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THE LAMINARY MODEL OF THE EXPLODED DESCARTES-PLANE

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ABSTRACT. Using exploded numbers, a formal explosion of the familiar Descartes-plane by the explosion of the coordinates of its points is easily imaginable. Moreover, the familiar Descartes-plane is a proper subset of this exploded Descartes-plane. By this model we can say that the exploded Descartes-plane exists.

1. PRELIMINARY

The concept of exploded real numbers was introduced in [1], with the following postulates and requirements:

Postulate of extension: The set of real numbers is a proper subset of the set of exploded real numbers. For any real number x there exists one exploded real number which is called exploded x or the exploded of x. Moreover, the set of exploded x is called the set of exploded real numbers.

Postulate of unambiguity: For any pair of real numbers x and y, their explodeds are equal if and only if x is equal to y.

Postulate of ordering: For any pair of real numbers x and y, exploded x is less than exploded y if and only if x is less than y.

Postulate of super-addition: For any pair of real numbers x and y, the super-sum of their explodeds is the exploded of their sum.

Postulate of super-multiplication: For any pair of real numbers x and y, the super-product of their explodeds is the exploded of their product.

Requirement of equality for exploded real numbers: If x and y are real numbers then x as an exploded real number equals to y as an exploded real number if they are equal in the traditional sense.

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Requirement of ordering for exploded real numbers: If x and y are real numbers then x as an exploded real number is less than y as an exploded real number if x is less than y in the traditional sense.

Requirement of monotonity of super-addition: If u and v are arbitrary exploded real numbers and u is less than v then, for any exploded real number w, u superplus w is less than v superplus w.

Requirement of monotonity of super-multiplication: If u and w are arbitrary exploded real numbers and u is less than v then, for any positive exploded real number w, u super-multiplied by w is less than v super-multiplied by w.

The field $(\overset{\sqcup}{R}, \circledast\!\!\!\!\oplus, \circledast\!\!\!\!\odot)$ of exploded real numbers is isomorphic with the field $(R, +, \cdot)$ of real numbers but super-operations are not extensions of traditional operations. Although, they are not different in the sense of abstract algebra, it is important that $R \subset \overset{\sqcup}{R}$. Using the explosion

$$\overset{\sqcup}{x} = \operatorname{area\,th} x \left(= \ln \frac{1+x}{1-x}\right); \quad |x| < 1, \tag{1.1}$$

we have that set of explodeds of $x \in (-1, 1) = \underset{\sqcup}{R}$ is just R. The exploded of $x \in (R - \underset{\sqcup}{R})$, denoted by the symbol $\overset{\sqcup}{x}$ was called invisible exploded real number. So, the set $\overset{\sqcup}{R}$ contains visible exploded real numbers, given by (1.1), and invisible exploded real numbers, which are symbols, merely.Considering the compression

$$\underset{\sqcup}{x} = \operatorname{th} x \left(= \frac{e^x - e^{-x}}{e^x + e^{-x}}\right); \quad x \in R, \tag{1.2}$$

we have $\overset{\sqcup}{(\underset{\sqcup}{x})} = x;\ x \in R$, $\underset{\sqcup}{(\overset{\sqcup}{x})} = x;\ x \in \underset{\sqcup}{R}$. By the Postulates of Extension and Unambiguity we may denote by $\underset{\sqcup}{u}$ the compressed of $u \in \overset{\sqcup}{R}$ independently of the fact that u is an visible or invisible exploded real number. Of course, $\underset{\sqcup}{u} \in R$ in both cases. Moreover, we can use the inversion identities

$$\overset{\sqcup}{(\underset{\sqcup}{u})} = u;\ u \in \overset{\sqcup}{R} \quad \text{and} \quad \underset{\sqcup}{(\overset{\sqcup}{x})} = x;\ x \in R, \tag{1.3}$$

too. Using (1.3), by Postulates of Super-addition and Super-multiplication

$$u \circledast\!\!\!\!\oplus v = \overset{\sqcup}{\overbrace{\underset{\sqcup}{u} + \underset{\sqcup}{v}}};\ u, v \in \overset{\sqcup}{R} \tag{1.4}$$

and

$$u \circledast\!\!\!\!\odot v = \overset{\sqcup}{\overbrace{\underset{\sqcup}{u} \cdot \underset{\sqcup}{v}}};\ u, v \in \overset{\sqcup}{R} \tag{1.5}$$

are obtained, so we are able to compute with invisible exploded real numbers, too. To answer the question whether invisible exploded real numbers exist, some models for the ordered field of exploded real numbers were given in [2] and [4].

The abstract exploded Descartes-plane was introduced in [3] by the following way:

$$\overset{\sqcup}{R^2} = \{(\overset{\sqcup}{x}, \overset{\sqcup}{y}) : (x, y) \in R^2\}. \tag{1.6}$$

Considering the operations

$$U \underset{\times}{\overset{\times}{\oplus}} V = (u_1 \underset{\times}{\overset{\times}{\oplus}} v_1, u_2 \underset{\times}{\overset{\times}{\oplus}} v_2);\ U = (u_1, u_2), V = (v_1, v_2) \in \overset{\sqcup}{R^2} \tag{1.7}$$

and

$$c \underset{\times}{\overset{\times}{\odot}} U = (c \underset{\times}{\overset{\times}{\odot}} u_1, c \underset{\times}{\overset{\times}{\odot}} u_2); c \in \overset{\sqcup}{R};\ U = (u_1, u_2) \in \overset{\sqcup}{R^2} \tag{1.8}$$

the set $\overset{\sqcup}{R}{}^2$ is a super-linear space. Moreover, by the super-inner product

$$U \underset{\times}{\overset{\times}{\odot}} V = (u_1 \underset{\times}{\overset{\times}{\odot}} v_1) \underset{\times}{\overset{\times}{\oplus}} (u_2 \underset{\times}{\overset{\times}{\oplus}} v_2); \tag{1.9}$$

$$U = (u_1, u_2), V = (v_1, v_2) \in \overset{\sqcup}{R^2},$$

which yields the super-norm $\|U\|_{\underset{R^2}{\sqcup}}$ and super-distance $d_{\underset{R^2}{\sqcup}}(U, V)$ in the usual way, we have that the set $\overset{\sqcup}{R^2}$ is a super-Euclidean space.

