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MODIFIED DYADIC DERIVATIVES AND INTEGRALS OF FRACTIONAL ORDER ON R_+

B.I. GOLUBOV

Dedicated to the 60th birthday of Professor W.R. Wade

Abstract. We give a brief review of the known results on pointwise and strong dyadic differentiation and integration of real functions. In section 3 some new results on modified dyadic fractional differentiation and integration are formulated.

Introduction

Following the concept of J.E. Gibbs [1] P.L. Butzer and H.J. Wagner [2] defined dyadic strong derivative D. After that they introduced dyadic pointwise derivative d and dyadic strong integral I (see $[3]-[5]$). Their definitions concerns to functions defined on dyadic group G or dyadic field K. Dyadic group G and dyadic field K are isomorphic to modified segment $[0,1]^*$ and modified positive half-line $R_+^* = [0,+\infty)^*$ respectively. The characters of dyadic group G and dyadic field K are Walsh-Paley functions $w_n(\cdot)$, $n \in Z_+ = \{0,\ 1,\ 2,\dots\}$ and generalized Walsh functions $\psi_y(\cdot)$, $y \in R_+$ respectively. P.L. Butzer and H.J. Wagner proved the equalities $D\,w_n = n\,w_n$ and $d\,w_n(x) = n\,w_n(x)$ for $n \in Z_+$, $x \in G$ and $d\,\psi_y(x) = |y|\psi_y(x)$ for $x, y \in K$.

C.W. Onneweer [6] introduced modified pointwise and strong dyadic derivatives for functions defined on dyadic group G or dyadic field K. He proved that the characters of dyadic group G or dyadic field K are differentiable in his sense and they are eigenfunctions of modified differential operator δ. For example, he proved the equalities

$$\delta(w_0)\,(x) \equiv 0, \quad \delta(w_n)\,(x) = 2^k w_n(x), \quad 2^k \le n < 2^{k+1}, \quad k \in Z_+, \quad x \in D.$$

In another article [7] C.W. Onneweer introduced modified fractional differentiation and integration on compact Vilenkin groups G_p of order $p \ge 2$ and proved fundamental theorem of dyadic calculus.

In this paper we give a brief outline of known results concerning dyadic derivatives and integrals.

We also define modified dyadic strong and pointwise integrals and derivatives of fractional order on R_+ and formulate some results concerning their properties.

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1. Notations and definitions

For a number $x \in R_+ \equiv [0, +\infty)$ we consider dyadic expansion

$$x = \sum_{n=-\infty}^{+\infty} 2^{-n-1} x_n,$$

where x_n equals to 0 or 1. Note that $x_n = 0$ for $n \le n(x)$, where $n(x) \in Z = \{0, \ \pm 1, \ \pm 2, \dots\}$. If x is dyadic rational, then we take its finite expansion, i.e. $x_n = 0$ for $n \ge n_0(x) > -\infty$. We define dyadic sum of two numbers $x, y \in R_+$ by the operation $\oplus$ as follows: $x \oplus y = z$, where $z_n = x_n + y_n \pmod 2$ for all $n \in Z$. Let us put $t(x, y) = \sum\limits_{n=-\infty}^{+\infty} x_n y_{-n-1}$ and define the generalized Walsh functions

$$\psi(x, y) \equiv \psi_y(x) = (-1)^{t(x,\, y)} \quad \text{for} \quad (x, y) \in R_+ \times R_+.$$

They were introduced by N.J. Fine [8]. It is evident that $\psi(x, y) = \psi(y, x)$, $\psi(x, y) = \pm 1$ for $x, y \in R_+$. The functions $w_n(x) \equiv \psi(x, n)$, $n \in Z_+$, are called the Walsh-Paley functions. They are 1-periodic on R_+. It is evident that $w_0(x) \equiv 1$ on R_+. The system $\{w_n(x)\}_{n=0}^{+\infty}$ is orthonormal on $[0, 1)$, i.e.

$$\int_0^1 w_m(x) w_m(x)\, dx = \delta_{m,n},$$

where $\delta_{m,n}$ is Kronecker symbol, i.e. $\delta_{m,n} = 0$ for $m \ne n$ and $\delta_{n,\,n} = 1$.

Let be given a function $f \in L[0, 1)$. We denote by $\sum\limits_{n=0}^{+\infty} \hat{f}(n) w_n(x)$ its Fourier series with respect to the Walsh-Paley system, where $\hat{f}(n) = \int_0^1 f(x) w_n(x)\, dx$, $n \in Z_+$, are Walsh-Fourier coefficients of the function f.

