Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
25(1) (2009), 85–104
www.emis.de/journals
ISSN 1786-0091

LIPSCHITZ ESTIMATES FOR MULTILINEAR COMMUTATOR OF LITTLEWOOD-PALEY OPERATOR

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Abstract. In this paper, we will study the continuity of multilinear commutator generated by Littlewood-Paley operator and Lipschitz functions on Triebel-Lizorkin space, Hardy space and Herz-Hardy space.

1. Introduction

Let T be the Calderón-Zygmund operator, Coifman, Rochberg and Weiss (see [4]) proves that the commutator $[b,T](f) = bT(f) - T(bf)$(where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when T is replaced by the fractional operators. In [8, 16], Janson and Paluszynski study these results for the Triebel-Lizorkin spaces and the case $b \in \mathrm{Lip}_\beta(R^n)$, where $\mathrm{Lip}_\beta(R^n)$ is the homogeneous Lipschitz space. The main purpose of this paper is to discuss the boundedness of multilinear commutator generated by Littlewood-Paley operator and b_j on Triebel-Lizorkin space, Hardy space and Herz-Hardy space, where $b_j \in \mathrm{Lip}_\beta(R^n)$.

2. Preliminaries and Definitions

Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, and write $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$, Q will denote a cube of R^n with side parallel to the axes. Let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. Denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)(0 < p \leq 1)$ has the atomic decomposition characterization(see [12][17][18]). For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(R^n)$ be the homogeneous Tribel-Lizorkin space. The Lipschitz space $\mathrm{Lip}_\beta(R^n)$ is the space

2000 *Mathematics Subject Classification.* 42B20, 42B25.
Key words and phrases. Littlewood-Paley operator; Multilinear commutator; Triebel-Lizorkin space; Herz-Hardy space; Herz space; Lipschitz space.

of functions f such that

$$||f||_{\mathrm{Lip}_\beta} = \sup_{\substack{x,y\in R^n \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty.$$

Lemma 1 ([16])**.** *For $0<\beta<1,\ 1<p<\infty$, we have*

$$\begin{aligned} ||f||_{\dot{F}_p^{\beta,\infty}} &\approx \left|\left|\sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |f(x)-f_Q|dx\right|\right|_{L^p} \\ &\approx \left|\left|\sup_{\cdot\in Q}\inf_c \frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |f(x)-c|dx\right|\right|_{L^p}. \end{aligned}$$

Lemma 2 ([16])**.** *For $0<\beta<1,\ 1\le p\le\infty$, we have*

$$\begin{aligned} ||f||_{\mathrm{Lip}_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |f(x)-f_Q|dx \\ &\approx \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}}\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^p dx\right)^{1/p}. \end{aligned}$$

Lemma 3 ([2])**.** *For $1\le r<\infty$ and $\beta>0$, let*

$$M_{\beta,r}(f)(x) = \sup_{x\in Q}\left(\frac{1}{|Q|^{1-\frac{\beta r}{n}}}\int_Q |f(y)|^r dy\right)^{1/r},$$

suppose that $r<p<n/\beta$, and $1/q=1/p-\beta/n$, then

$$||M_{\beta,r}(f)||_{L^q} \le C||f||_{L^p}.$$

Lemma 4 ([5])**.** *Let $Q_1\subset Q_2$, then*

$$|f_{Q_1}-f_{Q_2}| \le C||f||_{\mathrm{Lip}_\beta}|Q_2|^{\beta/n}.$$

Definition 1. Let $0<p\le 1$. A function $a(x)$ on R^n is called a H^p-atom , if

1) $\operatorname{supp} a\subset B(x_0,r)$ for some x_0 and for some $r>0$ (or for some $r\ge 1$),
2) $||a||_{L^\infty}\le |B(x_0,r)|^{-1/p}$,
3) $\int_{R^n} a(x)x^\gamma dx = 0$ for all γ with $0\le|\gamma|\le[n(1/p-1)]$.

Lemma 5. *(see [14][15]) Let $0<p\le 1$. A distribution f on R^n is in $H^p(R^n)$ if and only if f can be written as $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$ in the distributional sense, where each a_j is H^p-atom and λ_j are constants with $\sum_{j=-\infty}^{\infty}|\lambda_j|^p<\infty$. Moreover,*

$$||f||_{H^p} = \inf\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p},$$

where the infimum take over all decompositions of f as above.

Definition 2. Let $0<p,\ q<\infty$, $\alpha\in R$, $B_k=\{x\in R^n, |x|\le 2^k\}$, $A_k = B_k\backslash B_{k-1}$ and $\chi_k=\chi_{Ak}$ for $k\in \mathbf{Z}$.

1) The homogeneous Herz space is defined

$$\dot{K}_q^{\alpha,p}(R^n) = \{f \in L^q_{Loc}(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p\right]^{1/p};$$

2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f \in L^q_{Loc}(R^n) : ||f||_{K_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p + ||f\chi_{B_0}||_{L^q}^p\right]^{1/p}.$$

Definition 3. Let $\alpha \in R$, $0 < p, q < \infty$.

(1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{H\dot{K}_q^{\alpha,p}} = ||G(f)||_{\dot{K}_q^{\alpha,p}};$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{HK_q^{\alpha,p}} = ||G(f)||_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f, that is

$$G(f)(x) = \sup_{\varphi \in K_m} \sup_{|x-y|<t} |f * \varphi_t(y)|,$$

where $K_m = \{\varphi \in S(R^n) : \sup_{x \in R^n, |\alpha| \le m} (1 + |u|)^{m+n} |D^\alpha \varphi(u)| \le 1\}$, $\varphi_t(x) = t^{-n}\varphi(x/t)$ for $t > 0$, m is a positive integer and $S(R^n)$ is the Schwartz class (see [17, p. 88]).

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 4. Let $\alpha \in R$, $1 < q < \infty$. A function a on R^n is called a central (α, q)-atom (or a central (a, q)-atom of restrict type), if

1) $\operatorname{supp} a \subset B(0, r)$ for some $r > 0$ (or for some $r \ge 1$),
2) $||a||_{L^q} \le |B(0, r)|^{-\alpha/n}$,
3) $\int_{R^n} a(x) x^\eta dx = 0$ for all η with $|\eta| \le [\alpha - n(1 - 1/q)]$.

Lemma 6 ([6, 15])**.** *Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \geq n(1-1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$ (or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α, q)-atoms (or central (α, q)-atoms of restrict type) a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j|^p < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, and*

$$||f||_{H\dot{K}_q^{\alpha,p}} (\text{ or } \ ||f||_{HK_q^{\alpha,p}}) \approx \left(\sum_j |\lambda_j|^p \right)^{1/p}.$$

Definition 5 ([7])**.** Let $\alpha \in R$, $1 < p, q < \infty$.