If $X = (x, y) \in \underset{\sqcup}{R^2} (= \{(u, v) \in R^2 : -1 < u < 1; -1 < v < 1\})$ then (1.1) gives

$$\overset{\sqcup}{X} = (\overset{\sqcup}{x}, \overset{\sqcup}{y}) = (\operatorname{area\,th} x, \operatorname{area\,th} y) \in R^2, \tag{1.10}$$

otherwise the exploded point $\overset{\sqcup}{x}$ is invisible. By (1.1) we have

$$R^2 \subset \overset{\sqcup}{R^2}. \tag{1.11}$$

2. Laminary explosion

Our aim is to find a model of the super-Euclidean space $\overset{\sqcup}{R^2}$ in which invisible points become visible, too. For any $X = (x, y) \in R^2$ we give its *laminary exploded* by

$$\overset{\sqcup}{(x, y)}^{lam} = ((\operatorname{sgn} x) \operatorname{area\,th}\{|x|\}, (\operatorname{sgn} y) \operatorname{area\,th}\{|y|\}, d_X) \in R^3 \tag{2.1}$$

where $[x]$ is the greatest integer number which is less than or equal to x, $\{x\} = x - [x]$, and

$$d_X = (\operatorname{sgn} x)[|x|] + (\operatorname{sgn} y)\frac{[|y|]}{2([|y|] + 1)}. \tag{2.2}$$

With respect to the coordinates of $\overset{\sqcup}{(x, y)}^{lam}$ we mention the following lemmas:

Lemma 2.3. *(See* [2], *Theorem 1.1.) For any pair x, ξ of real numbers, the complex numbers*

$$(\operatorname{sgn} x)(\operatorname{area\,th}\{|x|\} + i[|x|]) = (\operatorname{sgn} \xi)(\operatorname{area\,th}\{|\xi|\} + i[|\xi|])$$

if and only if $x = \xi$.

Lemma 2.4. *(See* [4], *Theorem 2.) For any pair y, η of real numbers, the complex numbers*

$$(\operatorname{sgn} y)\left(\operatorname{area\,th}\{|y|\} + \frac{i}{2}\frac{[|y|]}{[|y|]+1}\right) = (\operatorname{sgn} \eta)\left(\operatorname{area\,th}\{|\eta|\} + \frac{i}{2}\frac{[|\eta|]}{2[|\eta|]+1}\right)$$

if and only if $y = \eta$.

Theorem 2.5 (Theorem of Unambiguity)**.** *For any pair of (x, y), $(\xi, \eta) \in R^2$, $\overbrace{(x,y)}^{lam} = \overbrace{(\xi,\eta)}^{lam}$ if and only if $(x, y) = (\xi, \eta)$.*

Proof of Theorem 2.5. Necessity. Assuming that $\overbrace{(x,y)}^{lam} = \overbrace{(\xi,\eta)}^{lam}$ by (2.1) and (2.2) we have

$$(\operatorname{sgn} x)(\operatorname{area\,th}\{|x|\}) = (\operatorname{sgn} \xi)(\operatorname{area\,th}\{|\xi|\}), \quad x, \xi \in R \tag{2.6}$$

$$(\operatorname{sgn} y)(\operatorname{area\,th}\{|y|\}) = (\operatorname{sgn} \eta)(\operatorname{area\,th}\{|\eta\}); \quad y, \eta \in R \tag{2.7}$$

and

$$\begin{aligned}(\operatorname{sgn} x)[|x|] + (\operatorname{sgn} y)\frac{[|y|]}{2([|y|]+1)}& \\ = (\operatorname{sgn} \xi)[|\xi|] + (\operatorname{sgn} \eta)&\frac{[|\eta|]}{2([|\eta|]+1)}; \quad x, y, \xi, \eta \in R.\end{aligned} \tag{2.8}$$

As

$$-\frac{1}{2} < (\operatorname{sgn} y)\frac{[|y|]}{2([|y|]+1)},\ (\operatorname{sgn} \eta)\frac{[|\eta|]}{2([|\eta|]+1)} < \frac{1}{2},$$

(2.8) yields

$$(\operatorname{sgn} x)[|x|] = (\operatorname{sgn} \xi)[|\xi|]; \quad x, \xi \in R. \tag{2.9}$$

By (2.6) and (2,9) Lemma 2.3 says that $x = \xi$. Considering (2.9), the equation (2.8) reduces to

$$(\operatorname{sgn} y)\frac{[|y|]}{2([|y|]+1)} = (\operatorname{sgn} \eta)\frac{[|y|]}{2([|\eta|]+1)},$$

which together with (2.7) by Lemma 2.4 gives that $y = \eta$.

Sufficiency. It is evident by (2.1) and (2.2). □

Theorem 2.10 (Theorem of Completeness)**.** *If the point $U=(x,y,d)$ belongs to the set*

$$S^* = \Big\{ (x,y,d) \in R^3 : n \cdot x \geq 0, m \cdot y \geq 0,$$
$$d = n + \frac{m}{2(|m|+1)};\ n,m = 0, \pm 1, \pm 2, \dots \Big\}$$

then $\overbrace{(n+\operatorname{th}x, m+\operatorname{th}y)}^{lam} = (x,y,d)$.

