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MORE ON INEQUALITIES OF SIMPSON TYPE

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Abstract. Some generalizations of a recent inequality of Simpson type are given. We also provide some sharp inequalities which improve previous results.

1. INTRODUCTION

In a recent paper [1], by appropriately choosing the Peano kernel

$$S_n(x) = \begin{cases} \frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{6(n-1)!} & \text{if } x \in [a, \frac{a+b}{2}], \\ \frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{6(n-1)!} & \text{if } x \in (\frac{a+b}{2}, b], \end{cases} \tag{1}$$

an inequality of Simpson type for an n-times continuously differentiable mapping is given as follows.

Theorem 1. *Let $f\colon [a,b] \to \mathbf{R}$ be an n-times continuously differentiable mapping, $n \geq 1$ and such that $\|f^{(n)}\|_\infty := \sup_{x\in(a,b)} |f^{(n)}(x)| < \infty$. Then*

$$\begin{aligned} \Big| \int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] \\ + \sum_{k=2}^{[\frac{n-1}{2}]} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}(\frac{a+b}{2}) \Big| \\ \leq \|f^{(n)}\|_\infty \times \begin{cases} \frac{4n^n(b-a)^{n+1}}{(n+1)!6^{n+1}} - \frac{(n-2)(b-a)^{n+1}}{3(n+1)!2^n} & \text{if } n < 3, \\ \frac{(n-2)(b-a)^{n+1}}{3(n+1)!2^n} & \text{if } n \geq 3, \end{cases} \end{aligned} \tag{2}$$

where $[\frac{n-1}{2}]$ denotes the integer part of $\frac{n-1}{2}$.

In [4], using the well-known pre-Grüss inequality (see [2]), Pečarić and Varošanec have obtained the following result:

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Theorem 2. *Let $f\colon [a,b] \to \mathbf{R}$ be a mapping such that the derivative $f^{(n)}$ $(n \geq 1)$ is integrable with $\gamma_n \leq f^{(n)}(x) \leq \Gamma_n$ for all $x \in [a,b]$, where $\gamma_n, \Gamma_n \in \mathbf{R}$ are constants. Then we have*

$$
\begin{aligned}
(3)\quad & \Big| \int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] \\
& \qquad + \sum_{k=2}^{m} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}(\frac{a+b}{2}) \Big| \\
& \qquad \leq \frac{(\Gamma_{2m+1} - \gamma_{2m+1})(b-a)^{2m+2}}{3(2m+1)!2^{2m+2}} \sqrt{\frac{16m^3 - 20m^2 + 4m + 3}{16m^2 + 16m + 3}}
\end{aligned}
$$

and

$$
\begin{aligned}
(4)\quad & \Big| \int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] \\
& \qquad + \sum_{k=2}^{m-1} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}(\frac{a+b}{2}) \\
& \qquad + \frac{(2m-2)(b-a)^{2m}}{3(2m+1)!2^{2m}}[f^{(2m-1)}(b) - f^{(2m-1)}(a)] \Big| \\
& \leq \frac{(\Gamma_{2m} - \gamma_{2m})(b-a)^{2m+1}}{3(2m+1)!2^{2m+1}} \sqrt{\frac{64m^5 - 176m^4 + 112m^3 + 16m^2 + 4m - 2}{16m^2 - 1}}
\end{aligned}
$$

valid for $m = 0, 1, 2, \ldots$.

The purpose of this paper is to further consider generalizations of the inequality (2) and also provides an improvement of the inequality (3).

For convenience, we shall first collect some technical results related to (1) which will be used in the proofs of our theorems.