For the function $f \in L(R_+)$ N.J. Fine [8] introduced its Walsh transform by the equality

$$F[f]\,(x) \equiv \tilde{f}(x) = \int\limits_{R_+} \psi(x, y)\, f(y)\, dy.$$

If $f \in L^p(R_+)$, $1 < p \le 2$, then its Walsh transform is defined as the limit as $n \to +\infty$ of the sequence $\int\limits_0^{2^n} f(y)\, \psi(x, y)\, dy$ in the norm of the space $L^q(R_+)$, where $1/p + 1/q = 1$.

For $f \in L(R_+)$, $g \in L^p(R_+)$, $1 \le p \le +\infty$, we set

$$(f * g)(x) \int\limits_{R_+} = f(x \oplus y) g(y)\, dy, x \in R_+,$$

i.e. $f * g$ is dyadic convolution of f and g. Let us note that $f * g \in L^p(R_+)$, $(f * \tilde{g}) = \tilde{f}\tilde{g}$.

The function $f \in L(R_+)$ is called $W-$continuous at the point $x \in R_+$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x \oplus y) - f(x)| < \varepsilon$ for $0 < y < \delta$ (see [9], Chapter 1).

Let us note that the Wash-Fourier transform $\tilde{f}$ of every function $f \in L(R_+)$ is W-continuous on R_+ (see [9], Theorem 6.1.5).

We call the point $x \in R_+$ dyadic Lebesgue point of local integrable function f, if f is defined at the point x and

$$\lim_{n \to +\infty} 2^n \int_0^{2^{-n}} |f(x \oplus t) - f(x)|\, dt = 0.$$

Almost all points of local integrable function are its dyadic Lebesgue points. If a function f is $W-$continuous at the point $x \in R_+$, then x is its dyadic Lebesgue point. (There is also a concept of Walsh-Lebesgue point of an integrable on $[0, 1)$ function, see [26], [27]).

Let us define the generalized Walsh-Dirichlet integral of the function $f \in L(R_+)$ by the equality

$$S_y(f)\,(x) = \int_0^y \tilde{f}(t)\psi(x,t)\,dt.$$

Theorem. *If $x \in R_+$ is dyadic Lebesgue point of the function $f \in L(R_+)$, then*

$$\lim_{n\to+\infty} S_{2^n}(f)(x) = f(x).$$

The statement of this theorem was proved at the page 430 in [10] for the points of $W-$continuity of the function f. But the proof is valid also for dyadic Lebesgue points.

It follows from this theorem that if $f, \tilde{f} \in L\,(R_+)$, then $f(x) = \int\limits_{R_+} \psi(x,y)\,\tilde{f}(y)\,dy$ almost everywhere (a.e.) on R_+.

Let $\Delta = \{\Delta_n^k\}$ denote the set of all dyadic intervals $\Delta_n^k \equiv [k2^{-n},\, (k+1)2^{-n})$, $k \in Z_+$, $n \in Z$. Let us introduce dyadic maximal function

$$M_d(f)(x) = \sup_{x\in I\in\Delta} \left| \frac{1}{|I|} \int_I f(t)\,dt \right|, \quad x \in R_+,$$

and dyadic Hardy space

$$H(R_+) = \{f \in L(R_+) :\ M_d(f) \in L(R_+)\}.$$

The norm on $H(R_+)$ is $\|f\|_{H(R_+)} = \|M_d(f)\|_{L(R_+)}$.

By similar way dyadic Hardy space $H([0, 1))$ is defined.

Below $C_W(R_+)$ is the space of uniformly $W-$continuous functions on R_+. The norm on the $C_W(R_+)$ is $\|f\|_{C_W(R_+)} = \sup\limits_{x\in R_+} |f(x)|$. The symbol $C_W[0, 1)$ will denote the space of uniformly $W-$continuous functions on $[0, 1)$ with the norm $\|f\|_{C_W[0,1)} = \sup\limits_{x\in[0,1)} |f(x)|$. For the sake of convenience we shall consider the spaces $C_W[0, 1)$ and $C_W(R_+)$ as the spaces $L^p[0, 1)$ or $L^p(R_+)$ respectively for $p = +\infty$.

2. The known concepts of dyadic derivatives and integrals

Dyadic derivatives. P.L. Butzer and H.J. Wagner [4] defined dyadic pointwise derivative as follows.

