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ON SUBMANIFOLDS OF CODIMENSION 2 IMMERSED IN A HSU – QUATERNION MANIFOLD

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Abstract. Integrability conditions of an almost quaternion manifold were studied by Yano and Ako [12]. Quaternion submanifolds of codimension 2 have been defined and studied by A. Hamoui [8] and others. In this paper, we have defined a Hsu-quaternion manifold and showed that a submanifold of codimension 2 of the Hsu-quaternion manifold admits Hsu – (F, U, V, u, v, η) –structure.

1. Introduction

A Hsu-quaternion manifold is the manifold M^{4n} admitting a set of tensor fields $\overset{*}{F}$, $\overset{*}{G}$, $\overset{*}{H}$ of type (1,1) satisfying following relations [9].

$$\overset{*}{F}{}^2 = a^r I_n, \quad \overset{*}{G}{}^2 = b^r I_n \, \overset{*}{H}{}^2 = c^r I_n; \quad 0 \le r \le n \text{ and } c^r = a^r b^r \tag{1.1}$$

I_n being identity operator; a, b, c complex numbers and r an integer such that

$$b^r \overset{*}{F} = \overset{*}{G}\overset{*}{H} = \overset{*}{H}\overset{*}{G} \tag{1.2a}$$

$$a^r \overset{*}{G} = \overset{*}{H}\overset{*}{F} = \overset{*}{F}\overset{*}{H} \tag{1.2b}$$

$$\overset{*}{H} = \overset{*}{F}\overset{*}{G} = \overset{*}{G}\overset{*}{F} \tag{1.2c}$$

Let M^{4n-2} be the submanifold of codimension 2 of the Hsu-quaternion manifold M^{4n}. Let B represent the differential of immersion $\tau\colon M^{4n-2} \to M^{4n}$. Suppose further that C and D are mutually orthogonal unit normals to M^{4n}. Let $\overset{*}{F}BX$, the transformation of BX by$\overset{*}{F}$, be expressed as

$$\overset{*}{F}BX = BFX + u(X)C + v(X)D \tag{1.3}$$

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where X is an arbitrary vector field, u, v 1-forms and F is tensor field of type (1,1) on M^{4n-2}

Corresponding to the (1,1) tensor fields $\overset{*}{F}$, $\overset{*}{G}$, $\overset{*}{H}$ we introduce the vector fields U, U', U'', V, V', V'', 1 – forms u, u', u'', v, v', v'' and a function η such that

$$\overset{*}{F} C = -BU + \eta D \tag{1.4a}$$

$$\overset{*}{F} D = -BV - \eta D \tag{1.4b}$$

similarly for the tensor fields $\overset{*}{G}$ and $\overset{*}{H}$ we can write transformation as follows

$$\overset{*}{G} BX = BGX + u'(X)C + v'(X)D \tag{1.5a}$$

$$\overset{*}{G} C = -BU' + \eta D \tag{1.5b}$$

$$\overset{*}{G} D = -BV' - \eta C \tag{1.5c}$$

and

$$\overset{*}{H} BX = BHX + u''(X)C + v''(X)D \tag{1.6a}$$

$$\overset{*}{H} C = -BU'' + \eta D \tag{1.6b}$$

$$\overset{*}{H} D = -BV'' - \eta C. \tag{1.6c}$$

A manifold Vm will be called to possess a Hsu – $(F,\ U,\ V,\ u,\ v,\ \eta)$ – structure if there exists a tensor field F of type (1,1), two vector fields U, V two 1 – forms u, v and a function η satisfying

$$F^2 = a^r I_n + u \otimes U + v \otimes V \tag{1.7a}$$

$$u \circ F = \eta v \tag{1.7b}$$

$$v \circ F = -\eta u \tag{1.7c}$$

$$F(U) = -\eta v \tag{1.7d}$$

$$F(V) = \eta U \tag{1.7e}$$

$$u(U) = v(V) = -\left(a^r I_n + \eta^2\right) \tag{1.7f}$$

$$u(V) = v(U) = 0 \tag{1.7g}$$

A manifold Vm will be called to possess Hsu – $(F,\ G,\ H,\ U,\ U',\ U'',\ V,\ V',\ V'',\ u,\ u',\ u'',\ v,\ v',\ v'',\ \eta)$ – structure if there exists tensor fields F, G, H each of type (1,1), vector fields U, U', U'', V, V', V''; 1 – form u, u', u'', v, v', v'' and a function η satisfying

