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A SIMPLE PROOF OF A WEIGHTED INEQUALITY FOR THE HARDY-LITTLEWOOD MAXIMAL OPERATOR IN $\mathbb{R}^n$

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Abstract. In this note we give a simple proof of the characterization of the weights for which the Hardy-Littlewood maximal operator maps L_p weighted space into weak L_p space.

1. Introduction.

The purpose of this note is to provide a simple proof of the weak (p,p) inequality

$$w\left(\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}\right) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p \, w(x)dx,$$

when the weight $w \in A_p$, $p > 1$.

Our proof avoids the Calderón-Zygmund decomposition (see [1]). In place of it we use the Vitali covering lemma and the fact that w as measure satisfies the doubling condition, i.e.

$$w\left(\lambda Q\right) \leq \lambda^{np}[w]_{A_p} w(Q)$$

(see Lemma 5.1. and Lemma 5.2.)

2. Definitions and notation

In this section we gather definitions and notation that will be used throughout the paper. We also include two lemmas that will play an important role in the proof of our main result.

A weight is a locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere. Therefore, weights are allowed to be zero or infinite only on a set of Lebesgue measure zero. Hence, if w is a weight and $1/w$ is locally integrable, then $1/w$ is also a weight.

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Given a weight w and a measurable set E, we will use the notation

$$w(E) = \int_E w(x)dx,$$

to denote the w-measure of the set E. Since weights are locally integrable functions $w(E) < \infty$ if E is a bounded set.

The weighted L_p space will be denote by $L_p(w)$.

3. The Hardy-Littlewood maximal operator

The uncentered Hardy-Littlewood maximal operator of a locally integrable function f on $\mathbb{R}^n$ is defined by

$$M(f)(x) = \sup_{x\in B} \frac{1}{m(B)} \int_B |f(y)|\, dy,$$

where the supremum is taken over all open balls $B(y,r)$ $(r>0)$ that contain the point x, and $m(\cdot)$ denotes $n-$dimensional Lebesgue measure. Lebesgue measure on $\mathbb{R}^n$ will also be denoted by dx.

Also recall the definition of the weighted uncentered Hardy-Littlewood maximal operator on $\mathbb{R}^n$ over balls

$$M_w(f)(x) = \sup_{x\in B} \frac{1}{w(B)} \int_B |f(y)|\, dy,$$

where w is any weight.

4. The A_p condition

Definition 4.1. Let $1 < p < \infty$. A weight w is said to be of class A_p if

$$\sup_{B \text{ balls in } \mathbb{R}^n} \left(\frac{1}{m(B)} \int_B w(x)dx\right) \left(\frac{1}{m(B)} \int_B w(x)^{-\frac{1}{p-1}} dx\right) < \infty. \tag{4.1}$$

The expression in (4.1) is called the A_p Muckehoupt characteristic constant of w and will be denoted by $[w]_{A_p}$.

Remark 1. The A_p condition first appeared, in somewhat different form, in a paper by M. Rosenblum (see [3]). The characterization of A_p when $n=1$ is due to B. Muckenhoupt (see [2]).

5. Properties of A_p weights

We summarize some basic properties of A_p in the following lemma.

Lemma 5.1. *Let $w \in A_p$ for some $1 \le p < \infty$. Then*

(1) $[\tau^z(w)]_{A_p} = [w]_{A_p}$, *where* $\tau^z(w)(x) = w(x-z)$, $z \in \mathbb{R}^n$.
(2) $[\lambda w]_{A_p} = [w]_{A_p}$ *for all* $\lambda > 0$.
(3) *For* $1 \le p < q < \infty$, *we have* $[w]_{A_q} \le [w]_{A_p}$.
(4) $\lim_{p\to 1^+} [w]_{A_p} = [w]_{A_1}$ *if* $w \in A_1$.