1) A measure function f is said to belong to homogeneous weak Herz space $W\dot{K}_q^{\alpha,p}(R^n)$, if

$$||f||_{W\dot{K}_q^{\alpha,p}} = \sup_{\lambda > 0} \lambda (\sum_{-\infty}^{+\infty} 2^{k\alpha p} |\{x \in E_k : |f(x)| > \lambda\}|^{p/q})^{1/p} < \infty;$$

2) A measure function f is said to belong to inhomogeneous weak Herz space $WK_q^{\alpha,p}(R^n)$, if

$$||f||_{WK_q^{\alpha,p}} = \sup_{\lambda > 0} \lambda (\sum_{k=1}^{+\infty} 2^{k\alpha p} |\{x \in E_k : |f(x)| > \lambda\}|^{p/q} + |\{x \in B_0 : |f(x) > \lambda|^{p/q}\}|)^{1/p} < \infty.$$

Definition 6. Let $\mu > 1$, $n > \delta > 0$ and ψ be a fixed function which satisfies the following properties:

1) $\int_{R^n} \psi(x) dx = 0$,
2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$,
3) $|\psi(x+y) - \psi(x)| \leq C|y|(1+|x|)^{-(n+2-\delta)}$ when $2|y| < |x|$.

Given a positive integer m and the locally integrable function $b_j (1 \leq j \leq m)$. The multilinear commutator of Littlewood-Paley operator is defined by

$$g_{\mu,\delta}^{\vec{b}}(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t + |x-y|} \right)^{n\mu} |F_t^{\vec{b}}(f)(x,y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x,y) = \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(z)) \right] \psi_t(y-z) f(z) dz.$$

When $m = 1$, set

$$g_{\mu,\delta}^{b}(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t + |x-y|} \right)^{n\mu} |F_t^{b}(f)(x,y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

where

$$F_t^b(f)(x,y) = \int_{R^n} (b(x) - b(z))\psi_t(y-z)f(z)dz$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Set $F_t(f)(x) = f * \psi_t(x)$, we also define that

$$g_{\mu,\delta}(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

which is the Littlewood-Paley operator (see [18]).

Let H be the space $H = \left\{h : ||h|| = (\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+1})^{1/2} < \infty\right\}$, then, for each fixed $x \in R^n$ $F_t(f)(x)$ may be viewed as a mapping from $[0,+\infty)$ to H, and it is clear that

$$g_{\mu,\delta}(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t(f)(y)\right\|$$

and

$$g_{\mu,\delta}^{\vec{b}}(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^{\vec{b}}(f)(x,y)\right\|.$$

Note that when $b_1 = \cdots = b_m$, $g_{\mu,\delta}^{\vec{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1, 2, 3, 4, 8, 10, 9, 11, 16]). Our main purpose is to establish the boundedness of the multilinear commutator on Triebel-Lizorkin space, Hardy space and Herz-Hardy space.

Lemma 7 ([9]). *Let $0 < \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$ and $w \in A_1$. Then $g_{\mu,\delta}$ is bounded from $L^p(w)$ to $L^q(w)$.*

Given some functions b_j $(j = 1,\dots,m)$ and a positive integer m and $1 \le j \le m$, we set $||\vec{b}||_{\mathrm{Lip}_\beta} = \prod_{j=1}^m ||b_j||_{\mathrm{Lip}_\beta}$ and denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1),\dots,\sigma(j)\}$ of $\{1,\dots,m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1,\dots,m\} \setminus \sigma$. For $\sigma = \{\sigma(1),\dots,\sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)},\dots,b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \dots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{\mathrm{Lip}_\beta} = ||b_{\sigma(1)}||_{\mathrm{Lip}_\beta} \dots ||b_{\sigma(j)}||_{\mathrm{Lip}_\beta}$.

3. Theorems and Proofs

Theorem 1. *Let $0 < \beta < min(1, 1/2m)$, $\mu > 3 + 1/n - 2\delta/n$,$1 < p < \infty$, $b_j \in \mathrm{Lip}_\beta(R^n)$ for $1 \le j \le m$ and $g_{\mu,\delta}^{\vec{b}}$ be the multilinear commutator of Littlewood-Paley operator as in Definition 6. Then*

(a) *$g_{\mu,\delta}^{\vec{b}}$ is bounded from $L^p(R^n)$ to $\dot{F}_p^{m\beta,\infty}(R^n)$ for $1 < p < n/\delta$ and $1/p - 1/q = \delta/n$.*

(b) *$g_{\mu,\delta}^{\vec{b}}$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $1/p - 1/q = (m\beta + \delta)/n$ and $1/p > (m\beta + \delta)/n$.*

Proof. (a). Fixed a cube $Q=(x_0,l)$ and $\tilde{x}\in Q$. Set $\vec{b}_Q=((b_1)_Q,\dots,(b_m)_Q)$, where $(b_j)_Q=|Q|^{-1}\int_Q b_j(y)dy,\ 1\le j\le m$. Write $f=f\chi_Q+f_2=f\chi_{R^n\setminus Q}=f_1+f_2$, we have

$$F_t^{\vec{b}}(f)(x,y)=\int_{R^n}\prod_{j=1}^{m}\left[(b_j(x)-(b_j)_Q)-(b_j(z)-(b_j)_Q)\right]\psi_t(y-z)f(z)dz$$
$$=\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma\int_{R^n}(\vec{b}(z)-\vec{b}_Q)_{\sigma^c}\psi_t(y-z)f(z)dz$$
$$=(b_1(x)-(b_1)_Q)\dots(b_m(x)-(b_m)_Q)F_t(f)(y)$$
$$+(-1)^mF_t((b_1-(b_1)_Q)\dots(b_m-(b_m)_Q)f)(y)$$
$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma\int_{\vec{R}^n}(\vec{b}(z)-\vec{b}_Q)_{\sigma^c}\psi_t(y-z)f(z)dz$$
$$=(b_1(x)-(b_1)_Q)\dots(b_m(x)-(b_m)_Q)F_t(f)(y)$$
$$+(-1)^mF_t((b_1-(b_1)_Q)\dots(b_m-(b_m)_Q)f_1)(y)$$
$$+(-1)^mF_t((b_1-(b_1)_Q)\dots(b_m-(b_m)_Q)f_2)(y)$$
$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}c_{m,j}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(y),$$

then

$$|g_{\mu,\delta}^{\vec{b}}(f)(x)-g_{\mu,\delta}(((b_1)_Q-b_1)\dots((b_m)_Q-b_m))f_2)(x_0)|$$
$$\le\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t^{\vec{b}}(f)(x,y)\right.$$
$$\left.-\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}F_t(((b_1)_Q-b_1)\dots((b_m)_Q-b_m)f_2)(y)\right\|$$
$$\le\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}(b_1(x)-(b_1)_Q)\dots(b_m(x)-(b_m)_Q)F_t(f)(y)\right\|$$
$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(y)\right\|$$
$$+\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t((b_1-(b_1)_Q)\dots(b_m-(b_m)_Q)f_1)(y)\right\|$$

$$+\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t(\prod_{j=1}^m(b_j-(b_j)_Q)f_2)(y)\right.$$
$$\left.-\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}F_t(\prod_{j=1}^m(b_j-(b_j)_Q)f_2)(y)\right\|$$
$$=I_1(x)+I_2(x)+I_3(x)+I_4(x).$$

