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ON CONCIRCULAR AND TORSE-FORMING VECTOR FIELDS ON COMPACT MANIFOLDS

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ABSTRACT. In this paper we modify the theorem by E. Hopf and found results and conditions, on which concircular, convergent and torse-forming vector fields exist on (pseudo-) Riemannian spaces. These results are applied for conformal, geodesic and holomorphically projective mappings of special compact spaces without boundary.

1. Introduction

Concircular and torse-forming vector fields on Riemannian manifolds and manifolds with affine connection were studied by K. Yano [21, 22]. We studied the theorem by E. Hopf (see [23], p. 26) about the existence of solutions of differential equations in partial derivations and we found some interesting result. If we modify this theorem we can prove that on a compact Riemannian manifold V_n with an indefinite metric there are no global torse-forming vector fields and we can also determine other examples for which these fields do not exist.

Immediately we can find these results in the theory of conformal, geodesic, holomorphically projective and almost geodesic mappings and transformations. For example, conformal transformations of Einstein spaces [2, 16, 12], geodesic, holomorphically projective mappings and transformations of semisymmetric, Ricci-semisymmetric and other spaces are connected with the existence of concircular vector fields on them, see [11, 12, 13, 17]. We generalize and intensify the results which were achieved in [18, 19, 20].

Examples of the existence of global concircular, convergent and torse-forming vector fields are introduced in [14, 15].

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2. Modifications of Theorem by E. Hopf

First we modify the Theorem by E. Hopf (see [23], p. 26).

Theorem 1. *Let M_n be a compact n-dimensional manifold and $\varphi \in C^2$ be a scalar function on M_n. Let at every point $P_0 \in M_n$ there exists a coordinate neighbourhood $U(x_1, x_2, \dots, x_n) \subset M_n$, on which there exist continuous functions $A^{ij}(x)$ and $B^i(x)$ at a point $P(x) \in U$ such that*

$$A^{\alpha\beta}(x)\frac{\partial^2\varphi(x)}{\partial x^\alpha \partial x^\beta} + B^\alpha(x)\frac{\partial\varphi(x)}{\partial x^\alpha} \geq 0 \quad (\leq 0), \tag{1}$$

holds all over U, and $A^{\alpha\beta}(x) z_\alpha z_\beta$ is a positive definite quadratic form on U.

Then $\varphi \equiv$ const on M_n.

Remarks.

- Signs "$\leq$" or "$\geq$" in the inequalities (1) for every neighbourhood are identical. Here and further we consider that the studied spaces are connected and boundless.
- Evidently we can replace partial derivatives by covariant derivations on spaces with affine connection and on Riemannian spaces [3, 23].
- In Theorem 1 it is not demanded that functions $A^{ij}(x)$ and $B^i(x)$ are defined on every coordinate neighbourhood by geometric objects globally on M_n as it is demanded in the theorem by S. Bochner and K. Yano [23], p. 26, and its applications [19].

Proof. On a compact manifold M_n there is possible to select a set of coordinate neighbourhoods U so that their union covers M_n and on each of them there hold equations (1); due to the determinacy we give the sign "$\geq$".

Because a function φ is continuous and a manifold M_n is compact, φ reaches its maximum at a point $P_0 \in U_0$ where U_0 is one of the neighbourhoods. Hence $\varphi(P) \leq \varphi(P_0)$ for all $P \in U_0$. On U_0 there holds every conditions of the theorem by E. Hopf (see [23], p. 26) and according to it holds $\varphi(P) = \varphi(P_0)$ for all $P \in U_0$.

Further we take a coordinate neighbourhood U_1 which covers U_0. Obviously $\varphi(P) \leq \varphi(P_0)$ for all $P \in U_1$, however $\varphi(P) = \varphi(P_0)$ for all $P \in U_1$. Similarly we can choose all selected neighbourhoods U. Because the number of these neighbourhoods is finite and M_n is connected, we verify that $\varphi(P) = \varphi(P_0)$ for all $P \in M_n$. □

This theorem is possible to prove as well as in case if functions A^{ij} and B^i depend also on $\varphi(x)$. We have the following theorem.

