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SOME HARDY SPACE ESTIMATES FOR MULTILINEAR SINGULAR INTEGRAL OPERATOR

ZHOU XIAOSHA AND LIU LANZHE

Abstract. In this paper, we establish the boundedness for some multilinear singular integral operators on Hardy and Herz type Hardy spaces. The operators include Calderon-Zygmund singular integral operators.

1. Introduction

Let $b \in BMO(R^n)$ and T be the Calderon-Zygmund operator. The commutator $[b, T]$ generated by b and T is defined by $[b, T]f(x) = b(x)Tf(x) - T(bf)(x)$. By a classical result of Coifman, Rochberg and Weiss(see [6]), we know that the commutator $[b, T]$ is bounded on $L^p(R^n)$ for $1 < p < \infty$. However, it was observed that $[b, T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$ and from $L^1(R^n)$ to $L^{1,\infty}(R^n)$ for $p \leq 1$. But, if $H^p(R^n)$ is replaced by a suitable atomic space $H^p_b(R^n)$(see [1, 14]), then $[b, T]$ is bounded from $H^p_b(R^n)$ to $L^p(R^n)$ for $p \in (n/(n+1), 1]$. In recent years, the theory of Herz space and Herz type Hardy space, as a local version of Lebesgue space and Hardy space, have been developed (see [8, 9, 11, 12]). The main purpose of this paper is to establish the boundedness properties of some multilinear operators related to certain non-convolution type singular integral operators on Hardy and Herz type Hardy spaces. The operators include Calderón-Zygmund singular integral operators.

2. Notations and Theorems

In this paper, we study the singular integral operators as following. Let $T : S \to S'$ be a linear operator and there exists a locally integrable function

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$K(x,y)$ on $R^n \times R^n \setminus \{(x,y) \in R^n \times R^n : x = y\}$ such that

$$Tf(x) = \int_{R^n} K(x,y)f(y)dy$$

for every bounded and compactly supported function f, where K satisfies: for fixed $\varepsilon > 0$ and $n > \delta \geq 0$,

$$|K(x,y)| \leq C|x-y|^{-n+\delta}$$

and

$$|K(y,x) - K(z,x)| + |K(x,y) - K(x,z)| \leq C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon+\delta}$$

if $2|y-z| \leq |x-z|$. Let m_i be positive integers$(i = 1, \ldots, l)$, $m_1 + \cdots + m_l = m$ and A_i be some functions on $R^n(i = 1, \ldots, l)$. The multilinear operator related to T is defined by

$$T^A(f)(x) = \int_{R^n} \frac{\prod_{i=1}^{l} R_{m_i+1}(A_i; x, y)}{|x-y|^m} K(x,y)f(y)dy,$$

where

$$R_{m_i+1}(A_i; x, y) = A_i(x) - \sum_{|\beta| \leq m_i} \frac{1}{\beta!} D^{\beta} A_i(y)(x-y)^{\beta}.$$

Note that when $m = 0$, T^A is just the multilinear commutator of T and A (see [16]). While when $m > 0$, T^A is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [3, 5, 4, 13]). In [7], the weighted $L^p(p > 1)$-boundedness of the multilinear operator related to some singular integral operator are obtained. The main purpose of this paper is to study the boundedness of the multilinear singular integral operator T^A on some Hardy and Herz-Hardy spaces.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [10])

$$f^{\#}(x) \approx \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^{\infty}(R^n)$ and $||f||_{BMO} = ||f^{\#}||_{L^{\infty}}$.