Thus,

$$\frac{1}{|Q|^{1+m\beta/n}}\int_Q|g^{\vec{b}}_{\mu,\delta}(f)(x)-g_{\mu,\delta}((b_1)_Q-b_1)\dots((b_m)_Q-b_m)f_2)(x_0)|dx$$
$$\le\frac{1}{|Q|^{1+m\beta/n}}\int_Q I_1(x)dx+\frac{1}{|Q|^{1+m\beta/n}}\int_Q I_2(x)dx$$
$$+\frac{1}{|Q|^{1+m\beta/n}}\int_Q I_3(x)dx+\frac{1}{|Q|^{1+m\beta/n}}\int_Q I_4(x)dx$$
$$=I+II+III+IV.$$

For I, by using Lemma 2, we have

$$I\le\frac{1}{|Q|^{1+m\beta/n}}\sup_{x\in Q}|b_1(x)-(b_1)_Q|\dots|b_m(x)-(b_m)_Q|\int_Q|g_{\mu,\delta}(f)(x)|dx$$
$$\le C||\vec{b}||_{\mathrm{Lip}_\beta}\frac{1}{|Q|^{1+m\beta/n}}|Q|^{m\beta/n}\int_Q|g_{\mu,\delta}(f)(x)|dx$$
$$\le C||\vec{b}||_{\mathrm{Lip}_\beta}M(g_{\mu,\delta}(f))(\tilde{x}).$$

For II, taking $1<r<p<q<n/\delta$, $1/q'+1/q=1, 1/s'+1/s=1$, $1/q=1/p-\delta/n$, $ps=r$ by using the Hölder's inequality and the boundedness of $g_{\mu,\delta}$ from L^p to L^q and Lemma 2, we get

$$II\le\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+m\beta/n}}\int_Q|(\vec{b}(x)-\vec{b}_Q)_\sigma||g_{\mu,\delta}((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)|dx$$
$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}\left(\frac{1}{|Q|}\int_Q|(\vec{b}(x)-\vec{b}_Q)_\sigma|^{q'}dx\right)^{1/q'}\times$$
$$\times\left(\frac{1}{|Q|}\int_{R^n}|g_{\mu,\delta}((\vec{b}-\vec{b}_Q)_{\sigma^c}f\chi_Q)(x)|^q dx\right)^{1/q}$$
$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}||\vec{b}_\sigma||_{\mathrm{Lip}_\beta}|Q|^{|\sigma|\beta/n}\frac{1}{|Q|^{1/q}}\times$$
$$\times\left(\int_{R^n}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}f(x)\chi_Q(x)|^p dx\right)^{1/p}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}||\vec{b}_\sigma||_{\mathrm{Lip}_\beta}|Q|^{|\sigma|\beta/n}|Q|^{(-1/q)+(1/ps')+(1-\delta ps/n)/ps}\times$$

$$\times\left(\frac{1}{|Q|}\int_Q|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|^{ps'}dx\right)^{1/ps'}\left(\frac{1}{|Q|^{1-\delta ps/n}}\int_Q|f(x)|^{ps}dx\right)^{1/ps}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}||\vec{b}_\sigma||_{\mathrm{Lip}_\beta}|Q|^{|\sigma|\beta/n}||\vec{b}_{\sigma^c}||_{\mathrm{Lip}_\beta}|Q|^{|\sigma^c|\beta/n}M_{\delta,r}(f)(\tilde{x})$$

$$\leq C||\vec{b}||_{\mathrm{Lip}_\beta}M_{\delta,r}(f)(\tilde{x}).$$

For III, we choose $1<r<p<q<n/\delta$, $1/q=1/p-\delta/n$, $r=ps$, by the boundness of $g_{\mu,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$ and Hölder's inequality with $1/s+1/s'=1$, we have

$$III\leq C\frac{1}{|Q|^{m\beta/n}}\left(\frac{1}{|Q|}\int_{R^n}|g_{\mu,\delta}(\prod_{j=1}^m(b_j-(b_j)_Q)f\chi_Q)(x)|^q dx\right)^{1/q}$$

$$\leq C\frac{1}{|Q|^{m\beta/n}}\frac{1}{|Q|^{1/q}}\left(\int_{R^n}|\prod_{j=1}^m(b_j(x)-(b_j)_Q)|^p|f(x)\chi_Q(x)|^p dx\right)^{1/p}$$

$$\leq C\frac{|Q|^{(-1/q)+1/ps'+(1-(\delta ps/n)/ps)}}{|Q|^{m\beta/n}}\left(\frac{1}{|Q|}\int_Q|\prod_{j=1}^m(b_j(x)-(b_j)_Q)|^{ps'}dx\right)^{1/ps'}\times$$

$$\times\left(\frac{1}{|Q|^{1-\delta ps/n}}\int_Q|f(x)|^{ps}dx\right)^{1/ps}$$

$$\leq C||\vec{b}||_{\mathrm{Lip}_\beta}M_{\delta,r}(f)(\tilde{x}).$$

For IV, by the Minkowski's inequality and by the inequality $a^{1/2}-b^{1/2}\leq(a-b)^{1/2}$ for $a\geq b\geq 0$, we have

$$\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t(\prod_{j=1}^m(b_j-(b_j)_Q)f_2)(y)\right.$$

$$\left.-\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}F_t(\prod_{j=1}^m(b_j-(b_j)_Q)f_2)(y)\right\|$$

$$\leq\left[\int\int_{R_+^{n+1}}\left(\int_{(Q)^c}\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}-\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}\right|\right.\right.$$

$$\left.\left.\times\prod_{j=1}^m|b_j(z)-(b_j)_Q||\psi_t(y-z)||f(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}$$

$$\leq C\int_{Q^c}\left[\int\int_{R^{n+1}_+}\left(\frac{t^{n\mu/2}|x-x_0|^{1/2}\prod_{j=1}^m |b_j(z)-(b_j)_Q||\psi_t(y-z)||f(z)|}{(t+|x-y|)^{(n\mu+1)/2}}\right)^2 \times\right.$$
$$\left.\times\frac{dydt}{t^{n+1}}\right]^{1/2}dz$$
$$\leq C\int_{Q^c}\prod_{j=1}^m|b_j(z)-(b_j)_Q||f(z)||x-x_0|^{1/2}\left(\int\int_{R^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{n\mu+1}\times\right.$$
$$\left.\times\frac{t^{-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}\right)^{1/2}dz.$$