Definition 2.1. Let be given the function $f \in L[0, 1)$ and a point $x \in [0, 1)$. If there exists finite limit

$$d^{(1)}f(x) = \lim_{n\to+\infty} \sum_{m=0}^{n} 2^{m-1} \left[f(x) - f(x \oplus 2^{-m-1})\right],$$

then $d^{(1)}f(x)$ is called dyadic derivative of the function f at the point x. The dyadic derivatives of higher order are defined by recurrence formulae

$$d^{(m)}f(x) = d^{(1)}(d^{(m-1)}f)\,(x), \quad m = 2, 3, \ldots$$

P.L. Butzer and H.J. Wagner proved that each Walsh-Paley function has dyadic derivative at each point $x \in [0, 1)$ and $d^{(1)}w_n(x) = n\,w_n(x)$ for $n \in Z_+$.

The notion of strong dyadic L^p-derivative was introduced by P.L. Butzer and H.J. Wagner [2] by the following way.

Definition 2.2. If for the function $f \in L^p[0,1)$, $1 \le p \le +\infty$, the limit

$$D^{(1)}(f)_{L^p} \equiv (L^p) - \lim_{n\to+\infty} \sum_{m=0}^{n} 2^{m-1}[f(\cdot) - f(\cdot + 2^{-m-1})]$$

exists in the norm of the space $L^p[0,1)$, then it is called $L^p[0,1)-$ derivative of the function f. The strong dyadic L^p-derivatives of higher order are defined by recurrence formula $D^{(m)}(f)_{L^p} = D^{(1)}((D^{(m-1)f})_{L^p})_{L^p}$, $m = 2, 3, \ldots$

It is proved in [2] that every Walsh function has strong dyadic $L^p[0,1)-$derivative of arbitrary order $r \in N$ for each $1 \le p \le +\infty$ and $D^{(r)}(w_n)_{L^p} = n^r w_n$ for $n \in Z_+$.

P.L. Butzer and H.J. Wagner [2] proved the following

Theorem 2.1. *If a function $f \in L^p[0,1)$, $1 \le p \le +\infty$, has strong dyadic $L^p[0,1)$-derivative $D^{(r)}(f)_{L^p} = g$, then $\hat{g}(n) = n^r \hat{f}(n)$, $n \in Z_+$, where $\hat{f}(n)$ are Walsh-Fourier coefficients of the function f.*

C.W. Onneweer [11] generalized the concepts of pointwise dyadic derivative and strong dyadic $L^p[0,1)$-derivative to functions defined on Vilenkin groups.

For functions f defined on R_+ the natural analogue of pointwise dyadic derivative $d^{(1)}f(x)$ is

$$df(x) = \lim_{n\to+\infty} \sum_{m=-n}^{n} 2^{m-1}(f(x) - f(x \oplus 2^{-m-1})).$$

(see [3]). P.L. Butzer and H.J. Wagner [3] proved that the generalized Walsh functions have dyadic derivative at each point. More precisely $d\psi_y(x) = y\psi_y(x)$, $x \in R_+$.

For the functions $f \in L^p[0,1)$, $1 \le p \le +\infty$ the strong dyadic $L^p(R_+)$-derivative is defined as follows:

$$D\,(f)_{L^p(R_+)} = \lim_{n\to\infty} \sum_{m=-n}^{n} 2^{m-1}[f(\cdot) - f(\cdot \oplus 2^{-m-1})],$$

where the limit is taken in the norm of the space $L^p(R_+)$ (see [5]). The notion of $L^p(R_+)$-derivative $D^{(r)}(f)_{L^p(R_+)}$ of higher order $r = 2, 3, \ldots$ is defined by recurrence formula.

It is known that if $f \in L^p(R_+)$, $p = 1$ or 2, and $D\,(f)_{L^p(R_+)}$ exists, then

$D\,(\tilde{f})_{L^p(R_+)}(x) = x\,\tilde{f}(x)$. (For $p = 1$ it was proved by P.L. Butzer and H.J. Wagner [3]; for $p = 2$ see J. Pál [12]). C.W. Onneweer [6] introduced *modified* pointwise and strong dyadic derivatives for functions defined on dyadic group G or dyadic field K. (The characters of dyadic field K are generalized Walsh functions $\psi_y(\cdot)$, $y \in R_+$, and the characters of the group G are Walsh-Paley functions w_n, $n \in Z_+$). He proved that the characters of dyadic group G or dyadic field K are differentiable in his sense at each point and they are eigenfunctions of modified differential operator δ. For example, he proved the equalities

$$\delta(w_0)\,(y) \equiv 0, \quad \delta(w_n)\,(y) = 2^k w_n(y), \quad 2^k \le n < 2^{k+1}, \quad k \in Z_+, \quad y \in D.$$

In another article [7] C.W. Onneweer introduced modified fractional differentiation and integration on compact groups of order $p \ge 2$ and proved fundamental theorem of dyadic calculus.