Proof of Theorem 2.10. As

$$\left| \frac{m}{2(|m|+1)} \right| < \frac{1}{2};\ m = 0, \pm 1, \pm 2,$$

the integer numbers n, m are unambiguously determined by d. Let us consider the two-dimensional point

$$X_U = (n+\operatorname{th}x, m+\operatorname{th}y) \tag{2.11}$$

and compute the first coordinate of laminary exploded $\overbrace{X_U}^{lam}$. By (2.1) we can write: If n is a positive integer number then $x \geq 0$, and

$$\operatorname{sgn}(n+\operatorname{th}x) \cdot \operatorname{area\,th}\{|n+\operatorname{th}x|\} = \operatorname{area\,th}\{n+\operatorname{th}x\} = \operatorname{area\,th}(\operatorname{th}x) = x.$$

If $n=0$ then x is an arbitrary real number, and

$$\begin{aligned}\operatorname{sgn}(n+\operatorname{th}x) \cdot \operatorname{area\,th}\{|n+\operatorname{th}x|\} &\\ = \operatorname{sgn}(\operatorname{th}x) \cdot \operatorname{area\,th}\{|\operatorname{th}x|\} &= \operatorname{sgn}(\operatorname{th}x) \cdot \operatorname{area\,th}|\operatorname{th}x| \\ &= \operatorname{area\,th}(|\operatorname{th}x| \cdot \operatorname{sgn}(\operatorname{th}x)) = \operatorname{area\,th}(\operatorname{th}x) = x.\end{aligned}$$

If n is a negative integer number then $x \leq 0$, and

$$\begin{aligned}&\operatorname{sgn}(n+\operatorname{th}x) \cdot \operatorname{area\,th}\{|n+\operatorname{th}x|\} \\ &\quad = -\operatorname{area\,th}\{-n-\operatorname{th}x\} = -\operatorname{area\,th}(-\operatorname{th}x) = \operatorname{area\,th}(\operatorname{th}x) = x.\end{aligned}$$

For the second coordinate of laminary exploded $\overbrace{X_U}^{lam}$,

$$\operatorname{sgn}(m+\operatorname{th}y) \cdot \operatorname{area\,th}\{|m+\operatorname{th}y|\} = y$$

is obtained in a similar way. Turning to the third coordinate of laminary exploded $\overbrace{X_U}^{lam}$, by (2.2) we write for the first member of d_{X_U}:

$$\operatorname{sgn}(n+\operatorname{th}x) \cdot [|n+\operatorname{th}x|] = \begin{cases} [n+\operatorname{th}x] = n; n = 1,2,3,4,, \\ (\operatorname{sgn} x) \cdot [|\operatorname{th}x|] = 0;\ n = 0, \\ -[-n-\operatorname{th}x] = n;\ n = -1,-2,-3. \end{cases}$$

Moreover, for the second one:

$$\operatorname{sgn}(m+\operatorname{th}y) \cdot \frac{[|m+\operatorname{th}y|]}{2([|m+\operatorname{th}y|]+1)} =$$

$$= \begin{cases} \frac{m}{2(|m|+1)}; \ m = 1, 2, 3, 4, , \\ (\operatorname{sgn} y) \cdot \frac{[|\mathrm{th} y|]}{2([|\mathrm{th} y|]+1)} = 0; \ m = 0, \\ -\frac{[-m-\mathrm{th} y]}{2([-m-\mathrm{th} y]+1)} = -\frac{-m}{2(-m+1)} = \frac{m}{2(|m|+1)}; \ m = -1, -2, -3, \end{cases}$$

so,

$$d_{X_U} = \operatorname{sgn}(n + \mathrm{th} x) \cdot [|n + \mathrm{th} x|] + \operatorname{sgn}(m + \mathrm{th} y) \cdot \frac{[|m + \mathrm{th} y|]}{2([m + \mathrm{th} y] + 1)} = n + \frac{m}{2(|m|+1)} = d$$

which completes our proof. □

By Theorems of Unambiguity and Completeness we give the *laminary model of the exploded two — dimensional space* as a set of laminary explodeds of the points of the two-dimensional Euclidean space:

$$(2.12) \qquad \overset{\sqcup\, lam}{R^2} = \Big\{ (x, y, d) \in R^3 : n \cdot x \geq 0, m \cdot y \geq 0, \ d = n + \frac{m}{2(|m|+1)}; n, m = 0, \pm 1, \pm 2, \dots \Big\}.$$

Moreover, by (2.11) for any $U = (x, y, d) \in \overset{\sqcup\, lam}{R^2}$ we define its *laminary compressed*:

$$(2.13) \qquad \underset{\sqcup\, lam}{U} = (n + \mathrm{th} x, m + \mathrm{th} y) \in R^2.$$

Clearly, the set

$$S^{**} = \{(x, y, 0) \in R^3 : x, y \in R\}$$

Is a subspace of the euclidian space R^3 with its traditional linear operations, inner product, norm and metric. We identify it with R^2, that is $R^2 \equiv S^{**}$. Casting a glance at (1.11) we have

$$(2.14) \qquad R^2 \subset \overset{\sqcup\, lam}{R^2} \subset R^3.$$

Theorem 2.10 with (2.13) yields the identity

$$(2.15) \qquad \overset{\sqcup\, lam}{\Big(\underset{\sqcup\, lam}{U} \Big)} = U; \quad U \in \overset{\sqcup\, lam}{R^2}.$$

Hence, denoting $U = \overset{\sqcup\, lam}{X}$; $X \in R^2$ Theorem 2.5 by (2.15) says that $\underset{\sqcup\, lam}{U} = X$ and so,

$$(2.16) \qquad \underset{\sqcup\, lam}{\Big(\overset{\sqcup\, lam}{X} \Big)} = X; \quad X \in R^2.$$

Definition 2.17. *For any pair of (x, y), $(\xi, \eta) \in R^2$ we say that the* laminary super-sum *of their laminary explodeds will be:*

$$\overset{\sqcup\, lam}{(x, y)} \underset{lam}{\circledast} \overset{\sqcup\, lam}{(\xi, \eta)} =$$

$= ((\operatorname{sgn}(x+\xi))\operatorname{area\,th}\{|x+\xi|\}, (sgn(y+\eta))\operatorname{area\,th}\{|y+\eta|\}, d_+) \in R^3$

with

$$d_+ = (\operatorname{sgn}(x+\xi))[|x+\xi|] + (\operatorname{sgn}(y+\eta))\frac{[|y+\eta|]}{2([|y+\eta|]+1)}.$$

Considering $X=(x,y)$; $\Psi=(\xi,\eta)\in R^2$, by (2.1) and (2.2) Definition 2.17 says

$$\overset{\sqcup lam}{X} \underset{lam}{\circledast} \overset{\sqcup lam}{\Psi} = \overset{\sqcup lam}{\overline{X+\Psi}};\quad X,\Psi\in R^2. \tag{2.18}$$