By elementary calculus, it is not difficult to get the following results:

$$
(5)\qquad \int_a^b S_n(x)\,dx = \begin{cases} 0, & n \text{ is odd,} \\ -\frac{(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \text{ is even.} \end{cases}
$$

$$
(6)\qquad \int_a^b |S_n(x)|\,dx = \begin{cases} \frac{5(b-a)^2}{36}, & n = 1, \\ \frac{(b-a)^3}{81}, & n = 2, \\ \frac{(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \geq 3. \end{cases}
$$

$$
(7)\qquad \int_a^b S_n^2(x)\,dx = \frac{(2n^3 - 11n^2 + 18n - 6)(b-a)^{2n+1}}{9(4n^2-1)(n!)^2 2^{2n}}.
$$

$$\max_{x\in[a,b]} |S_n(x)| = \begin{cases} \frac{b-a}{3}, & n=1,\\ \frac{(b-a)^2}{24}, & n=2,\\ \frac{(b-a)^3}{324}, & n=3,\\ \frac{(n-3)(b-a)^n}{3(n!)2^n}, & n\geq 4. \end{cases} \tag{8}$$

$$\max_{x\in[a,b]} |S_{2m}(x) - \frac{1}{b-a}\int_a^b S_{2m}(x)\,dx| = \begin{cases} \frac{(b-a)^2}{24}, & m=1,\\ \frac{(4m^2-6m-1)(b-a)^{2m}}{3(2m+1)!2^{2m}}, & m\geq 2. \end{cases} \tag{9}$$

In what follows, we will use the notations

$$D_n := \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}$$

and

$$T_n := \sum_{k=2}^{[\frac{n-1}{2}]} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}(\frac{a+b}{2}),$$

where $[\frac{n-1}{2}]$ denotes the integer part of $\frac{n-1}{2}$.

2. GENERALIZATIONS FOR DERIVATIVES THAT ARE ABSOLUTELY CONTINUOUS

Theorem 3. *Let $f\colon [a,b] \to \mathbf{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a,b]$. If $f^{(n)} \in L_\infty[a,b]$, then we have*

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] + T_n \right| \\ &\qquad \leq \|f^{(n)}\|_\infty \times \begin{cases} \frac{4n^n(b-a)^{n+1}}{(n+1)!6^{n+1}} - \frac{(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n<3,\\ \frac{(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n\geq 3, \end{cases} \end{aligned} \tag{10}$$

where $\|f^{(n)}\|_\infty := ess\sup_{x\in[a,b]} |f^{(n)}(x)|$ is the usual Lebesgue norm on $L_\infty[a,b]$.

The proof of inequality (10) is just like the proof of inequality (2) in [1] and so is omitted.

Theorem 4. *Let $f\colon [a,b] \to \mathbf{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a,b]$. If $f^{(n)} \in L_1[a,b]$, then we have*

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] + T_n \right| \\ &\qquad \leq \|f^{(n)}\|_1 \times \begin{cases} \frac{b-a}{3}, & n=1,\\ \frac{(b-a)^2}{24}, & n=2,\\ \frac{(b-a)^3}{324}, & n=3,\\ \frac{(n-3)(b-a)^n}{3(n!)2^n}, & n\geq 4, \end{cases} \end{aligned} \tag{11}$$

where $\|f^{(n)}\|_1 := \int_a^b |f^{(n)}(x)|\,dx$ *is the usual Lebesgue norm on* $L_1[a,b]$.

Proof. By using the identity

$$(12)\quad (-1)^n \int_a^b S_n(x) f^{(n)}(x)\,dx = \int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] + T_n,$$

we get

$$(13)\quad \left|\int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] + T_n\right| = \left|\int_a^b S_n(x) f^{(n)}(x)\,dx\right| \le \max_{x\in[a,b]} |S_n(x)| \int_a^b |f^{(n)}(x)|\,dx.$$

Consequently, the inequality (11) follows from (13) and (8). □

Theorem 5. *Let* $f\colon [a,b] \to \mathbf{R}$ *be a mapping such that the derivative* $f^{(n-1)}$ $(n \ge 1)$ *is absolutely continuous on* $[a,b]$. *If* $f^{(n)} \in L_2[a,b]$, *then we have*

$$(14)\quad \left|\int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] + T_n\right| \le \frac{(b-a)^{n+\frac{1}{2}}}{3(n!)2^n}\sqrt{\frac{2n^3 - 11n^2 + 18n - 6}{4n^2 - 1}}\|f^{(n)}\|_2,$$

where $\|f^{(n)}\|_2 := [\int_a^b |f^{(n)}(x)|^2\,dx]^{\frac{1}{2}}$ *is the usual Lebesgue norm on* $L_2[a,b]$.