Dyadic integrals. The *dyadic integral* for functions defined on the interval $[0,1)$ was introduced by P.L. Butzer and H.J. Wagner [2] as follows. Let us set

$$W_r(x) = 1 + \sum_{n=1}^{+\infty} \frac{w_n(x)}{n^r}, \quad r \in N.$$

It is evident that $W_r \in L[0,1)$, $r \in N$. If $f \in L^p[0,1)$, $1 \le p \le +\infty$, then there exists dyadic convolution

$$I_r(f) = (f * W_r)(x) \equiv \int_0^1 f(y)\, W_r(x \oplus y)\, dy, \quad r \in N \tag{$*$}$$

and $I_r(f) \in L^p[0,1)$. The function $I_r(f)$ is called *dyadic strong integral of order r of the function f in the space $L^p[0, 1)$.*

It follows from $(*)$ that for $f \in L^p[0,1)$, $1 \le p \le +\infty$ its dyadic integral $I_r(f)$ has Walsh-Paley series of the form

$$\hat{f}(0) + \sum_{n=0}^{+\infty} \frac{\hat{f}(n)}{n^r} w_n.$$

P.L. Butzer and H.J. Wagner [2] proved the following fundamental theorem of dyadic calculus.

Theorem 2.2. *Let $f \in L^p[0,1)$, $1 \le p \le +\infty$ and $\hat{f}(0) = 0$.*
a) *If there exists $L^p[0,1)$-derivative $D^{(r)}(f)_{L^p}$ of some order $r \in N$, then $I_r(D^{(r)}(f)_{L^p}) = f$.*
b) *One has $D^{(r)}(I_r(f))_{L^p} = f$ for all $r \in N$.*

J. Pál and P. Simon [13] generalized the concept of strong dyadic $L^p[0, 1)$-integral to functions defined on Vilenkin groups. They proved a generalization of the Theorems 2.1 (for $p = 1$) and 2.2, using the concept of strong dyadic $L^p[0, 1)$-derivative for functions defined on Vilenkin groups due to C.W. Onneweer.

F. Schipp [14] proved that dyadic strong integral has pointwise dyadic derivative a.e. More precisely the following theorem is valid.

Theorem 2.3. *If $f \in L[0, 1)$, then $d^{(1)}(I_1(f))(x) = f(x)$ a.e. on $[0, 1)$, where $I_1(f)$ is dyadic strong integral of first order of the function f in the space $L[0,1)$.*

For the functions $f \in L^p(R_+)$, $1 \le p \le +\infty$, the strong dyadic integral was defined by H. J. Wagner [5] as follows. For $n \in Z_+$ we set

$$W_n(x) = \lim_{k \to +\infty} \int_{2^{-n}}^{2^k} \frac{1}{t} \psi_x(t)\, dt, \quad x \in R_+.$$

It has been proved in [5] that this limit exists a.e. on R_+ and also in $L(R_+)$-metric. Therefore there exists dyadic convolution

$$(f * W_n)(x) = \int_{R_+} f(t) W_n(x \oplus t)\, dt, \quad n \in Z_+, \tag{$**$}$$

and $f * W_n \in L^p(R_+)$, if $f \in L^p(R_+)$, $1 \le p \le +\infty$.

Definition 2.3. If for a function $f \in L^p(R_+)$, $1 \le p \le +\infty$, the sequence $(**)$ converges in $L^p(R_+)$-metric to a function $g \in L^p(R_+)$ as $n \to +\infty$, then $g \equiv I(f)$ is called strong dyadic integral of the function f in the space $L^p(R_+)$ or shortly $L^p(R_+)$-integral of the function f.

The notion of $L^p(R_+)$-integral $I_r(f)$ of higher order $r = 2, 3, \ldots$ is defined by recurrence formula.

The following results were proved by H.J. Wagner [5]:

Theorem 2.4. *For two functions $f, g \in L(R_+)$ the equality $g = I(f)$ holds if and only if $\tilde{g}(0) = 0$ and $\tilde{g}(x) = \tilde{f}(x)/x$, $x > 0$, where $I(f)$ is $L(R_+)$-integral of the function f.*

Theorem 2.5. *Let be given a function $f \in L(R_+)$.*
a) *If $L(R_+)$-integral $I(f)$ exists, then $D(I(f))_{L(R_+)} = f$.*
b) *If $L(R_+)$-derivative $D(f)_{L(R_+)}$ exists and $\tilde{f}(0) = 0$, then $I(D(f)_{L(R_+)}) = f$.*

J. Pál and F. Schipp [15] proved the following theorem.

Theorem 2.6. *If a function $L(R_+)$ has strong dyadic integral $g = I(f)$ in the space $L(R_+)$, then $I(f)$ has pointwise dyadic derivative a.e. on R_+ and $d(I(f))(x) = f(x)$ a.e. on R_+.*

3. Modified dyadic integral and derivative of fractional order on R_+

Strong and pointwise derivatives and integrals of fractional order on R_+. . In this subsection we formulate our results most of which are analogues of the results of C.W. Onneweer [7] concerning the functions defined on compact groups G_p of order $p = 2, 3, \dots$

For $x > 0$ we set $h(x) = 2^{-n}$, $2^n \le x < 2^{n+1}$, $n \in Z$. It is evident that $x^{-1} \le h(x) < 2\,x^{-1}$.