Denoting $X = \underset{\sqcup lam}{U}$, $\Psi = \underset{\sqcup lam}{\Phi}$; $U,\Phi\in \overset{\sqcup lam}{R^2}$, (2.15) and (2.18) yield

$$U \underset{lam}{\circledast} \Phi = \overset{\sqcup lam}{\overline{\underset{\sqcup lam}{U} + \underset{\sqcup lam}{\Phi}}};\quad U,\Phi\in \overset{\sqcup lam}{R^2} \tag{2.19}$$

Clearly, by (2.18) and (2,19) we have

Theorem 2.20. *The laminary super-addition has the following properties:*

- *commutativity:* $U \underset{lam}{\circledast} \Phi = \Phi \underset{lam}{\circledast} U$; $U,\Phi\in \overset{\sqcup lam}{R}$
- *associativity:* $(U \underset{lam}{\circledast} V) \underset{lam}{\circledast} \Phi = U \underset{lam}{\circledast} (V \underset{lam}{\circledast} \Phi)$; $U,V,\Phi\in \overset{\sqcup lam}{R^2}$
- *for any* $U\in \overset{\sqcup lam}{R^2}$: $U \underset{lam}{\circledast} O = U$, *where* $O=(0,0,0)=\overset{\sqcup lam}{\overline{(0,0)}}$
- *for any* $U\in \overset{\sqcup lam}{R^2}$: $U \underset{lam}{\circledast} (-U) = O$. *(If* $U=(x,y,d)$ *then* $-U=(-x,-y,-d)$ *and see (2.12).)*

3. Explosion of axes

Having (2.1) and (2.2) we may speak of laminary exploded of real numbers in a double sense. Namely

$$\overset{\sqcup lam}{\gamma} = \overset{\sqcup lam}{\overline{(\gamma,0)}} = ((\operatorname{sgn}\gamma)\operatorname{area\,th}\{|\gamma|\}, 0, (\operatorname{sgn}\gamma)[|\gamma|]);\ \gamma\in R, \tag{3.1}$$

and

$$\overset{\square lam}{\gamma} = \overset{\sqcup lam}{\overline{(0,\gamma)}} = \left(0, (\operatorname{sgn}\gamma)\operatorname{area\,th}\{|\gamma|\}, (\operatorname{sgn}\gamma)\frac{[|\gamma|]}{2([|\gamma|]+1)}\right);\ \gamma\in R. \tag{3.2}$$

Explodeds $\overset{\sqcup lam}{\gamma}$ are situated on the *exploded x-axis*

$$\overset{\sqcup lam}{R} = \{(x,0,d)\in R^3 : n\cdot x\geq 0,\ d=n; n=0,\pm1,\pm2,\}, \tag{3.3}$$

while $\overset{\square lam}{\gamma}$ are on the *exploded y-axis*

$$\overset{\square lam}{R} = \left\{(0,y,d)\in R^3 : m\cdot y\geq 0, d=\frac{m}{2(|m|+1)};\ m=0,\pm1,\pm2,\right\} \tag{3.4}$$

By (3.1) and Lemma 2.3 we have that the mapping $\gamma \longleftrightarrow \overset{\sqcup lam}{\gamma}$ is mutually unambiguous between R and $\overset{\sqcup lam}{R}$. Moreover, by the definitions

(3.5)

$$\overset{\sqcup lam}{\gamma} \underset{lam}{\circledast} \overset{\sqcup lam}{\delta} = ((\mathrm{sgn}(y+\delta))\,\mathrm{area\,th}\{|\gamma+\delta|\}, 0, (\mathrm{sgn}(\gamma+\delta))[|\gamma+\delta|]);\ \gamma, \delta \in R$$

and

(3.6)

$$\overset{\sqcup lam}{\gamma} \underset{lam}{\odot} \overset{\sqcup lam}{\delta} = ((\mathrm{sgn}(\gamma \cdot \delta))\,\mathrm{area\,th}\{|\gamma \cdot \delta|\}0, (\mathrm{sgn}(\gamma+\delta))[|\gamma+\delta|]);\ \delta \in R$$

the isomorphism $(R, +, \cdot) \Longrightarrow (\overset{\sqcup lam}{R}, \underset{lam}{\oplus}, \underset{lam}{\odot})$ is obtained. Considering (3.1), definitions (3.5) and (3.6) yield the identities

$$\overset{\sqcup lam}{\gamma} \underset{lam}{\oplus} \overset{\sqcup lam}{\delta} = \overset{\sqcup lam}{\gamma+\delta};\ \gamma, \delta \in R \tag{3.7}$$

and

$$\overset{\sqcup lam}{\gamma} \underset{lam}{\odot} \overset{\sqcup lam}{\delta} = \overset{\sqcup lam}{\gamma \cdot \delta};\ \gamma, \delta \in R, \tag{3.8}$$

respectively. Practically, we can use the identities

$$\overset{\sqcup lam}{\gamma} \underset{lam}{\ominus} \overset{\sqcup lam}{\delta} = \overset{\sqcup lam}{\gamma - \delta};\ \gamma, \delta \in R$$

and

$$\overset{\sqcup lam}{\gamma} \underset{lam}{\odot} \overset{\sqcup lam}{\delta} = \overset{\sqcup lam}{\gamma : \delta};\ \gamma, \delta \neq 0 \in R,$$

too. Moreover, $\overset{\sqcup lam}{R^2}$ is an ordered field, with the ordering

$$\overset{\sqcup lam}{\gamma} < \overset{\sqcup lam}{\delta} \Longleftrightarrow \gamma < \delta; \gamma, \delta \in R.$$

We define the *laminary super-absolute value*:

$$|\overset{\sqcup lam}{\gamma}| = \begin{cases} \overset{\sqcup lam}{\gamma}, & \overset{\sqcup lam}{\gamma} > \overset{\sqcup lam}{0}\ (= (0,0,0)) \\ \overset{\sqcup lam}{0}, & \overset{\sqcup lam}{\gamma} = \overset{\sqcup lam}{0} \\ -(\overset{\sqcup lam}{\gamma})(= \overset{\sqcup lam}{-\gamma}), & \overset{\sqcup lam}{\gamma} < \overset{\sqcup lam}{0}. \end{cases}$$

By (3.1) we have the identity

$$|\overset{\sqcup lam}{\gamma}| = \overset{\sqcup lam}{|\gamma|};\quad \gamma \in R. \tag{3.9}$$