Theorem 2. *Let M_n be a compact n-dimensional manifold and $\varphi \in C^2$ be a scalar function on M_n. Let at every point $P_0 \in M_n$ there exist a coordinate neighbourhood $U(x_1, x_2, \dots, x_n) \subset M_n$, on which there exist continuous functions $A^{ij}(x, \varphi(x))$ and $B^i(x, \varphi(x))$ at a point $P(x) \in U$ such that*

$$A^{\alpha\beta}(x, \varphi(x))\frac{\partial^2\varphi(x)}{\partial x^\alpha \partial x^\beta} + B^\alpha(x, \varphi(x))\frac{\partial\varphi(x)}{\partial x^\alpha} \geq 0 \quad (\leq 0), \tag{2}$$

holds all over U. Let $A^{\alpha\beta}(x,\varphi(x))z_\alpha z_\beta$ be a positive definite quadratic form on U, then $\varphi \equiv \text{const}$ on M_n.

In case of Theorem 2 we have analogous remarks as we had for Theorem 1.

Proof. On a compact manifold M_n there is possible to select a set of coordinate neighbourhoods U so that their union covers M_n and on every of them there hold equations (2); due to the determinacy we give the sign "$\geq$".

Let $\overset{*}{\varphi}$ be a solution on M_n, for which formula (2) holds. On all domains U we set $A^{ij}(x) \equiv A^{ij}(x,\overset{*}{\varphi}(x))$ and $B^i(x) \equiv B^i(x,\overset{*}{\varphi}(x))$. Then inequalities (2) have forms (1) for a solution $\overset{*}{\varphi}$.

We search a solution φ on M_n which satisfies inequalities (1). According to Theorem 1, we have $\varphi \equiv \text{const}$ for all solutions φ.

Because $\overset{*}{\varphi}$ is a solution of (1), $\overset{*}{\varphi} \equiv \text{const}$ on M_n. □

3. Concircular and torse-forming vector fields

A vector field Φ defined on a space A_n with affine connection is called *torse-forming* if it holds:

$$\nabla_X \Phi = \nu\, X + \mu(X)\, \Phi \tag{3}$$

where ν is a function, $\mu(X)$ is a linear form, X is an arbitrary vector field, and ∇_X is a covariant derivation on A_n respectively to a vector X.

In local transcriptions we can write

$$\varphi^h_{,i} = \nu\, \delta^h_i + \mu_i\, \varphi^h \tag{4}$$

where φ^h and μ_i are components of Φ and $\mu(X)$, respectively, "," means a covariant derivation on A_n, and δ^h_i is the Kronecker symbol.

A torse-forming field we call *concircular* if $\mu(x) = 0$ and ν is an arbitrary function. If $\mu(X) = 0$ and $\nu = \text{const}$ this field is *convergent*.

If $\mu(X)$ is a gradient-like (i.e. it holds $\mu(X) = \nabla_X \overset{*}{\mu}$, where $\overset{*}{\mu}$ is a function) then a vector field $\Theta = \exp(-\overset{*}{\mu})\Phi$ satisfying equations $\nabla_X \Theta = \rho X$ where $\rho = \nu \exp(-\overset{*}{\mu})$ is concircular too. Vice versa, if Θ is concircular then every vector field which is collinear with Θ is torse-forming too.

On a Riemannian space V_n with a metric tensor $g(X,Y)$ we consider a linear form $\varphi(X) = g(X,\Phi)$. Locally this form $\varphi(X)$ is always gradient.

A torse-forming (including concircular and convergent) vector field will be called *gradient* if a linear form $\varphi(X)$ is gradient, i.e. that on V_n there exists a scalar function φ for which $\varphi(X) = \nabla_X \varphi$. A form $\mu(X)$ corresponding to gradient torse-forming fields Φ is collinear to a form $\varphi(X)$. Subsequently, we can write equations (3) for these vector fields as follows:

$$\nabla_X \nabla_Y \varphi = \nu\, g(X,Y) + \tau\, \nabla_X\varphi \nabla_Y \varphi, \tag{5}$$

where ν, τ are functions.