Definition 1. Let A_i be some function on R^n and m_i be positive integers$(i = 1, \ldots, l)$, $m_1 + \cdots + m_l = m$ and $0 < p \leq 1$. A bounded measurable function a on R^n is said to be a $(p, D^m A)$ atom if

i) $\operatorname{supp} a \subset Q = Q(x_0, r)$,
ii) $||a||_{L^\infty} \leq |Q|^{-1/p}$,
iii) $\int_{R^n} a(y)dy = \int_{R^n} a(y) \prod_{\nu=1}^{k} D^\beta A_\nu(y)dy = 0$ for $|\beta| = m_i, i = 1, \ldots, l$ and $k = 1, \ldots, l$;

A tempered distribution f is said to belong to $H^p_{D^m A}(R^n)$, if, in the Schwartz distributional sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x),$$

where a_j's are $(p, D^m A)$ atoms, $\lambda_j \in C$ and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Moreover, $||f||_{H^p_{D^m A}} \approx \left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p}$.

Definition 2. Let $0 < p, q < \infty$, $\alpha \in R$. For $k \in Z$, define $B_k = \{x \in R^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0.

The homogeneous Herz space is defined by

$$\dot{K}^{\alpha,p}_q(R^n) = \{f \in L^q_{loc}(R^n \setminus \{0\}) : ||f||_{\dot{K}^{\alpha,p}_q} < \infty\}, \tag{1}$$

where

$$||f||_{\dot{K}^{\alpha,p}_q} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||^p_{L^q}\right]^{1/p}.$$

The nonhomogeneous Herz space is defined by

$$K^{\alpha,p}_q(R^n) = \{f \in L^q_{loc}(R^n) : ||f||_{K^{\alpha,p}_q} < \infty\}, \tag{2}$$

where

$$||f||_{K^{\alpha,p}_q} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||^p_{L^q} + ||f\chi_{B_0}||^p_{L^q}\right]^{1/p}.$$

Definition 3. Let A_i be a function on R^n and m_i be positive integers ($i = 1, \ldots, l$), $m_1 + \cdots + m_l = m$, $\alpha \in R$, $0 < p < \infty$, $1 < q \leq \infty$. A function $a(x)$ on R^n is called a central $(\alpha, q, D^m A)$-atom (or a central $(a, q, D^m A)$-atom of restrict type), if

1) $\operatorname{supp} a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),
2) $||a||_{L^q} \leq |B(0, r)|^{-\alpha/n}$,
3) $\int_{R^n} a(x)dx = \int_{R^n} a(y) \prod_{i=1}^{k} D^\beta A_i(y)dy = 0$ for $|\beta| = m_i, i = 1, \ldots, l$ and $k = 1, \ldots, l$;

A tempered distribution f is said to belong to $H\dot{K}^{\alpha,p}_{q,D^m A}(R^n)$ (or $HK^{\alpha,p}_{q,D^m A}(R^n)$), if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=1}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, where a_j is a central $(\alpha, q, D^m A)$-atom (or a central $(\alpha, q, D^m A)$-atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$(or $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$), moreover, $||f||_{H\dot{K}^{\alpha,p}_{q,D^m A}}$(or $||f||_{HK^{\alpha,p}_{q,D^m A}}$)$\approx \left(\sum_j |\lambda_j|^p\right)^{1/p}$.

Now, we can state our results as following.

Theorem 1. *Let* $\max(n/(n+1), n/(n+\varepsilon-\delta)) < q \le 1$, $1/q = 1/p - \delta/n$, $D^\beta A_i \in BMO(R^n)$ *for all* β *with* $|\beta| = m_i$ *and* $i = 1, \dots, l$. *Suppose that* T^A *is bounded from* $L^s(R^n)$ *to* $L^r(R^n)$ *for any* $1 < s < n/\delta$ *and* $1/r = 1/s - \delta/n$. *Then* T^A *is bounded from* $H^p_{D^m A}(R^n)$ *to* $L^q(R^n)$.