Lemma 3.1. *If $\alpha > 0$ and $n \in Z$, then for each $x > 0$ there exists finite limit*

$$W_n^\alpha(x) = \lim_{m \to +\infty} \int_{2^{-n}}^{2^m} (h(y))^\alpha \; \psi_x(y)\, dy.$$

More precisely, $W_n^\alpha(x) = -2^{(\alpha-1)\,n}$ for $2^{n-1} \le x < 2^n$,

$$W_n^\alpha(x) = -2^{(\alpha-1)\,n} + 2(1 - 2^{-\alpha}) \sum_{i=0}^{k} 2^{(n-i)\,(\alpha-1)}$$

for $2^{n-k-2} \le x < 2^{n-k-1}$, $k = 0, 1, \dots$ and $W_n^\alpha(x) = 0$ for $x \ge 2^n$.

We shall write $f(x) \approx g(x)$, $x \to a$, if $f(x) = O(g(x))$, $x \to a$, and $g(x) = O(f(x))$, $x \to a$. Then we have the following corollary from the lemma 3.1.

Corollary 3.1. *1) If $0 < \alpha < 1$, $n \in Z$, then $W_n^\alpha(x) \approx x^{\alpha-1}$, $x \to +0$;*
2) $W_n^1(x) \approx \log_2(x^{-1})$, $x \to +0$; 3) if $\alpha > 1$, then $W_n^\alpha(x)$ is bounded on R_+;
4) $W_n^\alpha \in L(R_+)$ for all $\alpha > 0$, $n \in Z$.

Definition 3.1. If $\alpha > 0$, $f, g \in L^p(R_+)$, and $\lim\limits_{n \to +\infty} \|f * W_n^\alpha - g\|_{L^p(R_+)} = 0$, then the function $g = J_\alpha(f)$ is called modified strong dyadic integral (MSDI) of order α of the function f in the space $L^p(R_+)$.

Theorem 3.1. *Let $f, g \in L(R_+)$ and $\alpha > 0$. Then the function g is MSDI of order α of the function f in the space $L(R_+)$, if and only if $\tilde{g}(0) = 0$ and $\tilde{g}(x) = \tilde{f}(x)\,(h(x))^\alpha$ for $x > 0$.*

Let us set for $\alpha > 0$, $n \in Z$:

$$\Lambda_n^\alpha(x) = \int_0^{2^n} (h(t))^{-\alpha} \psi(x, t)\, dt, \quad x \in R_+ \,.$$

Lemma 3.2. *For $\alpha > 0$, $n \in Z$ we have $\Lambda_n^\alpha \in L(R_+) \cap L^\infty(R_+)$.*

Definition 3.2. If $\alpha > 0$, $f, \varphi \in L^p(R_+)$, $1 \le p \le +\infty$, and

$$\lim_{n \to +\infty} \|f * \Lambda_n^\alpha - \varphi\|_{L^p(R_+)} = 0,$$

then the function $\varphi = D^\alpha(f)$ is called modified strong dyadic derivative (MSDD) of order α of the function f in the space $L^p(R_+)$.

Theorem 3.2. *Let $\alpha > 0$ and $f, \varphi \in L^p(R_+)$, $1 \le p \le 2$. Then the function φ is MSDD of order α of the function f in the space $L^p(R_+)$ if and only if*

$$\tilde{\varphi}(x) = \tilde{f}(x)\,(h(x))^{-\alpha}$$

a.e. on R_+.

This theorem is a corollary from the R_+-version of a theorem of C.W. Onneweer (see [22], Theorem 3).

Theorem 3.3. *Let $\alpha > 0$ and the function $f \in L(R_+)$ has MSDD $D^\alpha(f)$ of order α in the space $L(R_+)$. If $\tilde{f}(0) = 0$, then the equality $J_\alpha(D^\alpha(f)) = f$ holds.*

Theorem 3.4. *Let $\alpha > 0$ and the function $f \in L(R_+)$ has MSDI $J_\alpha(f)$ of order α in the space $L(R_+)$. Then the equality $D^\alpha(J_\alpha(f)) = f$ is valid.*

The Theorems 3.3 and 3.4 are R_+-version of fundamental theorem of dyadic calculus (see Theorem 2.2 above).