For concircular and torse-forming fields it is naturally required that

$$\Phi, \varphi \in C^1; \quad \overset{*}{\varphi} \in C^2; \quad A_n \in C^0; \quad V_n \in C^1.$$

See a local expression

$$\Phi^h(x) \in C^1; \quad \varphi(x) \in C^1; \quad \overset{*}{\varphi}(x) \in C^2; \quad \Gamma^h_{ij}(x) \in C^0; \quad g_{ij}(x) \in C^1,$$

where Γ^h_{ij} are components of affine connection ∇ on A_n, g_{ij} are components of a metric tensor on Riemannian space V_n, C^r is the class of the continuity. Naturally, $\nu \in C^0$ and $\tau \in C^0$.

4. Concircular and torse-forming vector fields globally on compact manifolds

Below, we study some sufficient conditions on which torse-forming fields on compact spaces do not exist.

It holds

Theorem 3. *On compact pseudo-Riemannian spaces $V_n \in C^1$ there exist only vanishing torse-forming fields.*

Proof. Let on $V_n \in C^1$ there exist the mentioned vector fields, i.e. there exists a scalar function $\varphi \in C^2$ satisfying (5). We assume that φ ($\not\equiv$ const), ν and τ are continuous functions. We prove that this solution does not exist.

Remark that the pseudo-Riemannian space V_n has an indefinite metric g.

Let P_0 be some point on V_n. There exist such a coordinate neigbourhood $U^*(x)$ as

$$g_{ij}(P_0) = \operatorname{diag}(1, -1, e_3, \dots, e_n), \quad (e_i = \pm 1).$$

We get

$$A(x) \overset{\text{def}}{=} \frac{-1}{g_{11}(x)}\ (n\, g_{22}(x) + g_{33}(x) + \cdots + g_{nn}(x)).$$

Evidently, $A(P_0) \geq 2$. Therefore there exists a domain $U \subset U^*$ which includes a point P_0 for which $A(x) > 0$. Hence

$$A^{ij}(x) \overset{\text{def}}{=} \operatorname{diag}(A(x), n, 1, \dots, 1)$$

determine on a domain U the positive form $A^{\alpha\beta}(x) z_\alpha z_\beta$ as well as on this domain there holds

$$A^{\alpha\beta}(x)\, g_{\alpha\beta}(x) = 0.$$

The local expression of (5) has the following form

$$\varphi_{,ij} - \tau\, \varphi_{,i} \varphi_{,j} = \nu\, g_{ij}. \tag{6}$$

Contracting the last formula with A^{ij}, we get

$$A^{\alpha\beta}\, \varphi_{,\,\alpha\beta} - B^\alpha \varphi_\alpha = 0, \tag{7}$$

where $B^i \equiv \tau\, A^{i\beta} \varphi_{,\,\beta}$.

Evidently, according to Theorem 2 it holds $\varphi \equiv \mathrm{const}$ on V_n. □

Theorem 4. *On compact Riemannian spaces $V_n \in C^1$ there do not exist torse-forming fields satisfying condition $\nu \geq 0$ (or $\nu \leq 0$) on whole V_n.*

Proof. According to Theorem 3 it remains to study only the case if on V_n there is a positive definite metric.

We consider equations (5) respective to a function φ. On coordinate neighbourhoods U these equations have the form (6) and after the contraction (6) with $A^{ij} \equiv g^{ij}$ (where $\|g^{ij}\| = \|g_{ij}\|^{-1}$) we get

$$A^{\alpha\beta}\,\varphi_{\,,\,\alpha\beta} - B^{\alpha}\varphi_{\alpha} = n\,\nu, \tag{8}$$

where $B^i \equiv \tau\,A^{i\beta}\varphi_{\,,\,\beta}$. The right side of (8) is either every non-negative or non-positive. Considering a positive definite metric the conditions of Theorem 2 hold. Hence $\varphi \equiv \mathrm{const}$. □

Evidently, the conditions for concircular fields are contained in Theorems 3 and 4. The next follows from Theorem 4:

Lemma 1. *On compact Riemannian spaces $V_n \in C^1$ there do not exist gradient convergent vector fields.*

In fact, conditions of Theorem 4 in this case are $\nu = \mathrm{const}$ then they satisfy conditions of the Theorem 4.