Set $B=B(x,t)$, then

$$t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\mu+1}\frac{dy}{(t+|y-z|)^{2n+2-2\delta}}$$
$$\leq t^{-n}\left(\int_{B(x,t)}\left(\frac{t}{t+|x-y|}\right)^{n\mu+1}\frac{dy}{(t+|y-z|)^{2n+2-2\delta}}\right)$$
$$+t^{-n}\left(\sum_{k=1}^\infty\int_{2^kB\backslash 2^{k-1}B}\left(\frac{t}{t+|x-y|}\right)^{n\mu+1}\frac{dy}{(t+|y-z|)^{2n+2-2\delta}}\right)$$
$$\leq Ct^{-n}\left(\int_{B(x,t)}\frac{2^{2n+2-2\delta}dy}{(2t+|y-z|)^{2n+2-2\delta}}\right.$$
$$\left.+\sum_{k=1}^\infty\int_{2^kB\backslash 2^{k-1}B}\left(\frac{t}{t+2^{k-1}t}\right)^{n\mu+1}\frac{2^{(k+1)(2n+2-2\delta)}dy}{(2^{k+1}t+|y-z|)^{2n+2-2\delta}}\right)$$
$$\leq Ct^{-n}\left(\int_{B(x,t)}\frac{dy}{(t+|x-z|)^{2n+2-2\delta}}\right.$$
$$\left.+\sum_{k=1}^\infty 2^{(1-k)(n\mu+1)}\int_{2^kB}\frac{2^{k(2n+2-2\delta)}dy}{(t+|x-z|)^{2n+2-2\delta}}\right)$$
$$\leq Ct^{-n}\left(t^n+\sum_{k=1}^\infty 2^{-k(n\mu+1)}2^{k(2n+2-2\delta)}(2^kt)^n\right)\frac{1}{(t+|x-z|)^{2n+2-2\delta}}$$
$$\leq C\left(1+\sum_{k=1}^\infty 2^{k(3n-n\mu+1-2\delta)}\right)\frac{1}{(t+|x-z|)^{2n+2-2\delta}}$$
$$\leq\frac{C}{(t+|x-z|)^{2n+2-2\delta}}$$

and notice that $|x-z| \sim |x_0 - z|$ for $x \in Q$ and $z \in R^n \backslash Q$. We obtain

$$\int_{Q^c} \prod_{j=1}^{m} |b_j(z) - (b_j)_Q||f(z)||x - x_0|^{1/2} \times$$

$$\times \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\mu+1} \frac{t^{-n} dy dt}{(t+|y-z|)^{2n+2-2\delta}} \right)^{1/2} dz$$

$$\leq C \int_{Q^c} \prod_{j=1}^{m} |b_j(z) - (b_j)_Q||f(z)||x - x_0|^{1/2} \left(\int_0^\infty \frac{dt}{(t+|x-z|)^{2n+2-2\delta}} \right)^{1/2} dz$$

$$\leq C \sum_{k=1}^{\infty} \int_{2^k Q \backslash 2^{k-1} Q} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2-\delta)} |f(z)| \prod_{j=1}^{m} |b_j(z) - (b_j)_Q| dz$$

$$\leq C \sum_{k=1}^{\infty} 2^{-k/2} \left(\frac{1}{|2^k Q|^{1-\delta/n}} \int_{2^k Q} \left| \prod_{j=1}^{m} (b_j(y) - (b_j)_Q) \right|^{r'} dy \right)^{1/r'} \times$$

$$\times \left(\frac{1}{|2^k Q|^{1-\delta/n}} \int_{2^k Q} |f(y)|^r dy \right)^{1/r}$$

$$\leq C \sum_{k=1}^{\infty} 2^{k(m\beta-1/2)} ||\vec{b}||_{\mathrm{Lip}_\beta} |Q|^{m\beta/n} M_{\delta,r}(f)(\tilde{x})$$

$$\leq C ||\vec{b}||_{\mathrm{Lip}_\beta} |Q|^{m\beta/n} M_{\delta,r}(f)(\tilde{x}),$$

so

$$IV \leq C ||\vec{b}||_{\mathrm{Lip}_\beta} M_{\delta,r}(f)(\tilde{x}).$$

We put these estimates together, by using Lemma 1 and taking the supremum over all Q such that $\tilde{x} \in Q$, we obtain

$$\begin{aligned} ||g_{\mu,\delta}^{\vec{b}}(f)||_{\dot{F}_q^{m\beta,\infty}} &\leq C ||\vec{b}||_{\mathrm{Lip}_\beta} (||M(g_{\mu,\delta}(f)) + M_{\delta,r}(f)||_{L^q}) \\ &\leq C ||\vec{b}||_{\mathrm{Lip}_\beta} (||M(g_{\mu,\delta}(f))||_{L^q} + ||M_{\delta,r}(f)||_{L^q}) \\ &\leq C ||\vec{b}||_{\mathrm{Lip}_\beta} (||g_{\mu,\delta}(f)||_{L^q} + ||M_{\delta,r}(f)||_{L^q}) \\ &\leq C ||\vec{b}||_{\mathrm{Lip}_\beta} ||f||_{L^p}. \end{aligned}$$

This completes the proof of (a).
(b). By some argument as in proof of (a), we have

$$\frac{1}{|Q|} \int_Q |g_{\mu,\delta}^{\vec{b}}(f)(x) - g_{\mu,\delta}(((b_1)_Q - b_1) \ldots ((b_m)_Q - b_m) f_2)(x_0)| dx$$

$$\leq \frac{1}{|Q|} \int_Q I_1(x) dx + \frac{1}{|Q|} \int_Q I_2(x) dx + \frac{1}{|Q|} \int_Q I_3(x) dx + \frac{1}{|Q|} \int_Q I_4(x) dx$$

$$= V_1 + V_2 + V_3 + V_4.$$

For V_1, taking $1/s = 1/r - \delta/n$, by the boundedness of $g_{\mu,\delta}$ from $L^r(R^n)$ to $L^s(R^n)$, so we have

$$\begin{aligned}
V_1 &\le \frac{1}{|Q|} \sup_{x\in Q} |b_1(x) - (b_1)_Q| \dots |b_m(x) - (b_m)_Q| \int_Q |g_{\mu,\delta}(f)(x)|dx \\
&\le C||\vec{b}||_{\mathrm{Lip}_\beta} |Q|^{m\beta/n} \left(\frac{1}{|Q|} \int_Q |g_{\mu,\delta}(f)(x)|^s dx \right)^{1/s} \\
&\le C||\vec{b}||_{\mathrm{Lip}_\beta} |Q|^{m\beta/n - 1/s} \left(\int_Q |f(x)|^r dx \right)^{1/r} \\
&= C||\vec{b}||_{\mathrm{Lip}_\beta} \left(\frac{1}{|Q|^{1-(m\beta+\delta)r/n}} \int_Q |f(x)|^r dx \right)^{1/r} \\
&\le C||\vec{b}||_{\mathrm{Lip}_\beta} M_{m\beta+\delta,r}(f)(\tilde{x}).
\end{aligned}$$