Be careful, because $|\overset{\sqcup lam}{\gamma}| \neq \|\overset{\sqcup lam}{\gamma}\|_{R_3}$

Remark 3.10. *By (3.2) and Lemma 2.4 we have that the mapping* $\gamma \leftrightarrow \overset{\square_{lam}}{\gamma}$ *is mutually unambiguous between* R *and* $\overset{\square_{lam}}{R}$. *Moreover, by the definitions*

$$\overset{\square_{lam}}{\gamma} \underset{lam}{\oplus} \overset{\square_{lam}}{\delta} = \left(0, (\mathrm{sgn}(\gamma+\delta))\,\mathrm{area\,th}\{|\gamma+\delta|\}, (\mathrm{sgn}(\gamma+\delta))\frac{[|\gamma+\delta|]}{2([|\gamma+\delta|]+1)}\right); \qquad \gamma, \delta \in R$$

and

$$\overset{\square_{lam}}{\gamma} \underset{lam}{\odot} \overset{\square_{lam}}{\delta} = \left(0, (\mathrm{sgn}(\gamma\cdot\delta))\,\mathrm{area\,th}\{|\gamma\cdot\delta|\}, (\mathrm{sgn}(\gamma\cdot\delta))\frac{[|\gamma\cdot\delta|]}{2([|\gamma\cdot\delta|]+1)}\right); \qquad \gamma, \delta \in R$$

the isomorphism $(R, +, \cdot) \longleftrightarrow (\overset{\square_{lam}}{R}, \underset{lam}{\oplus}, \underset{lam}{\otimes})$ *is obtained.*

Definition 3.11. *For any pair of* $\gamma \in R$, $(x, y) \in R^2$ *we say that the* laminary super-product *of their laminary explodeds will be:*

$$\overset{\sqcup_{lam}}{\gamma} \underset{lam}{\odot} \overset{\sqcup_{lam}}{(x,y)} = ((\mathrm{sgn}(\gamma\cdot x))\,\mathrm{area\,th}\{|\gamma\cdot x|\}, (\mathrm{sgn}(\gamma\cdot y))\,\mathrm{area\,th}\{|\gamma\cdot y|\}, d_*) \in R^3$$

with

$$d_* = (\mathrm{sgn}(\gamma\cdot x))[|\gamma\cdot x|] + (\mathrm{sgn}(\gamma\cdot y))\frac{[|\gamma\cdot y|]}{2([|\gamma\cdot y|]+1)}.$$

As $\gamma\cdot(x,y) = (\gamma\cdot x, \gamma\cdot y)$ *by Definition 3.11, (2.1) and (2.2) say*

$$\overset{\sqcup_{lam}}{\gamma} \underset{lam}{\odot} \overset{\sqcup_{lam}}{(x,y)} = \overset{\sqcup_{lam}}{\gamma\cdot(x,y)}; \quad \gamma \in R, (x,y) \in R$$

and writing that $X = (x, y) \in R^2$

$$\overset{\sqcup_{lam}}{\gamma} \underset{lam}{\odot} \overset{\sqcup_{lam}}{X} = \overset{\sqcup_{lam}}{\gamma\cdot X}; \quad \gamma \in R, (x,y) \in R^2 \tag{3.12}$$

is obtained. Considering $X = (x, y)$; $\Psi = (\xi, \eta) \in R^2$, *by (3.12) and (2.18) we have*

Theorem 3.13. *The laminary super-multiplication has the following properties:*

$$\overset{\sqcup_{lam}}{1} \underset{lam}{\odot} \overset{\sqcup_{lam}}{X} = \overset{\sqcup_{lam}}{X}; l \in R;\ x \in R^2$$

$$(\overset{\sqcup_{lam}}{\gamma} \underset{lam}{\odot} \overset{\sqcup_{lam}}{\delta}) \underset{lam}{\odot} \overset{\sqcup_{lam}}{X} = \overset{\sqcup_{lam}}{\gamma} \underset{lam}{\odot} (\overset{\sqcup_{lam}}{\delta} \underset{lam}{\odot} \overset{\sqcup_{lam}}{X}); \qquad \gamma, \delta \in R; X \in R^2$$

$$(\overset{\sqcup lam}{\gamma} \underset{lam}{\oplus} \overset{\sqcup lam}{\delta}) \underset{lam}{\odot} \overset{\sqcup lam}{X} = (\overset{\sqcup lam}{\gamma} \underset{lam}{\odot} \overset{\sqcup lam}{X}) \underset{lam}{\oplus} (\overset{\sqcup lam}{\delta} \underset{lam}{\odot} \overset{\sqcup lam}{X}),$$
$$\gamma, \delta \in R; X \in R^2$$

$$\overset{\sqcup lam}{\gamma} \underset{lam}{\odot} (\overset{\sqcup lam}{X} \underset{lam}{\oplus} \overset{\sqcup lam}{\Psi}) = (\overset{\sqcup lam}{\gamma} \underset{lam}{\odot} \overset{\sqcup lam}{X}) \underset{lam}{\oplus} (\overset{\sqcup lam}{\gamma} \underset{lam}{\odot} \overset{\sqcup lam}{\Psi}),$$
$$\gamma \in R; X, \Psi \in R^2.$$

Theorems 2.20 and 3.13 say that $\overset{\sqcup lam}{R^2}$ is a *super-linear space* over the field $\overset{\sqcup lam}{R}$.