Lemma 2. *On compact Riemannian spaces $V_n \in C^1$ there do not exist covariantly non-constant convergent vector fields.*

Proof. On V_n there is a convergent vector field Φ satisfying $\nabla_X\Phi = \mu X$, μ is constant. We consider a function $\lambda \equiv g(\Phi,\Phi)$. It is easy to see, that $\nabla_X\nabla_Y\lambda = 2\mu g(X,Y)$. According to Lemma 1 it follows $\lambda \equiv \mathrm{const}$. Hence $\mu = 0$. Then it follows $\nabla_X\Phi = 0$. □

In [17] there are studied torse-forming vector fields Φ satisfying:

$$g(\Phi,\Phi) = e; \quad \nabla_X\Phi = \nu\,(X - e\,g(X,\Phi)\,\Phi), \tag{9}$$

where $e = \pm 1$.

These conditions hold on normalization non-vanishing non-isotropic torse-forming vector fields.

Therefore we will call a vector field Φ satisfying (9) a *normalized torse-forming field.* Naturally, the Theorems 1 and 3 hold as well as for these fields. It is shown that the Theorems 1 and 3 generalize the results introduced in [18].

5. Applications of achieved results

The above introduced results, we use for conformal, geodesic and holomorphically-projective mappings of compact manifolds without boundary.

(Pseudo-) Riemannian manifold V_n admits *conformal mappings* onto (pseudo-) Riemannian manifold V_n if their metrics are connected by the following condition $\bar{g} = \varrho \cdot g$ where ϱ is a function. If $\varrho \equiv$ const, then the mappings are *homothetic*. A conformal mapping $V_n \to V_n$ is called *conformal transformation* on V_n.

H. W. Brinkman, see [16], proved that if an Einstein space V_n admits a conformal mapping onto Einstein space $\bar{V_n}$, then a gradient (like a vector field) $\nabla\varrho$ is concircular.

The next Theorem follows from Theorem 3.

Theorem 5. *If a compact pseudo-Riemannian Einstein space V_n $(n \geq 3)$ admits a conformal mapping onto Einstein space $\bar{V_n}$, then this mapping is homothetic.*

Theorem 6. *If a compact pseudo-Riemannian Einstein space V_n $(n \geq 3)$ admits a conformal transformation, then this transformation is homothetic.*

Remark. Theorems 5 and 6 fail for classical Riemannian metrics (even if we replace ligt-line by usual completeness) – Möbius transformations of the standard round sphere and the stereographic map of the punctured sphere to the Euclidean space are conformal nonhomothetic mappings. One can construct other examples on warped Riemannian manifolds, see [6].

If semisymmetric, Ricci-semisymmetric, Kählerian, Ricci-flat, and flat (pseudo-) Riemannian spaces admit non-trivial geodesic mappings, then on these spaces there exists nonvanishing convergent vector field, see [9, 10, 8, 7, 11].

Analogically, if semisymmetric, Ricci-semisymmetric, and flat Kählerian spaces admit non-trivial holomorphically-projective mappings, then also on these spaces there exists nonvanishing convergent vector field, see [1, 10, 12].

According to Lemma 2, above mentioned compact special spaces do not admit global geodesic and holomorphically-projective mappings.

Finally, we remark that I. Hinterleitner and V. Kiosak introduced φ-*Ric* vector fields by the following equation $\varphi_{i,j} = \mu\, R_{ij}$, where μ is a constant, R_{ij} is the Ricci tensor, see [4]. One can use Theorem 1 for a gradient vector field φ_i.

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