For V_2, taking $1/s' + 1/s = 1$,$1/s = 1/r - \delta/n$, by using the Hölder's inequality and the boundedness of $g_{\mu,\delta}$ from $L^r(R^n)$ to $L^s(R^n)$, we get

$$\begin{aligned}
V_2 &\le \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} \frac{1}{|Q|} \int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma||g_{\mu,\delta}((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)|dx \\
&\le C \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} \left(\frac{1}{|Q|} \int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{s'} dx \right)^{1/s'} \times \\
&\quad \times \left(\frac{1}{|Q|} \int_{R^n} |g_{\mu,\delta}((\vec{b} - \vec{b}_Q)_{\sigma^c} f \chi_Q)(x)|^s dx \right)^{1/s} \\
&\le C \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} |Q|^{-1/s} \left(\frac{1}{|Q|} \int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{s'} dx \right)^{1/s'} \times \\
&\quad \times \left(\int_{R^n} |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c} f(x)\chi_Q(x)|^r dx \right)^{1/r} \\
&\le C \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} |Q|^{-1/s+1/r} ||\vec{b}_\sigma||_{\mathrm{Lip}_\beta} |Q|^{|\sigma|\beta/n} ||\vec{b}_{\sigma^c}||_{\mathrm{Lip}_\beta} |Q|^{|\sigma^c|\beta/n} \times \\
&\quad \times \left(\frac{1}{|Q|} \int_Q |f(x)|^r dx \right)^{1/r} \\
&\le C||\vec{b}||_{\mathrm{Lip}_\beta} \left(\frac{1}{|Q|^{1-(m\beta+\delta)r/n}} \int_Q |f(x)|^r dx \right)^{1/r} \\
&\le C||\vec{b}||_{\mathrm{Lip}_\beta} M_{m\beta+\delta,r}(f)(\tilde{x}).
\end{aligned}$$

For V_3, by the boundness of $g_{\mu,\delta}$ from $L^r(R^n)$ to $L^s(R^n)$ and Hölder's inequality we get

$$V_3 \le C\left(\frac{1}{|Q|}\int_{R^n}|g_{\mu,\delta}(\prod_{j=1}^m(b_j-(b_j)_Q)f\chi_Q)(x)|^s dx\right)^{1/s}$$
$$\le C|Q|^{-1/s}\left(\int_{R^n}|\prod_{j=1}^m(b_j(x)-(b_j)_Q)|^r|f(x)\chi_Q(x)|^r dx\right)^{1/r}$$
$$\le C|Q|^{-1/s}|Q|^{m\beta/n}|Q|^{1/r}||\vec{b}||_{Lip_\beta}\left(\frac{1}{|Q|}\int_Q|f(x)|^r dx\right)^{1/r}$$
$$\le C||\vec{b}||_{Lip_\beta}M_{m\beta+\delta,r}(f)(\tilde{x}).$$

For V_4, similar to the proof of IV in (a), we get

$$\int_{(Q)^c}\prod_{j=1}^m|b_j(z)-(b_j)_Q||f(z)||x-x_0|^{1/2}\times$$
$$\times\left(\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu+1}\frac{t^{-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}\right)^{1/2}dz$$
$$\le C\int_{(Q)^c}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}|f(z)|\prod_{j=1}^m|b_j(z)-(b_j)_Q|dz$$
$$\le C\sum_{k=1}^\infty 2^{-k/2}\left(\frac{1}{|2^kQ|^{1-\delta/n}}\int_{2^kQ}\left|\prod_{j=1}^m(b_j(y)-(b_j)_Q)\right|^{r'}dy\right)^{1/r'}\times$$
$$\times\left(\frac{1}{|2^kQ|^{1-\delta/n}}\int_{2^kQ}|f(y)|^r dy\right)^{1/r}$$
$$\le C\left(\frac{1}{|2^kQ|^{1-(m\beta+\delta)r/n}}\int_{2^kQ}|f(y)|^r dy\right)^{1/r}||\vec{b}||_{Lip_\beta}$$
$$\le C||\vec{b}||_{\mathrm{Lip}_\beta}M_{m\beta+\delta,r}(f)(\tilde{x}).$$

So we have

$$\frac{1}{|Q|}\int_Q|g_{\mu,\delta}^{\vec{b}}(f)(x)-g_{\mu,\delta}(((b_1)_Q-b_1)\dots((b_m)_Q-b_m)f_2)(x_0)|dx$$
$$\le C||\vec{b}||_{\mathrm{Lip}_\beta}M_{m\beta+\delta,r}(f)(\tilde{x}).$$

Thus,

$$(g_{\mu,\delta}^{\vec{b}}(f))^\# \le C||\vec{b}||_{\mathrm{Lip}_\beta}M_{m\beta+\delta,r}(f).$$

By using Lemma 3 and the boundedness of $g_{\mu,\delta}$, we have

$$||g^{\vec{b}}_{\mu,\delta}(f)||_{L^q} \le C||(g^{\vec{b}}_{\mu,\delta}(f))^{\#}||_{L^q}$$
$$\le C||\vec{b}||_{\mathrm{Lip}_\beta}||M_{m\beta+\delta,r}(f)||_{L^q} \le C||\vec{b}||_{\mathrm{Lip}_\beta}||f||_{L^p}.$$

This completes the proof of (b) and the theorem. □

Theorem 2. *Let* $0 < \beta \le 1$, $\mu > 3 + 1/n - 2\delta/n$, $n/(n+m\beta) < p \le 1$, $1/q = 1/p - (m\beta+\delta)/n$, $b_j \in \mathrm{Lip}_\beta(R^n)$ *for* $1 \le j \le m$*. Then* $g^{\vec{b}}_{\mu,\delta}$ *is bounded from* $H^p(R^n)$ *to* $L^q(R^n)$.

Proof. It suffices to show that there exists a constant $C > 0$ such that for every H^p-atom a,

$$||g^{\vec{b}}_{\mu,\delta}(a)||_{L^q} \le C.$$

Let a be a H^p-atom, that is that a supported on a cube $Q = Q(x_0, r)$, $||a||_{L^\infty} \le |Q|^{-1/p}$ and $\int_{R^n} a(x)x^\gamma dx = 0$ for $|\gamma| \le [n(1/p-1)]$.

When $m = 1$ see [9]. Now, consider the case $m \ge 2$. Write

$$||g^{\vec{b}}_{\mu,\delta}(a)(x)||_{L^q} \le \left(\int_{|x-x_0|\le 2r} |g^{\vec{b}}_{\mu,\delta}(a)(x)|^q dx\right)^{1/q}$$
$$+ \left(\int_{|x-x_0|>2r} |g^{\vec{b}}_{\mu,\delta}(a)(x)dx|^q\right)^{1/q} = I + II.$$

For I, choose $1 < p_1 < n/(m\beta+\delta)$ and q_1 such that $1/q_1 = 1/p_1 - (m\beta+\delta)/n$. By the boundedness of $g^{\vec{b}}_{\mu,\delta}$ from $L^{p_1}(R^n)$ to $L^{q_1}(R^n)$ (see Theorem 1), the size condition of a and Hölder's inequality, we get

$$I \le C||g^{\vec{b}}_{\mu,\delta}(a)||_{L^{q_1}} r^{n(1/q-1/q_1)} \le C||\vec{b}||_{\mathrm{Lip}_\beta}||a||_{L^{p_1}} r^{n(1/q-1/q_1)}$$
$$\le C||\vec{b}||_{\mathrm{Lip}_\beta} r^{n(-1/p+1/p_1)} r^{n(1/q-1/q_1)} \le C||\vec{b}||_{\mathrm{Lip}_\beta}.$$