4. Laminary super-Euclidean space

Definition 4.1. *For any pair of $X = (x, y)$; $\Psi = (\xi, \eta) \in R^2$ we say that the* laminary super-inner product *of their laminary explodeds will be:*

$$\overset{\sqcup lam}{X} \underset{lam}{\odot} \overset{\sqcup lam}{\Psi} = (\overset{\sqcup lam}{x} \underset{lam}{\odot} \overset{\sqcup lam}{\xi}) \underset{lam}{\oplus} (\overset{\sqcup lam}{y} \underset{lam}{\odot} \overset{\sqcup lam}{\eta}),$$
$$\overset{\sqcup lam}{X}, \overset{\sqcup lam}{\Psi} \in \overset{\sqcup lam}{R^2}$$

Using (3.7) and (3.8) we have the identity

$$\overset{\sqcup lam}{X} \underset{lam}{\odot} \overset{\sqcup lam}{\Psi} = \overset{\sqcup lam}{X \cdot \Psi}; \; X, \Psi \in R^2. \tag{4.2}$$

Using (2.18), (3.12) and (4.2) we have

Theorem 4.3. *The laminary super-inner product has the following properties:*

$$\overset{\sqcup lam}{X} \underset{lam}{\odot} \overset{\sqcup lam}{\Psi} = \overset{\sqcup lam}{\Psi} \underset{lam}{\odot} \overset{\sqcup lam}{X}; \overset{\sqcup lam}{X}, \overset{\sqcup lam}{\Psi} \in \overset{\sqcup lam}{R^2}$$

$$\overset{\sqcup lam}{\gamma} \underset{lam}{\odot} (\overset{\sqcup lam}{X} \underset{lam}{\odot} \overset{\sqcup lam}{\Psi}) = (\overset{\sqcup lam}{\gamma} \underset{lam}{\odot} \overset{\sqcup lam}{X}) \underset{lam}{\odot} \overset{\sqcup lam}{\Psi};$$
$$\overset{\sqcup lam}{\gamma} \in \overset{\sqcup lam}{R}; \overset{\sqcup lam}{X}, \overset{\sqcup lam}{\Psi} \in \overset{\sqcup lam}{R^2}$$

$$(\overset{\sqcup lam}{X} \underset{lam}{\oplus} \overset{\sqcup lam}{\Psi}) \underset{lam}{\odot} \overset{\sqcup lam}{\Psi} = (\overset{\sqcup lam}{X} \underset{lam}{\odot} \overset{\sqcup lam}{\Psi}) \underset{lam}{\oplus} (\overset{\sqcup lam}{\Psi} \underset{lam}{\odot} \overset{\sqcup lam}{\Psi}),$$
$$\overset{\sqcup lam}{X}, \overset{\sqcup lam}{\Psi}, \overset{\sqcup lam}{\Phi} \in \overset{\sqcup lam}{R^2}$$

$$\overset{\sqcup lam}{X} \underset{lam}{\odot} \overset{\sqcup lam}{X} \geq \overset{\sqcup lam}{0} = 0, 0, 0 \text{ see (3.1).}$$

Theorem 4.3 says that $\overset{\sqcup lam}{R^2}$ *is a* super-euclidian space.

In the usual way we have that $\overset{\sqcup^{lam}}{R^2}$ is a *super-normed space*, with the *laminary super-norm*

$$\| \overset{\sqcup^{lam}}{X} \|_{\overset{\sqcup^{lam}}{R^2}} = \overset{\sqcup^{lam}}{(\|X\|_{R^2})} \; ; \quad X \in R^2. \tag{4.4}$$

By (4.4), (3.1) and Lemma 2.3 we get the property

$$\| \overset{\sqcup^{lam}}{X} \|_{\overset{\sqcup^{lam}}{R^2}} = \overset{\sqcup^{lam}}{0} \iff \overset{\sqcup^{lam}}{X} = \overset{\sqcup^{lam}}{O} \; (= O). \tag{4.5}$$

By (3.12), (4.4) and (3.9) we get the property

$$\| \overset{\sqcup^{lam}}{\gamma} \underset{lam}{\odot} \overset{\sqcup^{lam}}{X} \|_{\overset{\sqcup^{lam}}{R^2}} = | \overset{\sqcup^{lam}}{\gamma} | \underset{lam}{\odot} \| \overset{\sqcup^{lam}}{X} \|_{\overset{\sqcup^{lam}}{R^2}} \; ; \tag{4.6}$$

$$\overset{\sqcup^{lam}}{\gamma} \in \overset{\sqcup^{lam}}{R}, \; \overset{\sqcup^{lam}}{X} \in \overset{\sqcup^{lam}}{R^2}.$$

By (2.18), (4.4) and (3.7) we get the property

$$\| \overset{\sqcup^{lam}}{X} \underset{lam}{\oplus} \overset{\sqcup^{lam}}{\Psi} \|_{\overset{\sqcup^{lam}}{R^2}} \leq \| \overset{\sqcup^{lam}}{X} \|_{\overset{\sqcup^{lam}}{R^2}} \underset{lam}{\oplus} \| \overset{\sqcup^{lam}}{\Psi} \|_{\overset{\sqcup^{lam}}{R^2}} \; ; \tag{4.7}$$

$$\overset{\sqcup^{lam}}{X}, \; \overset{\sqcup^{lam}}{\Psi} \in \overset{\sqcup^{lam}}{R^2}.$$

Moreover, $\overset{\sqcup^{lam}}{R^2}$ is a *super-metrical space*, with the *laminary super-distance*

$$d_{\overset{\sqcup^{lam}}{R^2}} (\overset{\sqcup^{lam}}{X}, \overset{\sqcup^{lam}}{\Psi}) = \overset{\sqcup^{lam}}{d_{R^2}(X, \Psi)} \; ; X, \Psi \in R^2. \tag{4.8}$$

Using (4.8), (3.1), Lemma 2.3 and Theorem 2.5 we get the property

$$d_{\overset{\sqcup^{lam}}{R^2}} (\overset{\sqcup^{lam}}{X}, \overset{\sqcup^{lam}}{\Psi}) = \overset{\sqcup^{lam}}{0} \iff \overset{\sqcup^{lam}}{X} = \overset{\sqcup^{lam}}{\Psi}. \tag{4.9}$$

Clearly,

$$d_{\overset{\sqcup^{lam}}{R^2}} (\overset{\sqcup^{lam}}{X}, \overset{\sqcup^{lam}}{\Psi}) = d_{\overset{\sqcup^{lam}}{R^2}} (\overset{\sqcup^{lam}}{\Psi}, \overset{\sqcup^{lam}}{X}). \tag{4.10}$$