Theorem 3.5. *Let $\alpha > 0$, $\beta > 0$ and $f \in L(R_+)$. Then $D^\alpha(D^\beta(f)) = D^{\alpha+\beta}(f)$ (respectively $J^\alpha(J^\beta(f)) = J^{\alpha+\beta}(f)$), if the left side of this equality exists.*

Theorem 3.6. *The functions $a_{m,n}(x) = \psi(x, m2^{-n)})X_{[0,2^n)}(x)$, $m \in N$, $n \in Z$, for each $\alpha > 0$ are eigenfunctions of the operators J_α and D^a with eigenvalues $2^{-r\alpha}$ and $2^{r\,\alpha}$ respectively. Here X_E is indicator function of the set E and $r = r\,(m,n) \in Z$ is uniquely determined by the imbedding $[m2^{-n}, (m+1)2{-}n] \subset [2^r, 2^{r+1})$.*

Let us denote by $L_{J_\alpha}(R_+)$ or $L_{D^\alpha}(R_+)$ the natural domain of the operator J_α or D^α respectively, i.e. the set of all functions $f \in L(R_+)$ for which $J_\alpha(f)$ or $D^\alpha(f)$ respectively exists. It is evident that $L_{J_\alpha}(R_+)$ and $L_{D^\alpha}(R_+)$ are linear subspaces in $L(R_+)$.

It follows from the Theorem 3.6 that

$$J_\alpha(a_{1,\,n}) = 2^{n\,\alpha}a_{1,\,n}, \quad D^\alpha(a_{1,\,n}) = 2^{-n\,\alpha}a_{1,\,n}, \quad n \in Z, \quad \alpha > 0.$$

Therefore we have

Corollary 3.2. *The linear operators $J_\alpha \colon L_{J_\alpha}(R_+) \to L(R_+)$ and $D^\alpha \colon L_{D^\alpha}(R_+) \to L(R_+)$ are unbounded for each $\alpha > 0$.*

Let us define the pointwise dyadic derivative of fractional order. According to the lemma 3.2 we have $\Lambda_n^\alpha \in L^\infty(R_+) \cap L(R_+)$ for $\alpha > 0$, $n \in Z$. Therefore the dyadic convolution $(\Lambda_n^\alpha * f)\,(x)$ exists at each point $x \in R_+$ for all $\alpha > 0$, $n \in Z$, if $f \in L(R_+)$ or $f \in L^\infty(R_+)$. Taking into account this fact we may to introduce the following definition.

Definition 3.3. Let $\alpha > 0$, $x \in R_+$ and $f \in L(R_+)$ or $f \in L^\infty(R_+)$. If there exists finite limit $d^\alpha(f)\,(x) \equiv \lim\limits_{n \to +\infty} (\Lambda_n^\alpha * f)(x)$, then we shall say that the function f has the modified dyadic derivative (MDD) $d^\alpha(f)(x)$ of order α at the point x.

Theorem 3.7. *For each $\alpha > 0$ and fixed $y \in R_+$ the Walsh generalized function $\psi_y(\cdot)$ has MDD of order α at each point $x \in R_+$. More precisely, $d^\alpha(\psi_0)(x) \equiv 0$ on R_+ and $d^\alpha(\psi_y)(x) = (h(y))^{-\alpha}\psi_y(x)$ for $x \in R_+$, $y > 0$.*

For the case $\alpha = 1$ these results were published in [21].

Theorem 3.8. *If $\alpha > 0$ and the function $f \in L(R_+)$ is such that $(h(x))^{-\alpha}\tilde{f}(x) \in L(R_+)$, then at each point $x \in R_+$ it has MDD of order α equal to $\int_0^{+\infty}(h(y))^{-\alpha}\tilde{f}(y)\psi(x,\,y)\,dy$.*

Theorem 8 is an analogue of the following theorem of P.L. Butzer and H.J. Wagner [4].

Theorem 3.9. *Under the assumption* $\sum\limits_{n=0}^{\infty} n|\alpha_n| < +\infty$ *the series* $\sum\limits_{n=0}^{\infty} a_n w_n(x)$ *is absolutely and uniformly convergent on* $[0, 1)$ *to a function* f*, which has dyadic derivative* $d^{(1)}f(x)$ *for all* $x \in [0, 1)$ *and* $d^{(1)}f(x) = \sum\limits_{n=0}^{+\infty} n\, a_n w_n(x)$.