For II, let $\tau, \tau' \in N$ such that $\tau + \tau' = m$, and $\tau' \ne 0$. We get

$$|F^{\vec{b}}_t(a)(x,y)| \le$$
$$\le |(b_1(x)-b_1(x_0))\dots(b_m(x)-b_m(x_0))\int_B (\psi_t(y-z)-\psi_t(y-x_0))a(z)dz|$$
$$+\sum_{j=1}^m \sum_{\sigma\in C^m_j} |(\vec{b}(x)-\vec{b}(x_0))_{\sigma^c}\int_B (\vec{b}(z)-\vec{b}(x_0))_\sigma \psi_t(y-z)a(z)dz|$$
$$\le C||\vec{b}||_{\mathrm{Lip}_\beta}|x-x_0|^{m\beta}\cdot\int_B |\psi_t(y-z)-\psi_t(y-x_0)||a(z)|dz$$
$$+C||\vec{b}||_{\mathrm{Lip}_\beta}\sum_{\tau+\tau'=m}|x-x_0|^{\tau\beta}\int_B |z-x_0|^{\tau'\beta}|\psi_t(y-z)||a(z)|dz$$
$$\le C||\vec{b}||_{\mathrm{Lip}_\beta}\frac{|x-x_0|^{m\beta}t}{(t+|y-x_0|)^{n+2-\delta}}\int_B |x_0-z||a(z)|dz$$

$$+C||\vec{b}||_{\mathrm{Lip}_\beta}\sum_{\tau+\tau'=m}|x-x_0|^{\tau\beta}\frac{t}{(t+|y-x_0|)^{n+1-\delta}}\int_B|z-x_0|^{\tau'\beta}|a(z)|dz$$
$$\leq C||\vec{b}||_{\mathrm{Lip}_\beta}\frac{t}{(t+|y-x_0|)^{n+2-\delta}}\cdot r^{1+n(1-1/p)}\cdot|x-x_0|^{m\beta}$$
$$+C||\vec{b}||_{\mathrm{Lip}_\beta}\frac{t}{(t+|y-x_0|)^{n+1-\delta}}\cdot\sum_{\tau+\tau'=m}r^{\tau'\beta+n(1-1/p)}\cdot|x-x_0|^{\tau\beta}.$$

Thus,

$$|g^{\vec{b}}_{\mu,\delta}(a)(x)|\leq$$
$$\leq C||\vec{b}||_{\mathrm{Lip}_\beta}\left(\int_0^\infty\left(\frac{t}{(t+|x-x_0|)^{n+2-\delta}}\right)^2\frac{dt}{t}\right)^{1/2}\cdot r^{1+n(1-1/p)}\cdot|x-x_0|^{m\beta}$$
$$+C||\vec{b}||_{\mathrm{Lip}_\beta}\left(\int_0^\infty\left(\frac{t}{(t+|x-x_0|)^{n+1-\delta}}\right)^2\frac{dt}{t}\right)^{1/2}\times$$
$$\times\sum_{\tau+\tau'=m}r^{\tau'\beta+n(1-1/p)}\cdot|x-x_0|^{\tau\beta}$$
$$\leq C||\vec{b}||_{\mathrm{Lip}_\beta}|x-x_0|^{-(n+1-\delta)}\cdot r^{1+n(1-1/p)}\cdot|x-x_0|^{m\beta}$$
$$+C||\vec{b}||_{\mathrm{Lip}_\beta}|x-x_0|^{-(n-\delta)}\cdot\sum_{\tau+\tau'=m}r^{\tau'\beta+n(1-1/p)}\cdot|x-x_0|^{\tau\beta},$$

so

$$II\leq C||\vec{b}||_{\mathrm{Lip}_\beta}\cdot r^{1+n(1-1/p)}\left(\int_{|x-x_0|>2r}|x-x_0|^{-(n+1-\delta-m\beta)q}dx\right)^{1/q}$$
$$+C||\vec{b}||_{\mathrm{Lip}_\beta}\cdot\sum_{\tau+\tau'=m}r^{\tau'\beta+n(1-1/p)}\left(\int_{|x-x_0|>2r}|x-x_0|^{-(n-\delta-\tau\beta)q}dx\right)^{1/q}$$
$$=J_1+J_2,$$

we get

$$J_1\leq C||\vec{b}||_{\mathrm{Lip}_\beta}\cdot r^{1+n(1-1/p)}\left(\sum_{k=1}^\infty\int_{2^{k+1}Q\backslash 2^kQ}(2^kr)^{-(n+1-\delta-m\beta)q}dx\right)^{1/q}$$
$$\leq C||\vec{b}||_{\mathrm{Lip}_\beta}\cdot r^{1+n(1-1/p)}\left(\sum_{k=1}^\infty 2^{-kq(n+1-\delta-m\beta)}r^{-q(n+1-\delta-m\beta)}2^{(k+1)n}r^ndx\right)^{1/q}$$
$$\leq C||\vec{b}||_{\mathrm{Lip}_\beta}\sum_{k=1}^\infty 2^{-k(n+1-\delta-m\beta-n/q)}r^{1+n(1-1/p)-(n+1-\delta-m\beta)+n/q}$$
$$\leq C||\vec{b}||_{\mathrm{Lip}_\beta}.$$

For J_2, similar to J_1, we have $J_2 \leq C||\vec{b}||_{\mathrm{Lip}_\beta}$.
Combining the estimates for I and II, then leads to the desired result. □

Theorem 3. *Let* $0 < \beta \leq 1$, $\mu > 3 + 1/n - 2\delta/n$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = (m\beta + \delta)/n$, $n(1 - 1/q_1) \leq \alpha < n(1 - 1/q_1) + \varepsilon$, $\varepsilon < \min(1, m\beta)$, $b_j \in \mathrm{Lip}_\beta(R^n)$ *for* $1 \leq j \leq m$. *Then* $g^{\vec{b}}_{\mu,\delta}$ *is bounded from* $H\dot{K}^{\alpha,p}_{q_1}(R^n)$ *to* $\dot{K}^{\alpha,p}_{q_2}(R^n)$.

Proof. By Lemma 7, let $f \in H\dot{K}^{\alpha,p}_{q_1}(R^n)$ and $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, $\operatorname{supp} a_j \subset B_j = B(0, 2^j)$, a_j be a central $(\alpha, q_1)-$atom, and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$.

$$||g^{\vec{b}}_{\mu,\delta}(f)||^p_{\dot{K}^{\alpha,p}_{q_2}} \leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| ||g^{\vec{b}}_{\mu,\delta}(a_j)\chi_k||_{L^{q_2}} \right)^p + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j| ||g^{\vec{b}}_{\mu,\delta}(a_j)\chi_k||_{L^{q_2}} \right)^p = I + II.$$