Theorem 2. *Let* $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = \delta/n$, $n(1-1/q_1) \le \alpha < \min(n(1-1/q_1)+1, n(1-1/q_1)+\varepsilon)$ *and* $D^\beta A_i \in BMO(R^n)$ *for all* β *with* $|\beta| = m_i$ *and* $i = 1, \dots, l$. *Suppose that* T^A *is bounded from* $L^s(R^n)$ *to* $L^r(R^n)$ *for any* $1 < s < n/\delta$ *and* $1/r = 1/s - \delta/n$. *Then* T^A *is bounded from* $H\dot{K}^{\alpha,p}_{q_1,D^m A}(R^n)$ *to* $\dot{K}^{\alpha,p}_{q_2}(R^n)$.

Remark 1. Theorem 2 is also hold for nonhomogeneous Herz and Herz type Hardy space.

3. Proofs of Theorems

To prove the theorems, we need the following lemma.

Lemma 1 (see [4]). *Let* A *be a function on* R^n *and* $D^\beta A \in L^q(R^n)$ *for* $|\beta| = m$ *and some* $q > n$. *Then*

$$|R_m(A; x, y)| \le C|x-y|^m \sum_{|\beta|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\beta A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ *is the cube centered at* x *and having side length* $5\sqrt{n}|x-y|$.

Proof of Theorem 1: It suffices to prove that there exists a constant $C > 0$ such that for every $(p, D^m A)$ atom a,

$$||T^A(a)||_{L^q} \le C.$$

Let a be a $(p, D^m A)$ atom supported on a cube $Q = Q(x_0, d)$. We write

$$\int_{R^n} |T^A(a)(x)|^q dx = \int_{2Q} |T^A(a)(x)|^q dx + \int_{(2Q)^c} |T^A(a)(x)|^q dx = I + II.$$

For I, taking $r, s > 1$ with $q < s < n/\delta$ and $1/r = 1/s - \delta/n$, by Holder's inequality and the (L^s, L^r)-boundedness of T^A, we get

$$I \le C||T^A(a)||^q_{L^r} |Q(x_0, 2d)|^{1-q/r} \le C||a||^q_{L^s} |Q|^{1-q/r} \le C|Q|^{-q/p+q/s+1-q/r} \le C.$$

To obtain the estimate of II, we need to estimate $T^A(a)(x)$ for $x \in (2Q)^c$. Without loss of generality, we may assume $l = 2$. Let $\tilde{A}_i(x) = A_i(x) - \sum_{|\beta|=m_i} \frac{1}{\beta!}(D^\beta A_i)_Q x^\beta$. Then $R_{m_i}(A_i; x, y) = R_{m_i}(\tilde{A}_i; x, y)$ and $D^\beta \tilde{A}_i = D^\beta A_i -$

$(D^\beta A_i)_Q$ for $|\beta| = m_i$. We write, by the vanishing moment of a,

$$T^A(a)(x) = \int_{R^n} \left[\frac{K(x,y)}{|x-y|^m} - \frac{K(x,x_0)}{|x-x_0|^m} \right] R_{m_1}(\tilde{A}_1; x, y) R_{m_2}(\tilde{A}_2; x, y) a(y) dy$$

$$+ \int_{R^n} \frac{K(x,x_0)}{|x-x_0|^m} [R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x, x_0)] R_{m_2}(\tilde{A}_2; x, y) a(y) dy$$

$$+ \int_{R^n} \frac{K(x,x_0)}{|x-x_0|^m} [R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x, x_0)] R_{m_1}(\tilde{A}_1; x, x_0) a(y) dy$$

$$- \sum_{|\beta_2|=m_2} \frac{1}{\beta_2!} \int_{R^n} \left[\frac{K(x,y)(x-y)^{\beta_2}}{|x-y|^m} - \frac{K(x,x_0)(x-x_0)^{\beta_2}}{|x-x_0|^m} \right] \times$$

$$\times R_{m_1}(\tilde{A}_1; x, y) D^{\beta_2} \tilde{A}_2(y) a(y) dy$$

$$- \sum_{|\beta_2|=m_2} \frac{1}{\beta_2!} \int_{R^n} \frac{K(x,x_0)(x-x_0)^{\beta_2}}{|x-x_0|^m} [R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x, x_0)] \times$$

