_n = (1-\theta)^{-1}(\beta_n\|s_n\| + \|q_n\| + d_n)$, $c_n = 0$ and $t_n = (1-\theta)\alpha_n$ for all $n \ge 0$. According to (2.8), (3.1), (3.3) and Lemma 2.1, we conclude that $\lim_{n\to\infty} z_n = z$. □

Remark 3.1. Under the assumptions of Theorem 3.1, we know that the generalized nonlinear complementarity problem (2.1) has a unique solution $u = P_K(z) = \lim_{n\to\infty} P_K(z_n)$, where z is the unique fixed point of G and $\{z_n\}_{n\ge0}$ satisfies (2.7).

Remark 3.2. Theorem 3.1 extends Theorem 4.3 of Noor [10] and Theorem 3.1 of Noor and Al-Said [14] in the following ways:

(i) the strongly nonlinear complementarity problem in [10] and the generalized strongly nonlinear complementarity problem in [14] are replaced by the more general generalized nonlinear complementarity problem;

(ii) Algorithm 3.3 in [10] and Algorithm 3.1 in [14] are replaced by the more general Algorithm 3.1;

(iii) the conditions (3.1)-(3.6) are weaker than the conditions used in [10] and [14].

Theorem 3.2. *Let N and θ be as in Theorem 3.1 and (3.1) hold. Suppose that*

$$\lim_{n\to\infty} \|p_n\| = 0 \tag{3.13}$$

an there exists a constant $s \in (0,1)$ such that

$$\alpha_n \geq s \text{ for all } n \geq 0. \tag{3.14}$$

Let $\{A_n\}_{n\geq 0}$ be an arbitrary sequence in H and define $\{\varepsilon_n\}_{n\geq 0} \subset [0,+\infty)$ by

$$\begin{aligned}
&\varepsilon_n = \|A_{n+1} - [(1-\alpha_n)A_n^+ + \alpha_n(B_n^+ - \frac{1}{2}\rho N(AB_n^+, BB_n^+)) + p_n]\|,\\
&B_n^+ = P_K(B_n),\\
&B_n = (1-\beta_n)A_n^+ + \beta_n(C_n^+ - \frac{1}{2}\rho N(AC_n^+, BC_n^+)) + q_n, C_n^+ = P_K(C_n),\\
&C_n = (1-\gamma_n)A_n^+ + \gamma_n(A_n^+ - \frac{1}{2}\rho N(AA_n^+, BA_n^+)) + s_n, A_n^+ = P_K(A_n)
\end{aligned} \tag{3.15}$$

for all $n \geq 0$. If there exists a constant $\rho > 0$ satisfying (3.4) and one of (3.5) and (3.6), then the mapping G defined by (2.5) and (2.6) has a unique fixed point $z \in H$, the sequence $\{z_n\}_{n\geq 0}$ generated by Algorithm 2.1 converges strongly to z and $\lim_{n\to\infty} A_n = z$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof. Let $d_n = \|p_n\|\alpha_n^{-1}$ for all $n \geq 0$. Then (3.13) and (3.14) yield that (3.3) holds. It follows from Theorem 3.1 that G has a unique fixed point $z \in H$ and $\lim_{n\to\infty} z_n = z$. As in the proof of Theorem 3.1, by (3.11) and (3.15) we obtain that

$$\begin{aligned}
&\|(1-\alpha_n)A_n^+ + \alpha_n(B_n^+ - \frac{1}{2}\rho N(AB_n^+, BB_n^+)) + p_n - z\|\\
&\leq (1-\alpha_n)\|A_n^+ - z^+\| + \alpha_n\|B_n^+ - z^+\\
&\quad - \frac{1}{2}\rho(N(AB_n^+, BB_n^+) - N(Az^+, BB_n^+))\|\\
&\quad + \frac{1}{2}\alpha_n\rho\|N(Az^+, BB_n^+) - N(Az^+, Bz^+)\| + \|p_n\|\\
&\leq (1-\alpha_n)\|A_n^+ - z^+\| + \alpha_n\theta\|B_n^+ - z^+\| + \|p_n\|\\
&= (1-\alpha_n)\|P_K(A_n) - P_K(z)\| + \alpha_n\theta\|P_K(B_n) - P_K(z)\| + \|p_n\|\\
&\leq (1-\alpha_n)\|A_n - z\| + \alpha_n\theta\|B_n - z\| + \|p_n\|,
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
&\|B_n - z\| \leq (1-\beta_n)\|A_n - z\| + \beta_n\theta\|C_n - z\| + \|q_n\|,\\
&\|C_n - z\| \leq (1-\gamma_n)\|A_n - z\| + \gamma_n\theta\|A_n - z\| + \|s_n\|.
\end{aligned} \tag{3.17}$$

Substituting (3.17) into (3.16), by (3.14) we have

$$\begin{aligned}
&\|(1-\alpha_n)A_n^+ + \alpha_n(B_n^+ - \frac{1}{2}\rho N(AB_n^+, BB_n^+)) + p_n - z\|\\
&\leq (1-\alpha_n(1-\theta))\|A_n - z\| + \alpha_n(\beta_n\|s_n\| + \|q_n\|) + \|p_n\|\\
&\leq (1-s(1-\theta))\|A_n - z\| + \beta_n\|s_n\| + \|q_n\| + \|p_n\|.
\end{aligned} \tag{3.18}$$

Suppose that $\lim_{n\to\infty} A_n = z$. Then (3.1), (3.13), (3.14) and (3.18) ensure that

$$\begin{aligned}
\varepsilon_n &\leq \|A_{n+1} - z\| + \|(1-\alpha_n)A_n^+ + \alpha_n(B_n^+ - \frac{1}{2}\rho N(AB_n^+, BB_n^+)) + p_n - z\|\\
&\leq \|A_{n+1} - z\| + (1-s(1-\theta))\|A_n - z\| + \beta_n\|s_n\| + \|q_n\| + \|p_n\| \to 0
\end{aligned}$$

as $n \to \infty$. That is, $\lim_{n\to\infty} \varepsilon_n = 0$.

Conversely, suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. Then (3.14), (3.15) and (3.18) imply that

$$\begin{aligned} \|A_{n+1} - z\| &\le \varepsilon_n + \|(1-\alpha_n)A_n^+ + \alpha_n(B_n^+ \\ &\qquad - \frac{1}{2}\rho N(AB_n^+, BB_n^+)) + p_n - z\| \\ &\le (1 - s(1-\theta))\|A_n - z\| + \beta_n\|s_n\| + \|q_n\| + \|p_n\| + \varepsilon_n. \end{aligned} \tag{3.19}$$

Put $a_n = \|A_n - z\|$, $b_n = s^{-1}(1-\theta)^{-1}(\beta_n\|s_n\| + \|q_n\| + \|p_n\| + \varepsilon_n)$, $c_n = 0$ and $t_n = s(1-\theta)$ for all $n \ge 0$. It follows from (3.1), (3.13),(3.19) and Lemma 2.1 that $\lim_{n\to\infty} A_n = z$. □

Remark 3.3. Theorem 3.2 reveals that the iterative sequence generated by Algorithm 2.1 is G-stable, where G is defined by (2.5).

Acknowledgment. The authors would like to thank the referee for his many helpful comments and suggestions towards the improvement of this paper.

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Received March 06, 2003; October 13, 2003 revised.

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