For II, by the boundedness of $g^{\vec{b}}_{\mu,\delta}$ on (L^{q_1}, L^{q_2}) (see Theorem 1), we have

$$\begin{aligned} II &\leq C||\vec{b}||^p_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\sum_{j=k-1}^{\infty} |\lambda_j| ||a_j||_{L^{q_1}})^p \\ &\leq C||\vec{b}||^p_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{\infty} (\sum_{j=k-1}^{\infty} |\lambda_j| \cdot 2^{(k-j)\alpha})^p \\ &\leq C||\vec{b}||^p_{\mathrm{Lip}_\beta} \begin{cases} \sum\limits_{j=-\infty}^{\infty} |\lambda_j|^p \sum\limits_{k=-\infty}^{j+1} 2^{(k-j)\alpha p}, \ 0 < p \leq 1 \\ \sum\limits_{j=-\infty}^{\infty} (\sum\limits_{k=-\infty}^{j+1} |\lambda_j|^p \cdot 2^{(k-j)\alpha p/2})(\sum\limits_{k=-\infty}^{j+1} 2^{(k-j)\alpha p'/2})^{p/p'}, \ 1 < p < \infty \end{cases} \\ &\leq C||\vec{b}||^p_{\mathrm{Lip}_\beta} \sum_{j=-\infty}^{\infty} |\lambda_j|^p. \end{aligned}$$

For I, we have

$$\begin{aligned} &|F^{\vec{b}}_t(a_j))(x,y)| \leq \\ &\leq |(b_1(x) - b_1(0)) \cdots (b_m(x) - b_m(0)) \int_{B_j} (\psi_t(y-z) - \psi_t(y)) a_j(z) dz| \\ &\quad + \sum_{j=1}^{\infty} \sum_{\sigma \in C^m_j} |(\vec{b}(x) - \vec{b}(0))_{\sigma^c} \int_{B_j} (\vec{b}(z) - \vec{b}(0))_\sigma \psi_t(y-z) a_j(z) dz| \\ &\leq C||\vec{b}||_{\mathrm{Lip}_\beta} |x|^{m\beta} \int_{B_j} |\psi_t(y-z) - \psi_t(y)||a_j(z)| dz \end{aligned}$$

$$+C\|\vec{b}\|_{\mathrm{Lip}_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}\int_{B_j}|z|^{\tau'\beta}|\psi_t(y-z)||a_j(z)|dz$$
$$\leq C\|\vec{b}\|_{\mathrm{Lip}_\beta}\frac{|x|^{m\beta}t}{(t+|y|)^{n+2-\delta}}\cdot 2^{j(1+n(1-1/q_1)-\alpha)}$$
$$+C\|\vec{b}\|_{\mathrm{Lip}_\beta}\sum_{\tau+\tau'=m}\frac{|x|^{\tau\beta}t}{(t+|y|)^{n+1-\delta}}\cdot 2^{j(\tau'\beta+n(1-1/q_1)-\alpha)}.$$

Thus,

$$g_{\mu,\delta}^{\vec{b}}(a_j)(x)\leq$$
$$\leq C\|\vec{b}\|_{\mathrm{Lip}_\beta}|x|^{m\beta}\cdot 2^{j(1+n(1-1/q_1)-\alpha)}\cdot\left(\int_0^\infty\left(\frac{t}{(t+|x|)^{n+2-\delta}}\right)^2\frac{dt}{t}\right)^{1/2}$$
$$+C\|\vec{b}\|_{\mathrm{Lip}_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}\cdot 2^{j(\tau'\beta+n(1-1/q_1)-\alpha)}\times$$
$$\times\left(\int_0^\infty\left(\frac{t}{(t+|x|)^{n+1-\delta}}\right)^2\frac{dt}{t}\right)^{1/2}$$
$$\leq C\|\vec{b}\|_{\mathrm{Lip}_\beta}|x|^{m\beta}|x|^{-(n+1-\delta)}\cdot 2^{j(1+n(1-1/q_1)-\alpha)}$$
$$+C\|\vec{b}\|_{\mathrm{Lip}_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}|x|^{-(n-\delta)}\cdot 2^{j(\tau'\beta+n(1-1/q_1)-\alpha)},$$

and

$$\|g_{\mu,\delta}^{\vec{b}}(a_j)\chi_k\|_{L^{q_2}}\leq$$
$$\leq C\|\vec{b}\|_{\mathrm{Lip}_\beta}\cdot 2^{j(1+n(1-1/q_1)-\alpha)}\cdot\left(\int_{B_k\setminus B_{k-1}}|x|^{-(n-(m\beta+\delta)+1)q_2}dx\right)^{1/q_2}$$
$$+C\|\vec{b}\|_{\mathrm{Lip}_\beta}\sum_{\tau+\tau'=m}2^{j(\tau'\beta+n(1-1/q_1)-\alpha)}\cdot\left(\int_{B_k\setminus B_{k-1}}|x|^{-(n-\tau\beta-\delta)q_2}dx\right)^{1/q_2}$$
$$\leq C\|\vec{b}\|_{\mathrm{Lip}_\beta}(2^{j(1+n(1-1/q_1)-\alpha)-k(1+n(1-1/q_1))}$$
$$+\sum_{\tau+\tau'=m}2^{j(\tau'\beta+n(1-1/q_1)-\alpha)-k(\tau'\beta+n(1-1/q_1))})$$
$$\leq C\|\vec{b}\|_{\mathrm{Lip}_\beta}2^{-k\alpha}\left(2^{(j-k)(1+n(1-1/q_1)-\alpha)}+2^{(j-k)(m\beta+n(1-1/q_1)-\alpha)}\right),$$

so

$$I \le C||\vec{b}||^p_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \cdot 2^{(j-k)(1+n(1-1/q_1)-\alpha)} \right)^p + C||\vec{b}||^p_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \cdot 2^{(j-k)(m\beta+n(1-1/q_1)-\alpha)} \right)^p .$$

When $0 < p \le 1$,

$$\begin{aligned} I &\le C||\vec{b}||^p_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p \cdot 2^{p(j-k)(1+n(1-1/q_1)-\alpha)} \right) \\ &\quad + C||\vec{b}||^p_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p \cdot 2^{p(j-k)(m\beta+n(1-1/q_1)-\alpha)} \right) \\ &\le C||\vec{b}||^p_{\mathrm{Lip}_\beta} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+2}^{\infty} 2^{p(j-k)(1+n(1-1/q_1)-\alpha)} \right. \\ &\quad \left. + \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+2}^{\infty} 2^{p(j-k)(m\beta+n(1-1/q_1)-\alpha)} \right) \\ &\le C||\vec{b}||^p_{\mathrm{Lip}_\beta} \sum_{j=-\infty}^{\infty} |\lambda_j|^p. \end{aligned}$$

When $p > 1$,

$$\begin{aligned} I &\le C||\vec{b}||^p_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p 2^{p(j-k)(1+n(1-1/q_1)-\alpha)/2} \right) \times \\ &\quad \times \left(\sum_{j=-\infty}^{k-2} 2^{p'(j-k)(1+n(1-1/q_1)-\alpha)/2} \right)^{p/p'} \\ &\quad + C||\vec{b}||^p_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p 2^{p(j-k)(m\beta+n(1-1/q_1)-\alpha)/2} \right) \times \\ &\quad \times \left(\sum_{j=-\infty}^{k-2} 2^{p'(j-k)(m\beta+n(1-1/q_1)-\alpha)/2} \right)^{p/p'} \\ &\le C||\vec{b}||^p_{\mathrm{Lip}_\beta} \sum_{j=-\infty}^{\infty} |\lambda_j|^p. \end{aligned}$$

From I and II, we have

$$||g^{\vec{b}}_{\mu,\delta}(f)||_{\dot{K}^{\alpha,p}_{q_2}} \le C||\vec{b}||_{\mathrm{Lip}_\beta} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \le C||f||_{H\dot{K}^{\alpha,p}_{q_1}}$$

This completes the proof of Theorem 3. □

When $\alpha = n(1-1/q_1)+\varepsilon$, $\varepsilon < \min(1, m\beta)$, this kind of boundedness fails. Now, we give an estimate of weak type.