$$\times D^{\beta_2} \tilde{A}_2(y) a(y) dy$$

$$- \sum_{|\beta_1|=m_1} \frac{1}{\beta_1!} \int_{R^n} \left[\frac{K(x,y)(x-y)^{\beta_1}}{|x-y|^m} - \frac{K(x,x_0)(x-x_0)^{\beta_1}}{|x-x_0|^m} \right] \times$$

$$\times R_{m_2}(\tilde{A}_2; x, y) D^{\beta_1} \tilde{A}_1(y) a(y) dy$$

$$- \sum_{|\beta_1|=m_1} \frac{1}{\beta_2!} \int_{R^n} \frac{K(x,x_0)(x-x_0)^{\beta_1}}{|x-x_0|^m} [R_{m_1}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x, x_0)] \times$$

$$\times D^{\beta_1} \tilde{A}_1(y) a(y) dy$$

$$+ \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \frac{1}{\beta_1! \beta_2!} \int_{R^n} \left[\frac{K(x,y)(x-y)^{\beta_1+\beta_2}}{|x-y|^m} - \frac{K(x,x_0)(x-x_0)^{\beta_1+\beta_2}}{|x-x_0|^m} \right] \times$$

$$\times D^{\beta_1} \tilde{A}_1(y) D^{\beta_2} \tilde{A}_2(y) a(y) dy$$

$$= II_1(x) + II_2(x) + II_3(x) + II_4(x) + II_5(x) + II_6(x) + II_7(x) + II_8(x).$$

By Lemma and the following inequality (see [17])

$$|b_{Q_1} - b_{Q_2}| \le C \log(|Q_2|/|Q_1|) ||b||_{BMO} \quad \text{for } Q_1 \subset Q_2,$$

we know that, for $y \in Q$ and $x \in 2^{k+1}Q \setminus 2^k Q$,

$$|R_{m_i}(\tilde{A}_i; x, y)| \le C|x-y|^{m_i} \sum_{|\beta|=m_i} (||D^\beta A_i||_{BMO} + |(D^\beta A_i)_{Q(x,y)} - (D^\beta A_i)_Q|)$$

$$\le Ck|x-y|^{m_i} \sum_{|\beta|=m_i} ||D^\beta A_i||_{BMO}.$$

Note that $|x-y| \sim |x-x_0|$ for $y \in Q$ and $x \in R^n \setminus 2Q$, we obtain, by the condition on K,

$|II_1(x)| \le$

$$\leq C\int_{R^n}\left(\frac{|y-x_0|}{|x-x_0|^{m+n+1-\delta}}+\frac{|y-x_0|^{\varepsilon}}{|x-x_0|^{m+n+\varepsilon-\delta}}\right)\prod_{i=1}^{2}|R_{m_i}(\tilde{A}_i;x,y)||a(y)|dy$$

$$\leq C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)\int_Q k^2\left(\frac{|y-x_0|}{|x-x_0|^{n+1-\delta}}+\frac{|y-x_0|^{\varepsilon}}{|x-x_0|^{n+\varepsilon-\delta}}\right)|a(y)|dy$$

$$\leq C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)k^2\left(\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta}}\right).$$

For $II_2(x)$, by the formula (see [5]):

$$R_{m_i}(\tilde{A}_i;x,y)-R_{m_i}(\tilde{A}_i;x,x_0)=\sum_{|\gamma|<m_i}\frac{1}{\gamma!}R_{m_i-|\gamma|}(D^{\gamma}\tilde{A}_i;x,x_0)(x-y)^{\gamma}$$

and Lemma, we have

$$|R_{m_i}(\tilde{A}_i;x,y)-R_{m_i}(\tilde{A}_i;x,x_0)|\leq C\sum_{|\gamma|<m_i}\sum_{|\beta|=m_i}|x-x_0|^{m_i-|\gamma|}|x-y|^{|\gamma|}||D^{\beta}A_i||_{BMO},$$

thus

$$|II_2(x)|\leq C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)\int_Q k\frac{|y-x_0|}{|x-x_0|^{n+1-\delta}}|a(y)|dy$$

$$\leq C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)k\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}.$$