Pointwise and strong dyadic term by term differentiation of Walsh series was investigated by W.R. Wade [23], V.A. Skvorčov and W.R. Wade [24], C.H. Powell and W.R. Wade [25].

Let us define the pointwise dyadic integral of fractional order. According to the corollary 3.1 we have $W_n^{\alpha} \in L(R_+)$. Therefore the dyadic convolution $(W_n^{\alpha} * f)(x)$ exists at each point $x \in R_+$ for all $\alpha > 0$, $n \in Z$, if $f \in L^{\infty}(R_+)$. Taking into account we may to introduce the following definition.

Definition 3.4. If $\alpha > 0$, $x \in R_+$ and for a function $f \in L^{\infty}(R_+)$ there exists finite limit $j_{\alpha}(f)(x) \equiv \lim\limits_{n\to+\infty}(f * W_n^{\alpha})(x)$, then we say that the function f has modified dyadic integral (MDI) of order α at the point x equal to $j_{\alpha}(f)(x)$.

Theorem 3.10. *For each* $\alpha > 0$ *and fixed* $y \in R_+$ *the generalized Walsh function* $\psi_y(\cdot)$ *has MDI of order* α *at each point* $x \in R_+$*. More precisely,* $j_{\alpha}(\psi_0)(x) \equiv 0$ *on* R_+ *and* $j_{\alpha}(\psi_y)(x) = (h(y))^{\alpha}\psi_y(x)$ *for* $x \in R_+$, $y > 0$.

Definition 3.5. Let $\alpha > 0$ and $x \in R_+$. If for the function $f \in L(R_+)$ there exists finite limit

$$d^{(\alpha)}(f)(x) \equiv \lim_{n\to+\infty} \int_0^{2^n} (h(y))^{-\alpha}\tilde{f}(y)\psi(x,y)\,dy,$$

then we shall call it dyadic α-derivative of the function f at the point x.

If $f \in L(R_+)$ and $(h)^{-\alpha}\tilde{f} \in L(R_+)$, then dyadic α-derivative of the function f exists at every point $x \in R_+$ and $d^{(\alpha)}(f)(x) = \int_{R_+}(h(y))^{-\alpha}\tilde{f}(y)\psi(x,y)\,dy$.

Definition 3.6. Let be given $\alpha > 0$, $x \in R_+$ and $f \in L(R_+)$. If there exists finite limit

$$j_{(\alpha)}(f)(x) \quad \equiv \quad \lim_{n\to+\infty} \int_{2^{-n}}^{2^n} (h(y))^{\alpha}\tilde{f}(y)\psi(x,y)\,dy,$$

then we shall call it dyadic α-integral of the function f at the point x.

If $f \in L(R_+)$ and $(h)^{\alpha}\tilde{f} \in L(R_+)$, then dyadic $\alpha-$integral of the function f exists at every point $x \in R_+$ and $j_{(\alpha)}(f)(x) = \int_{R_+}(h(y))^{\alpha}\tilde{f}(y)\psi(x,y)\,dy$.

The following theorem may be considered as a dyadic analogue of the classical theorem of Lebesgue on pointwise differentiation of Lebesgue integral.

Theorem 3.11. *Let* $\alpha > 0$ *and the function* $f \in L(R_+)$ *has MSDI* $J_{\alpha}(f)$ *of order* α *in the space* $L(R_+)$*. Then at each Lebesgue point* $x \in R_+$ *of the function* f*, hence a.e. on* R_+*, the equality* $d^{(\alpha)}(J_{\alpha}(f))(x) = f(x)$ *holds.*

Theorem 3.12. *Let* $\alpha > 0$ *and the function* $f \in L(R_+)$ *has MSDD* $D^{(\alpha)}(f)$ *of order* α *in the space* $L(R_+)$*. Then at each Lebesgue point* $x \in R_+$ *of the function* f*, hence a.e. on* R_+*, the equality* $j_{(\alpha)}(D^{(\alpha)}(f))(x) = f(x)$ *holds.*

Dyadic integral in dyadic Hardy space. The following theorem of Hardy [16] is well known.

Theorem 3.13. *If the function* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ *belongs to the Hardy space* $H(|z| < 1)$ *on the unit disc* $|z| < 1$ *of the complex plane* C *and* $f(e^{i\,t})$ *is its* *boundary function on the unit circle* $|z| = 1$, *then*

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq \frac{1}{2} \int_0^{2\pi} |f(e^{it})| \, dt.$$

An analogue of this theorem has been proved by E. Hille and J.D. Tamarkin [17].