(1.8a) $$GH = b^r F - u'' \otimes U' - v'' \otimes V'$$

(1.8b) $$u' \circ H = b^r u - \eta v''$$

(1.8c) $$v' \circ H = b^r v - \eta u''$$

(1.8d) $$b^r U = GU'' + \eta V'$$

(1.8e) $$u' \circ U'' = -\eta^2$$

(1.8f) $$v' \circ V'' = \eta^2$$

(1.8g) $$v' \circ U'' = -b^r \eta$$

(1.8h) $$GV'' = b^r V + \eta U'$$

(1.8i) $$u' \circ V'' = b^r \eta$$

2. Submanifolds of Hsu-quaternion manifold

In this section, we shall prove some theorems on the submanifolds M^{4n-2} of codimension 2 of Hsu-quaternion manifold M^{4n}.

Theorem 1. *The submanifold M^{4n-2} of codimension 2 of Hsu-quaternion manifold M^{4n} admits a Hsu – (F, U, V, u, v, η) –structure.*

Proof. Applying $\overset{*}{F}$ to (1.3) and (1.4a), (1.4b) and making use of (1.1) we obtain

$$a^r BX = BF^2X + u(FX)C + v(FX)D + u(X)FC + v(X)FD$$

using (1.4a), (1.4b) and equating of tangential and normal vector fields, we get

(2.1a) $$F^2X = a^r X + u(X)U + v(X)V$$

(2.1b) $$u(FX) = \eta v(X)$$

(2.1c) $$v(FX) = -\eta u(X)$$

Since C, D are mutually independent. Again operation of $\overset{*}{F}$ on (1.4a) and using (1.1) yields

$$a^r C = -\{BFU + u(U)C + v(U)D\} + \eta\{-BV - \eta C\}$$

Equating of tangential and normal fields gives

(2.2a) $$F(U) = -\eta V$$

(2.2b) $$u(U) = -\left(a^r I_n + \eta^2\right)$$

(2.2c) $$v(U) = 0$$

similarly applying $\overset{*}{F}$ to (1.4b) and equating tangential and normal vector fields, we obtain

(2.3a) $$F(V) = \eta U$$

(2.3b) $$v(V) = -\left(a^r I_n + \eta^2\right)$$

(2.3c) $$u(V) = 0$$

□

The theorem follows by the virtue of equations (2.1), (2.2) and (2.3).

Corollary 1. *The submanifold M^{4n-2} of codimension 2 of Hsu-quaternion manifold M^{4n} also admits similar structures with respect of tensor field $\overset{*}{G}$ and $\overset{*}{H}$.*

Theorem 2. *An orientable submanifold of codimension 2 of almost Hsu-quaternion manifold admits a F, G, H 3-structure expressed as*

$$(F, G, H, U, U', U'', V, V', V'', u, u', u'', v, v', v'', \eta)$$

Proof. Operating (1.2a) with $\overset{*}{F}$ both sides, we get

$$b^r \overset{*}{F} BX = \overset{*}{G} \overset{*}{H} BX$$

which in view of (1.3) and (2.1a) yields

$$BGHX + u'(HX)C + v'(XH)D + u'(X)(-BU' + \eta D) + v''(X)(-BV' - \eta C)$$
$$= b^r \{BFX + u(X)C + v(X)D\}$$