Similarly,

$$|II_3(x)|\leq C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)k\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}.$$

For $II_4(x)$, similar to the proof of $II_1(x)$ and $II_2(x)$, we get

$$|II_4(x)|\leq C\sum_{|\beta_1|=m_1}||D^{\beta_1}A_1||_{BMO}\sum_{|\beta_2|=m_2}k\left(\frac{|Q|^{1/n-1/p}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n-1/p}}{|x-x_0|^{n+\varepsilon-\delta}}\right)\times$$

$$\times\int_Q|D^{\beta_2}A_2(y)-(D^{\beta_2}A_2)_Q|dy$$

$$\leq C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)k\left(\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta}}\right).$$

Similarly,

$$|II_5(x)| \le C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)k\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}.$$

$$|II_6(x)| \le C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)k\left(\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta}}\right).$$

$$|II_7(x)| \le C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)k\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}.$$

For $II_8(x)$, taking $1<r_1,r_2<\infty$ such that $1/r_1+1/r_2=1$, then, by Holder's inequality,

$$\begin{aligned}
&|II_8(x)| \le \\
&\le C\sum_{|\beta_1|=m_1,|\beta_2|=m_2}\int_{R^n}\left|\frac{K(x,y)(x-y)^{\beta_1+\beta_2}}{|x-y|^m}-\frac{K(x,x_0)(x-x_0)^{\beta_1+\beta_2}}{|x-x_0|^m}\right|\times \\
&\quad\times|D^{\beta_1}\tilde{A}_1(y)||D^{\beta_2}\tilde{A}_2(y)||a(y)|dy \\
&\le C\sum_{|\beta_1|=m_1}\left(\int_Q|D^{\alpha_1}A_1(y)-(D^{\beta_1}A_1)_Q|^{r_1}dy\right)^{1/r_1} \\
&\quad\times\sum_{|\beta_2|=m_2}\left(\int_Q|D^{\alpha_2}A_2(y)-(D^{\beta_2}A_2)_Q|^{r_2}dy\right)^{1/r_2}\left(\frac{|Q|^{1/n-1/p}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n-1/p}}{|x-x_0|^{n+\varepsilon-\delta}}\right) \\
&\le C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)\left(\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta}}\right).
\end{aligned}$$

Thus, recall that $\max(n/(n+1), n/(n+\varepsilon-\delta)) < q \le 1$,

$$\begin{aligned}
II &\le \sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|T^A(a)(x)|^q dx \\
&\le C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}k^{2q}\times \\
&\quad\times\left(\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta}}\right)^q dx \\
&\le C\left[\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)\right]^q\sum_{k=1}^{\infty}k^{2q}[2^{knq(1/p-1/n-1)}+2^{knq(1/p-\varepsilon/n-1)}]
\end{aligned}$$

$$\leq C\left[\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)\right]^q.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2: Without loss of generality, we may assume $l=2$. Let $f\in H\dot{K}^{\alpha,p}_{q_1,D^mA}(R^n)$ and $f(x)=\sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 3. We write

$$\begin{aligned}||T^A(f)||^p_{\dot{K}^{\alpha,p}_{q_2}} &\leq \sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|||T^A(a_j)\chi_k||_{L^{q_2}}\right)^p\\ &+\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||T^A(a_j)\chi_k||_{L^{q_2}}\right)^p\\ &= J+JJ.\end{aligned}$$