Theorem 4. *Let $0<\beta\le 1$, $0<p\le 1$, $1<q_1,q_2<\infty$ $1/q_2 = 1/q_1-(m\beta+\delta)/n$, $b_j \in \mathrm{Lip}_\beta(R^n)$ for $1\le j\le m$. Then $g^{\vec{b}}_{\mu,\delta}$ maps $H\dot{K}^{n(1-1/q_1)+\varepsilon,p}_{q_1}(R^n)$ continuously into $W\dot{K}^{n(1-1/q_1)+\varepsilon,p}_{q_2}(R^n)$, where $0<\varepsilon<\min(1,m\beta)$.*

Proof. We write $f=\sum_{k=-\infty}^{\infty}\lambda_k a_k$, where each a_k is a central $(n(1-1/q_1)+\varepsilon, q_1)$ atom supported on B_k and $\sum_{k=-\infty}^{\infty}|\lambda_k|^p<\infty$. Write

$$||g^{\vec{b}}_{\mu,\delta}||_{W\dot{K}^{n(1-1/q_1)+\varepsilon,p}_{q_2}} \le$$

$$\le \sup_{\lambda>0}\lambda\{\sum_{l=-\infty}^{\infty}2^{l(n(1-1/q_1)+\varepsilon)p}|\{x\in E_l : |g^{\vec{b}}_{\mu,\delta}(\sum_{k=l-3}^{\infty}\lambda_k a_k)(x)|>\lambda/2\}|^{p/q_2}\}^{1/p}$$

$$+\sup_{\lambda>0}\lambda\{\sum_{l=-\infty}^{\infty}2^{l(n(1-1/q_1)+\varepsilon)p}|\{x\in E_l : |g^{\vec{b}}_{\mu,\delta}(\sum_{k=-\infty}^{l-4}\lambda_k a_k)(x)|>\lambda/2\}|^{p/q_2}\}^{1/p}$$

$$= G_1+G_2.$$

By the (L^{q_1}, L^{q_2}) boundedness of $g^{\vec{b}}_{\mu,\delta}$ and an estimate similar to that for I_1 in Theorem 3, we get

$$G_1^p \le C\sum_{l=-\infty}^{\infty}2^{lp(n(1-1/q_1)+\varepsilon)}||g^{\vec{b}}_{\mu,\delta}(\sum_{k=l-3}^{\infty}\lambda_k a_k)(x)\chi_l||^p_{q_2}\le C||\vec{b}||^p_{\mathrm{Lip}_\beta}\sum_{k=-\infty}^{\infty}|\lambda_k|^p.$$

To estimate G_2, let us now use the estimate

$$g^{\vec{b}}_{\mu,\delta}(a_k)(x)\le C||\vec{b}||_{\mathrm{Lip}_\beta}|x|^{m\beta}|x|^{-(n+1-\delta)}\cdot 2^{k(1+n(1-1/q_1)-\alpha)}$$
$$+C||\vec{b}||_{\mathrm{Lip}_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}|x|^{-(n-\delta)}\cdot 2^{k(\tau'\beta+n(1-1/q_1)-\alpha)},$$

which we get in the proof of Theorem 3. Note that when $x\in E_l$,

$$\alpha = n(1-1/q_1)+\varepsilon,$$

$$\lambda < 2\sum_{k=-\infty}^{l-4}|\lambda_k||g^{\vec{b}}_{\mu,\delta}(a_k)(x)|\le$$

$$\le C||\vec{b}||_{\mathrm{Lip}_\beta}\sum_{k=-\infty}^{l-4}|\lambda_k|(2^l)^{m\beta+\delta-n-1}\sum_{k=-\infty}^{l-4}(2^k)^{1+n(1-1/q_1)-\alpha}$$

$$+C||\vec{b}||_{\mathrm{Lip}_\beta}\sum_{k=-\infty}^{l-4}|\lambda_k|\sum_{\tau+\tau'=m}(2^l)^{\tau\beta+\delta-n}\sum_{k=-\infty}^{l-4}(2^k)^{\tau'\beta+n(1-1/q_1)-\alpha}$$

$$\leq C||\vec{b}||_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{l-4} |\lambda_k| \dot{(} 2^l)^{m\beta+\delta-n-\varepsilon}$$

$$\leq C||\vec{b}||_{\mathrm{Lip}_\beta} 2^{l(m\beta+\delta-n-\varepsilon)} (\sum_{k=-\infty}^{\infty} |\lambda_k|^p)^{1/p},$$

for $\lambda > 0$, let l_λ be the maximal positive integer satisfying

$$2^{l_\lambda(n+\varepsilon-m\beta-\delta)} \leq C||\vec{b}||_{\mathrm{Lip}_\beta} \lambda^{-1} (\sum_{k=-\infty}^{\infty} |\lambda_k|^p)^{1/p},$$

then if $l > l_\lambda$, we have

$$|\{x \in E_l : |g_{\mu,\delta}^{\vec{b}}(\sum_{k=-\infty}^{l-4} \lambda_k a_k)| > \lambda/2\}| = 0,$$

so we obtain

$$\begin{aligned} G_2 &\leq \sup_{\lambda>0} \lambda \{ \sum_{l=-\infty}^{l_\lambda} 2^{l(n(1-1/q_1)+\varepsilon)p} (2^l)^{np/q_2} \}^{1/p} \\ &\leq \sup_{\lambda>0} \lambda \{ \sum_{l=-\infty}^{l_\lambda} (2^l)^{(n+\varepsilon-m\beta-\delta)} \} \\ &\leq \sup_{\lambda>0} \lambda 2^{l_\lambda((n+\varepsilon-m\beta-\delta)} \\ &\leq C||\vec{b}||_{\mathrm{Lip}_\beta} (\sum_{k=-\infty}^{\infty} |\lambda_k|^p)^{1/p}. \end{aligned}$$

Now, combining the above estimates for G_1 and G_2, we obtain

$$||g_{\mu,\delta}^{\vec{b}}(f)||_{W\dot{K}_{q_2}^{n(1-1/q_1)+\varepsilon,p}} \leq C||\vec{b}||_{\mathrm{Lip}_\beta} (\sum_{k=-\infty}^{\infty} |\lambda_k|^p)^{1/p}.$$

Theorem 4 follows by taking the infimum over all central atomic decompositions. □

Acknowledgement

The authors would like to express their gratitude to the referee for his comments.

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Received June 3, 2008.

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