Equating of tangential and normal tensor fields gives

(2.4a) $$GHX = b^r FX - u^{''}(X)U' - v^{''}(X)V'$$

(2.4b) $$u'(HX) = b^r u(X) - \eta v^{''}(X)$$

(2.4c) $$v'(HX) = b^r v(X) - \eta u^{''}(X)$$

Also,

$$b^r \overset{*}{F} C = \overset{*}{G} \overset{*}{H} C$$

which in view of (1.4a) and (1.6) becomes

$$b^r(-BU + \eta D) = G(-BU^{''} + \eta D)$$

Making use of and on(1.5a) equating of tangential and normal vector fields, we get

(2.5a) $$b^r U = GU^{''} + \eta V$$

(2.5b) $$u'(U'') = -\eta^2$$

(2.5c) $$v'(U'') = -b^r\eta$$

and the equation $\overset{*}{G}\,\overset{*}{H}\,D = b^r\,\overset{*}{F}\,D$, yields in a similar manner the following results

(2.6a) $$GV'' = b^rV + \eta U'$$

(2.6b) $$u'(V'') = \eta b^r$$

(2.6c) $$v'(V'') = \eta^2$$

thus we have

(2.7a) $$GH = b^rF - u''(X)U' - v''(X)V'$$

(2.7b) $$v'oH = b^rv - \eta u''$$

(2.7c) $$u'oH = b^ru - \eta v''$$

(2.7d) $$GU'' = b^rV + \eta V'$$

(2.7e) $$GV'' = b^rV + \eta U'$$

(2.7f) $$u' \circ U'' = -\eta^2$$

(2.7g) $$v' \circ V'' = \eta^2$$

(2.7h) $$v' \circ U'' = -\eta b^r$$

(2.7i) $$u' \circ V'' = \eta b^r$$

similarly, we obtain the rest of the relations

(2.8a) $$HF = a^rG - u \otimes U'' - u \otimes V''$$

(2.8b) $$FG = H - u' \otimes U - v' \otimes V$$

Further more we have

$$\overset{*}{G}\,\overset{*}{H}\,BX = \overset{*}{H}\,\overset{*}{G}\,BX$$

(2.9a)

$$BGHX + u'(HX)C + v'(HX)D + u''(X)(-BU' + \eta D) + v''(X)(-BV' - \eta C)$$
$$= BHGX + u''(GX)C + v''(GX)D + u'(X)(-BU + \eta D) + v(X)(-BV'' - \eta C).$$

Equating of tangential and normal vector fields gives

(2.10a) $$(GH - HG)X = u''(X)U' + v''(X)V' - u'(X)U'' - v'(X)V''$$

(2.10b) $$u'(HX) - u''(GX) = \eta v''(X) - \eta v'(X)$$

(2.10c) $$v'(HX) - v''(GX) = \eta u'(X) - \eta u''(X)$$

Again

$$\overset{*}{G}\,\overset{*}{H}\,C = \overset{*}{H}\,\overset{*}{G}\,C \text{ and}$$

$$\overset{*}{G}\,\overset{*}{H}\,D = \overset{*}{H}\,\overset{*}{G}\,D$$

which yields in a similar manner

(2.11a) $$GU'' + \eta V' - HU' - \eta V'' = 0$$

(2.11b) $$u'(U'') = u''(U')$$

(2.11c) $$v'(U'') = v''(U')$$

and

(2.12a) $$GV''' + \eta U' - HV' - \eta U'' = 0$$

(2.12b) $$u'(V'') = u''(V')$$

(2.12c) $$v'(V'') = v''(V').$$

We can also prove that

(2.13a) $$HF - FH = u \otimes U'' + v \otimes V'' - u'' \otimes U - v'' \otimes V$$

(2.13b) $$u'' \circ F - u \circ H = \eta v - \eta v''$$

(2.13c) $$v'' \circ F - v \circ H = \eta u'' - \eta u$$

(2.13d) $$HU - FU'' + \eta V'' - \eta V = 0$$

(2.13e) $$u'(U) = u(U'')$$

(2.13f) $$v(U) = v(U'')$$

(2.13g) $$HV - FV'' + \eta U - \eta U'' = 0$$

(2.13h) $$u''(V) = u(V)$$

(2.13i) $$v''(V) = v(V'')$$

the theorem is proved by virtue of equation (2.4) to (2.13) □

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