Proof. By using the identity (12), we get

$$(15)\quad \left|\int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] + T_n\right| = \left|\int_a^b S_n(x) f^{(n)}(x)\,dx\right| \le \|S_n\|_2 \|f^{(n)}\|_2.$$

Consequently, the inequality (14) follows from (15) and (7). □

3. SOME SHARP INEQUALITIES AND RELATED RESULTS

Theorem 6. *Let* $f\colon [a,b] \to \mathbf{R}$ *be a mapping such that the derivative* $f^{(n)}$ $(n \ge 1)$ *is integrable with* $\gamma_n \le f^{(n)}(x) \le \Gamma_n$ *for all* $x \in [a,b]$, *where* $\gamma_n, \Gamma_n \in \mathbf{R}$ *are constants. Then we have*

$$(16)\quad \left|\int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] + T_n\right|$$

$$\leq \frac{\Gamma_n - \gamma_n}{2} \times \begin{cases} \frac{5(b-a)^2}{36}, & n = 1, \\ \frac{(b-a)^3}{81}, & n = 2, \\ \frac{(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \geq 3 \textit{ and odd.} \end{cases}$$

$$(17) \quad \left| \int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] + T_n \right|$$

$$\leq (D_n - \gamma_n) \times \begin{cases} \frac{(b-a)^2}{3}, & n = 1, \\ \frac{(b-a)^3}{24}, & n = 2, \\ \frac{(b-a)^4}{324}, & n = 3, \\ \frac{(n-3)(b-a)^{n+1}}{3(n!)2^n}, & n \geq 5 \textit{ and odd.} \end{cases}$$

$$(18) \quad \left| \int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] + T_n \right|$$

$$\leq (\Gamma_n - D_n) \times \begin{cases} \frac{(b-a)^2}{3}, & n = 1, \\ \frac{(b-a)^3}{24}, & n = 2, \\ \frac{(b-a)^4}{324}, & n = 3, \\ \frac{(n-3)(b-a)^{n+1}}{3(n!)2^n}, & n \geq 5 \textit{ and odd.} \end{cases}$$

$$(19) \quad \left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a) + f(b)] + T_{2m} + \frac{(2m-2)(b-a)^{2m+1}}{3(2m+1)!2^{2m}} D_{2m} \right|$$

$$\leq (D_{2m} - \gamma_{2m}) \times \begin{cases} \frac{(b-a)^3}{24}, & m = 1, \\ \frac{(4m^2-6m-1)(b-a)^{2m+1}}{3(2m+1)!2^{2m}}, & m \geq 2. \end{cases}$$

$$(20) \quad \left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a) + f(b)] + T_{2m} + \frac{(2m-2)(b-a)^{2m+1}}{3(2m+1)!2^{2m}} D_{2m} \right|$$

$$\leq (\Gamma_{2m} - D_{2m}) \times \begin{cases} \frac{(b-a)^3}{24}, & m = 1, \\ \frac{(4m^2-6m-1)(b-a)^{2m+1}}{3(2m+1)!2^{2m}}, & m \geq 2. \end{cases}$$

where m is any positive integer.

Proof. For n odd and $n = 2$, by (5) and (12) we get

$$(21) \quad (-1)^n \int_a^b S_n(x)[f^{(n)}(x) - C]\,dx$$

$$= \int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] + T_n,$$

where $C \in \mathbf{R}$ is a constant.

If we choose $C = \frac{\gamma_n+\Gamma_n}{2}$, then we have

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)] + T_n\right| \leq \frac{\Gamma_n-\gamma_n}{2}\int_a^b |S_n(x)|\,dx, \tag{22}$$

and hence the inequality (16) follows from (6).

If we choose $C = \gamma_n$, then we have

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)] + T_n\right| \leq \max_{x\in[a,b]}|S_n(x)|\int_a^b |f^{(n)}(x)-\gamma_n|\,dx, \tag{23}$$

and hence the inequality (17) follows from (8).

Similarly we can prove that the inequality (18) holds.