By (4.8) and (3.7) we get the property

$$\begin{aligned} d_{\overset{\sqcup^{lam}}{R^2}} (\overset{\sqcup^{lam}}{X}, \overset{\sqcup^{lam}}{\Phi}) & \\ \leq d_{\overset{\sqcup^{lam}}{R^2}} (\overset{\sqcup^{lam}}{X}, \overset{\sqcup^{lam}}{\Psi}) & \underset{lam}{\oplus} d_{\overset{\sqcup^{lam}}{R^2}} (\overset{\sqcup^{lam}}{\Psi}, \overset{\sqcup^{lam}}{\Phi}); \\ & \overset{\sqcup^{lam}}{X}, \; \overset{\sqcup^{lam}}{\Psi}, \; \overset{\sqcup^{lam}}{\Phi} \in \overset{\sqcup^{lam}}{R^2}. \end{aligned} \tag{4.11}$$

5. Explosion by quadrants

Let us divide into parts the set R^2 by the *quadrant-compositions*

$$Q_{(p,q)} = \{(x,y) \in R^2 : p \le |x| < p+1; q \le |y| < q+1; p,q = 0,1,\dots\}. \tag{5.1}$$

Each quadrant-composition contains four quadrants. In detail:

$$\begin{aligned}
&\textit{Left-before quadrant} = \{(x,y) \in R^2 : -p-1 < x \le -p; -q-1 < y \le -q\},\\
&\textit{Left-behind quadrant} = \{(x,y) \in R^2 : -p-1 < x \le -p; q \le y < q+1\},\\
&\textit{Right-before quadrant} = \{(x,y) \in R^2 : p \le x < p+1; -q-1 < y \le -q\},\\
&\textit{Right-behind-quadrant} = \{(x,y) \in R^2 : p \le x < p+1; q \le y < q+1\}.
\end{aligned}$$

For a fixed pair (p,q) of non-negative integer numbers (2.1) and (2.2) yield:

$$\overbracket{left-bef}^{lam} = \Big\{(u,v,d) \in R^3 : u \in (-\infty,0]; v \in (-\infty,0]; \; d = -p - \frac{q}{2(q+1)}\Big\}.$$

$$\overbracket{left-beh}^{lam} = \Big\{(u,v,d) \in R^3 : u \in (-\infty,0]; v \in [0,\infty); \; d = -p + \frac{q}{2(q+1)}\Big\},$$

$$\overbracket{right-bef}^{lam} = \Big\{u,v,d) \in R^3 : u \in [0,\infty); v \in (-\infty,0]; \; d = p - \frac{q}{2(q+1)}\Big\},$$

$$\overbracket{right-beh}^{lam} = \Big\{u,v,d) \in R^3 : u \in [0,\infty); v \in [0,\infty); d = p + \frac{q}{2(q+1)}\Big\},$$

where the used abbreviations are clear. Each is a "quarter plane" in an appropriate two-dimensional plane of the Euclidean space R^3. It means that each exploded of any quadrant of any quadrant-composition is visible by the traditional two-dimensional space. So, the invisible points of $\overbracket{R^2}$ become visible by the laminary model $\overbracket{R^2}^{lam}$.

By (5.1) it is easy to see, that

$$\overbracket{Q_{(0,0)}}^{lam} = R^2 (= \{(u,v,0) \in R^3 : -\infty < u < \infty, -\infty < v < \infty\}) \tag{5.2}$$

holds.

Moroeover, each $\overset{\sqcup}{Q_{(0,q);q\neq 0}}^{lam}$, $\overset{\sqcup}{Q_{(p,0);p\neq 0}}^{lam}$ is a union of two disjunct two-dimensional “half-planes”. If $p \neq 0$; $q \neq 0$ then $\overset{\sqcup}{Q_{(p,q)}}^{lam}$ is a union of four disjunct two-dimensional “quarter-planes”.

Example 5.3. *Exploding the points of the circle with centre $O = (0,0)$ and radius $\sqrt{2}$ having the equation*

$$\|X\|_{R^2} = \sqrt{2} \quad X = (x,y) \in R^2, \tag{5.4}$$

the super-circle *with centre* $\overset{\sqcup}{O} = (\overset{\sqcup}{0}, \overset{\sqcup}{0}) = (0,0) = O$ *and (super-) radius* $\overset{\vdash\!\dashv}{\sqrt{2}}$, *having the equation*

$$\| \overset{\sqcup}{X} \|_{\underset{R^2}{\sqcup}} = \overset{\vdash\!\dashv}{\sqrt{2}}\,; \ \overset{\sqcup}{X} = (\overset{\sqcup}{x}, \overset{\sqcup}{y}) \in \overset{\sqcup}{R^2} \tag{5.5}$$

is obtained. By (5.4) it is clear that if X is a point of the circle then $X \notin \underset{\sqcup}{R^2}$, so each point of super-circle is invisible in the exploded two-dimensional space. Our task is to present the super-circle in the laminary model of exploded two-dimensional space given by (2.12). In $\overset{\sqcup}{R^2}^{lam}$ the super circle with centre $\overset{\sqcup}{O}^{lam} = \overset{\sqcup}{(0,0)}^{lam} = (0,0,0)$, and radius

$$\overset{\vdash\!\dashv}{\sqrt{2}}^{lam} = (\operatorname{area th}(\sqrt{2}-1), 0, 1) \approx (0,4406866793; 0; 1) \in R^3$$

has the equation

$$\| \overset{\sqcup}{X}^{lam} \|_{\underset{R^2}{\sqcup}^{lam}} = \overset{\vdash\!\dashv}{\sqrt{2}}^{lam}; \quad \overset{\sqcup}{X}^{lam} = \overset{\sqcup}{(x,y)}^{lam} \in R^3. \tag{5.6}$$

Considering $\overset{\sqcup}{X}^{lam} = (u,v,d) \in \overset{\sqcup}{R^2}^{lam}$ we have to find a connection between the coordinates u and v while the third coordinate d has a certain fixed value. By (5.4) we have that the circle is situated on the union $Q_{(1,0)} \cup Q_{(0,1)} \cup Q_{(1,1)}$ so, the super-circle is situated on the union

$$\overset{\sqcup}{Q_{1,0)}}^{lam} \cup \overset{\sqcup}{Q_{(0,1)}}^{lam} \cup \overset{\sqcup}{Q_{(1,1)}}^{lam}.$$