(5) *The following is an equivalent characterization of the A_p characteristic constant of w*

$$[w]_{A_p} = \sup_{B} \sup_{f \in L_p(w): m(B \cap \{|f|=0\})=0} \left\{ \frac{\left(\frac{1}{m(B)} \int_B |f(x)|\, dx \right)^p}{\frac{1}{w(B)} \int_B |f(x)|^p\, w(x)dx} \right\}.$$

(6) *The measure $w(x)dx$ is doubling: precisely, for all $\lambda > 1$ and all cubes B we have*

$$w(\lambda B) \le \lambda^{np}[w]_{A_p} w(B),$$

where $\lambda B(x,r) = \{y \in \mathbb{R}^n : |x-y| < \lambda r\}$, $(\lambda > 0)$.

The following lemma is due to Vitali, and it will play an important role in the proof of our main result.

Lemma 5.2. *Let E be an open set in $\mathbb{R}^n$ and let $\{Q_j\}$ be a family of cubes covering E. Then there exists a countable subfamily $\{Q_{j_k}\}$ of disjoints cubes such that*

$$E \subset \bigcup_k 5Q_{j_k}.$$

6. Main Result

We like to point out that the proof of the main result in this paper (theorem 6.1) is direct and fairly elementary. Instead of the one that use the heavy tool provide by Calderón-Zygmund decomposition. Indeed, we use only Hölder's inequality and Vitali's covering Lemma (see lemma 5.2 in this paper) and the fact that w as measure satisfies the doubling condition (see the introduction in the present paper)

Theorem 6.1. *For $1 \le p < \infty$, then weak (p,p) inequality*

$$w\left(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}\right) \le \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p\, w(x)dx$$

holds if and only if $w \in A_p$.

Proof. Using Hölder's inequality we see that

$$\begin{aligned}
\left(\frac{1}{m(Q)} \int_Q |f| dx \right)^p &= \left(\frac{1}{m(Q)} \int_Q |f(x)| w^{\frac{1}{p}} w^{-\frac{1}{p}} dx \right)^p \\
&\le \left[\frac{1}{m(Q)} \right]^p \left(\int_Q |f(x)|^p w(x) dx \right) \left(\int_Q w^{-\frac{q}{p}}(x) dx \right)^{\frac{p}{q}} \\
&\le \left(\frac{1}{w(Q)} \int_Q |f(x)|^p w(x) dx \right) \frac{w(Q)}{m(Q)} \left(\frac{1}{m(Q)} \int_Q w^{-\frac{1}{p-1}}(x) dx \right)^{p-1} \\
&\le \left(\frac{1}{w(Q)} \int_Q |f(x)|^p w(x) dx \right) [w]_{A_p},
\end{aligned} \tag{6.1}$$

since

$$M_w(f)(x) = \sup_{x\in Q} \frac{1}{w(Q)} \int_Q |f(x)| w(x) dx.$$

Fix $\lambda > 0$, from (6.1) we get

$$\{x \in \mathbb{R}^n : M(f)(x) > \lambda\} \subset \left\{ x \in \mathbb{R}^n : M_w(f^p)(x) > \frac{\lambda^p}{C} \right\}.$$

Thus,

$$w\left(\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}\right) \leq w\left(\left\{ x \in \mathbb{R}^n : M_w(f^p)(x) > \frac{\lambda^p}{C} \right\} \right),$$

let $A_\lambda = \left\{ x \in \mathbb{R}^n : M_w(f^p)(x) > \frac{\lambda^p}{C} \right\}$ for each $x \in A_\lambda$ there exists a cube Q_x such that

$$\frac{\lambda^p}{C} < \frac{1}{w\left(Q_x\right)} \int_{Q_x} |f|^p \, w(x) dx \tag{6.2}$$

from (6.2) it is easy to see that $A_\lambda \subset \cup_{x\in A_\lambda} Q_x$, since $M_w(f^p)(x)$ is lower semicontinuous, then A_λ is an open set. Thus, by Lemma 5.2 one can have a subfamily $\left\{Q_{x_j}\right\}$ of disjoints cubes such that

$$A_\lambda \subset \bigcup_j 5Q_{x_j},$$

then

$$\begin{aligned} w(A_\lambda) &\leq w\left(\bigcup_j 5Q_{x_j} \right) \\ &\leq \sum_j w\left(5Q_{x_j}\right) \\ &\leq 5^{pn}[w]_{A_p} \sum_j w\left(Q_{x_j}\right) \end{aligned}$$

□

Thus, theorem 5.3 follows from (5.2)

References

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