By (5) and (12) we can also get

$$\begin{aligned}\Bigg|\int_a^b f(x)\,dx - \frac{b-a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)] \\ + T_{2m} + \frac{(2m-2)(b-a)^{2m+1}}{3(2m+1)!2^{2m}}D_{2m}\Bigg| \\ = \left|\int_a^b [S_{2m}(x) - \frac{1}{b-a}\int_a^b S_{2m}(x)\,dx][f^{(2m)}(x)-C]\,dx\right|,\end{aligned} \tag{24}$$

where $C \in \mathbf{R}$ is a constant.

If we choose $C = \gamma_{2m}$, then we have

$$\begin{aligned}\left|\int_a^b f(x)\,dx - \frac{b-a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)] + T_{2m} + \frac{(2m-2)(b-a)^{2m+1}}{3(2m+1)!2^{2m}}D_{2m}\right| \\ \leq \max_{x\in[a,b]}\left|S_{2m}(x) - \frac{1}{b-a}\int_a^b S_{2m}(x)\,dx\right|\int_a^b |f^{(2m)}(x)-\gamma_{2m}|\,dx\end{aligned} \tag{25}$$

and hence the inequality (19) follows from (9).

Similarly we can prove that the inequality (20) holds. □

Remark 1. It is not difficult to find that the inequality (16) improves the inequality (3). Indeed, the inequality (16) is sharp in the sense that we can choose f to attain the equality in (16). e.g., for $n = 1$, we construct the function $f(x) = \int_a^x j(y)\,dy$, where

$$j(x) = \begin{cases} \gamma_1, & a \leq x < \frac{5a+b}{6}, \\ \Gamma_1, & \frac{5a+b}{6} \leq x < \frac{a+b}{2}, \\ \gamma_1, & \frac{a+b}{2} \leq x < \frac{a+5b}{6}, \\ \Gamma_1, & \frac{a+5b}{6} \leq x \leq b, \end{cases}$$

for $n = 2$, we construct the function $f(x) = \int_a^x (\int_a^y j(z)\,dz)\,dy$, where

$$j(x) = \begin{cases} \gamma_2, & a \leq x < \frac{2a+b}{3}, \\ \Gamma_2, & \frac{2a+b}{3} \leq x < \frac{2b+a}{3}, \\ \gamma_2, & \frac{2b+a}{3} \leq x \leq b, \end{cases}$$

and for $n \geq 3$ and odd, we construct the function

$$f(x) = \int_a^x (\int_a^{y_n} (\cdots \int_a^{y_2} j(y_1)\,dy_1 \cdots)\,dy_{n-1})\,dy_n,$$

where

$$j(x) = \begin{cases} \gamma_n, & a \leq x < \frac{a+b}{2}, \\ \Gamma_n, & \frac{a+b}{2} \leq x \leq b. \end{cases}$$

Remark 2. If in the inequality (16) we choose $n = 1, 2, 3$, then we get

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] \right| \leq \frac{5}{72}(\Gamma_1 - \gamma_1)(b-a)^2,$$
$$\left| \int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] \right| \leq \frac{1}{162}(\Gamma_2 - \gamma_2)(b-a)^3,$$
$$\left| \int_a^b f(x)\,dx - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] \right| \leq \frac{1}{1152}(\Gamma_3 - \gamma_3)(b-a)^4,$$

which improve the results in Theorem 5 of [3] as well as Theorem 12 of [4].

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References

[1] Z. Liu. An inequality of Simpson type. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 461(2059):2155–2158, 2005.

[2] M. Matić, J. Pečarić, and N. Ujević. Improvement and further generalization of inequalities of Ostrowski-Grüss type. *Comput. Math. Appl.*, 39(3-4):161–175, 2000.

[3] C. E. M. Pearce, J. Pečarić, N. Ujević, and S. Varošanec. Generalizations of some inequalities of Ostrowski-Grüss type. *Math. Inequal. Appl.*, 3(1):25–34, 2000.

[4] J. Pečarić and S. Varošanec. Harmonic polynomials and generalization of Ostrowski inequality with applications in numerical integration. *Nonlinear Anal.*, 47:2365–2374, 2001.

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