Theorem 3.14. *If the function* $f(z)$ *belongs to the Hardy space* $H(R^2_+)$ *on the upper half-plane* $R^2_+ = \{z \in C : \operatorname{Im} z > 0\}$ *and* $\hat{f}(x)$ *is Fourier transform of its boundary function* $f(x)$ *on real axis, then the following inequality holds*

$$\int\limits_{R_+} \frac{|\hat{f}(x)|}{x} \, dx \leq \frac{1}{2} \int_{-\infty}^{+\infty} |f(x)| \, dx.$$

An extension of the Theorem 3.14 on Hardy space $H^p(R)$, $0 < p \leq 1$, is also known.

Theorem 3.15. *If* $f \in H^p(R)$, *then*

$$\int_{R_+} |\hat{f}(x)|^p x^{p-2} \, dx \leq C_p \|f\|^p_{H^p(R)}.$$

(See [18], *p.342).*

Problem. *What are the least constants in right-hand sides of the inequalities of the Theorems 3.13–3.15?*

N.R. Ladhawala [19] proved a dyadic analogue of the Theorem 3.13 in the following form.

Theorem 3.16. *If the function* f *belongs to dyadic Hardy space* $H([0,1))$, *then*

$$\sum_{n=1}^{+\infty} \frac{|\hat{f}(n)|}{n} \leq 12\sqrt{2}\|f\|_H,$$

where $\hat{f}(n)$ *are Walsh–Fourier coefficients of the function* f.

A dyadic analogue of the Theorem 3.14 was proved in [20]:

Theorem 3.17. *If* $f \in H(R_+)$, *then the following inequality holds*

$$\int_{R_+} \frac{|\tilde{f}(x)|}{x} \, dx \leq 50\sqrt{2}\|f\|_{H(R_+)}.$$

Problem. *1) What is the least constant in right-hand side of the former inequality? 2) To extend this inequality on dyadic Hardy space* $H^p(R_+)$, $0 < p < 1$, *i.e. to prove dyadic analogue of the Theorem 3.15.*

Definition 3.7. Let us define the functions $(f * W_n^\alpha)^*(x)$, $n \in Z_+$, by the equality

$$\begin{aligned}(f * W_n^\alpha)^*(x) &\equiv \int_{2^{-n} \leq t \leq 2^n} (f * W_n^\alpha)\tilde{}(t) \psi_x(t) \, dt \\ &= \int_{2^{-n} \leq t \leq 2^n} \tilde{f}(t)(h(t))^\alpha \psi_x(t) \, dt, \quad x \in R_+,\end{aligned}$$

where $f \in L(R_+)$. If there exists the limit

$$J^*_\alpha(f)(x) \equiv \lim_{n \to +\infty} (f * W_n^\alpha)^*(x),$$

which is uniform on R_+, then we say that the function f has uniform modified dyadic integral (UMDI) of order α on R_+.

As a corollary from the Theorem 3.17 we obtain:

Theorem 3.18. *Each function $f \in H(R_+)$ has UMDI of first order on R_+. More precisely, the operator $J_1^* : H(R_+) \to C_w(R_+)$ is bounded and*

$$\|J_1^*(f)\|_{C_W(R_+)} \leq 100\sqrt{2}\|f\|_{H(R_+)}.$$

In [21] the following theorem is proved.

Theorem 3.19. *The functions $\psi(x, m2^{-n})\mathrm{X}_{[0,\,2^n)}(x)$, $m \in N$, $n \in Z$, belong to the space $H(R_+)$ and their linear hull L is dense in this space.*

If a function $f \in H(R_+)$ has modified dyadic strong integral $J_1(f)$ in the space $L(R_+)$, then $J_1(f)(x) = J_1^*(f)(x)$ a.e. on R_+. But the functions

$$\psi(x, m2^{-n})\mathrm{X}_{[0,\,2^n)}(x), \qquad m \in N, \quad n \in Z,$$

have modified strong dyadic integral of first order in the space $L(R_+)$ (see Theorem 3.6 above). Therefore by setting $\tilde{J}_1(f) \equiv J_1(f)$ we can deduce from the Theorems 3.1, 3.17 and 3.19 the following result.

Theorem 3.20. *The operator $\tilde{J}_1 : L \to L(R_+)$ is bounded and his operator norm does not exceed $100\sqrt{2}$. Therefore it can be extended continuously on the space $H(R_+)$ without changing its operator norm.*

This theorem may be considered as a dyadic analogue of the Theorem 3.14.

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Department of Higher Mathematics,
Moscow Engineering Physics Institute,
115409, Moscow, Kashirskoe shosse, 31,
Russia
E-mail address: golubov@mail.mipt.ru