Selecting the points

$$A = (1,-1);\ B = (\sqrt{2},0);\ C = (1,1);\ D = (0,\sqrt{2});$$
$$E = (-1,1);\ F = (-\sqrt{2},0);\ G = (-1,-1);\ H = (0,-\sqrt{2}),$$

we observe that their laminary explodeds are:

$$\overset{\llcorner\lrcorner\, lam}{A} = \left(0, 0, \frac{3}{4}\right) \in \overset{\llcorner\lrcorner\, lam}{Q_{(1,1)}} ;$$
$$\overset{\llcorner\lrcorner\, lam}{B} = (\operatorname{area}\operatorname{th}(\sqrt{2}-1), 0, 1) \in \overset{\llcorner\lrcorner\, lam}{Q_{(1,0)}} ;$$
$$\overset{\llcorner\lrcorner\, lam}{C} = \left(0, 0, \frac{5}{4}\right) \in \overset{\llcorner\lrcorner\, lam}{Q_{(1,1)}} ;$$
$$\overset{\llcorner\lrcorner\, lam}{D} = \left(0, \operatorname{area}\operatorname{th}(\sqrt{2}-1), \frac{1}{4}\right) \in \overset{\llcorner\lrcorner\, lam}{Q_{(0,1)}} ;$$

$$\overset{\llcorner\lrcorner\, lam}{E} = \left(0, 0, -\frac{3}{4}\right) \in \overset{\llcorner\lrcorner\, lam}{Q_{(1,1)}} ;$$
$$\overset{\llcorner\lrcorner\, lam}{F} = (\operatorname{area}\operatorname{th}(-\sqrt{2}+1), 0, -1) \in \overset{\llcorner\lrcorner\, lam}{Q_{(1,0)}} ;$$
$$\overset{\llcorner\lrcorner\, lam}{G} = \left(0, 0, -\frac{5}{4}\right) \in \overset{\llcorner\lrcorner\, lam}{Q_{(1,1)}} ;$$
$$\overset{\llcorner\lrcorner\, lam}{H} = \left(0, \operatorname{area}\operatorname{th}(-\sqrt{2}+1), -\frac{1}{4}\right) \in \overset{\llcorner\lrcorner\, lam}{Q_{(0,1)}} .$$

Moreover, by (2.1), (2.2) and (5.4) we have the following four cases:

Case (a): *circle* $\cap$ *right* $Q_{(1,0)}$

$$\text{(5.7)} \quad \text{super}-circle \cap \overset{\llcorner\lrcorner\, lam}{rightQ_{(1,0)}}$$
$$= \{(u,v,d) \in R^3 : (\mathrm{th}u+1)^2 + \mathrm{th}^2 v = 2; d = 1\},$$

Case (b): *circle* $\cap$*beh* $Q_{(0,1)}$

$$\text{(5.8)} \quad \text{super} - circle \cap \overset{\llcorner\lrcorner\, lam}{behQ_{(0,1)}}$$
$$= \left\{(u,v,d) \in R^3 : \mathrm{th}^2 u + (\mathrm{th}v+1)^2 = 2; d = \frac{1}{4}\right\},$$

Case (c): *circle* $\cap$ *left* $Q_{(1,0)}$

$$\text{(5.9)} \quad \text{super} - circle \cap \overset{\llcorner\lrcorner\, lam}{Q_{(1,0)}}$$
$$= \{u,v,d) \in R^3 : (\mathrm{th}u-1)^2 + \mathrm{th}^2 v = 2; d = -1\},$$

Case (d): *circle* $\cap$ *bef* $Q_{(0,1)}$

$$\text{(5.10)} \quad \text{super} - circle \cap \overset{\llcorner\lrcorner\, lam}{befQ_{(0,1)}} =$$
$$= \left\{(u,v,d) \in R^3 : \mathrm{th}^2 u + (\mathrm{th}v-1)^2 = 2; d = -\frac{1}{4}\right\},$$

Instead of (5.4) we may use the equation-system

$$\begin{array}{ll}(5.11) & \begin{array}{l} x=\sqrt{2}\cos\varphi \\ y=\sqrt{2}\sin\varphi \end{array} \quad ,-\frac{\pi}{4}\le\varphi<\frac{7\pi}{4},\end{array}$$

too. Now, instead of (5.7)-(5.10), by (2.1), (2.2) and (5.11) for the cases (a)-(d)

$$\text{super}-circle\cap\ \overline{rightQ_{(1,0)}}^{lam}$$

$$=\left\{(u,v,d)\in R^3:\ \begin{array}{l} u=\operatorname{area}\operatorname{th}(\sqrt{2}\cos\varphi-1)\\ v=\operatorname{area}\operatorname{th}(\sqrt{2}\sin\varphi)\\ d=1\end{array}\quad ;-\frac{\pi}{4}<\varphi<\frac{\pi}{4}\right\},$$

$$\text{super}-circle\cap\ \overline{behQ_{(0,1)}}^{lam}$$

$$=\left\{(u,v,d)\in R^3:\ \begin{array}{l} u=\operatorname{area}\operatorname{th}(\sqrt{2}\cos\varphi)\\ v=\operatorname{area}\operatorname{th}(\sqrt{2}\sin\varphi-1)\\ d=\frac{1}{4}\end{array}\quad ;\frac{\pi}{4}<\varphi<\frac{3\pi}{4}\right\},$$

$$\text{super}-circle\cap\ \overline{leftQ_{(1,0)}}^{lam}$$

$$=\left\{(u,v,d)\in R^3:\ \begin{array}{l} u=\operatorname{area}\operatorname{th}(\sqrt{2}\cos\varphi+1)\\ v=\operatorname{area}\operatorname{th}(\sqrt{2}\sin\varphi)\\ d=-1\end{array}\quad ;\frac{3\pi}{4}<\varphi<\frac{5\pi}{4}\right\},$$

and

$$\text{super}-circle\cap\ \overline{befQ_{(0,1)}}^{lam}$$

$$=\left\{(u,v,d)\in R^3:\ \begin{array}{l} u=\operatorname{area}\operatorname{th}(\sqrt{2}\cos\varphi)\\ v=\operatorname{area}\operatorname{th}(\sqrt{2}\sin\varphi+1)\\ d=-\frac{1}{4}\end{array}\quad ;\frac{5\pi}{4}<\varphi<\frac{7\pi}{4}\right\},$$

are obtained, respectively.

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