For JJ, by the (L^{q_1},L^{q_2})-boundedness of T^A, we get

$$\begin{aligned}JJ &\leq C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||a_j||_{L^{q_1}}\right)^p\\ &\leq C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|2^{-j\alpha}\right)^p\\ &\leq \begin{cases} C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|^p2^{-j\alpha p}\right), & 0<p\leq 1\\ C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|^p2^{-j\alpha p/2}\right)\left(\sum_{j=k-2}^{\infty}2^{-j\alpha p'/2}\right)^{p/p'}, & p>1\end{cases}\\ &\leq \begin{cases} C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p}\right), & 0<p\leq 1\\ C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p/2}\right)\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p'/2}\right)^{p/p'}, & p>1\end{cases}\\ &\leq C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\leq C||f||^p_{H\dot{K}^{\alpha,p}_{q_1,D^mA}}.\end{aligned}$$

For J, similar to the proof of Theorem 1, we get, for $x\in C_k$, $j\leq k-3$,

$$\begin{aligned}&|T^A(a_j)(x)|\leq\\ &\leq C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)\left(\frac{|B_j|^{1/n}}{|x|^{n+1-\delta}}+\frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\delta}}\right)\int_{R^n}|a_j(y)|dy\\ &+C\sum_{|\beta_1|=m_1}||D^{\beta_1}A_1||_{BMO}\left(\frac{|B_j|^{1/n}}{|x|^{n+1-\delta}}+\frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\delta}}\right)\sum_{|\beta_2|=m_2}\int_{R^n}|a_j(y)||D^{\beta_2}\tilde{A}_2(y)|dy\end{aligned}$$

$$+ C \sum_{|\beta_2|=m_2} ||D^{\beta_2}A_2||_{BMO} \left(\frac{|B_j|^{1/n}}{|x|^{n+1-\delta}} + \frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\delta}} \right) \sum_{|\beta_1|=m_1} \int_{R^n} |a_j(y)||D^{\beta_1}\tilde{A}_1(y)|dy$$

$$+ C \left(\frac{|B_j|^{1/n}}{|x|^{n+1-\delta}} + \frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\delta}} \right) \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \int_{R^n} |a_j(y)||D^{\beta_1}\tilde{A}_1(y)||D^{\beta_2}\tilde{A}_2(y)|dy$$

$$\leq C \left(\frac{2^{j(1+n(1-1/q_1)-\alpha)}}{|x|^{n+1-\delta}} + \frac{2^{j(\varepsilon+n(1-1/q_1)-\alpha)}}{|x|^{n+\varepsilon-\delta}} \right).$$

To be simply, denote $W(j,k) = 2^{(j-k)(1+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)}$, then

$$J \leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j|^p \left[\frac{2^{j(1+n(1-1/q_1)-\alpha)}}{2^{k(n+1-\delta)}} + \frac{2^{j(\varepsilon+n(1-1/q_1)-\alpha)}}{2^{k(n+\varepsilon-\delta)}} \right]^p \right) 2^{knp/q_2}$$

$$\leq \begin{cases} C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} W(j,k)^p, & 0 < p \leq 1 \\ C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left[\sum_{k=j+3}^{\infty} W(j,k)^{p/2} \right] \left[\sum_{k=j+3}^{\infty} W(j,k)^{p'/2} \right]^{p/p'}, & p > 1 \end{cases}$$

$$\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \leq C ||f||^p_{H\dot{K}^{\alpha,p}_{q_1,D^m A}}.$$

These yield the desired result and finish the proof of Theorem 2. □

4. Examples

In this section we shall apply Theorem 1 and 2 of the paper to the Calderón-Zygmund singular integral operator.

Let T be the Calderón-Zygmund operator (see [2, 10, 15, 18], the multilinear operator related to T is defined by

$$T^A(f)(x) = \int_{R^n} \frac{\prod_{i=1}^{l} R_{m_i+1}(A_i; x, y)}{|x-y|^m} K(x,y) f(y) dy.$$

In particular, the multilinear commutator related to T is (see [11])

$$T^A(f)(x) = \int_{R^n} \left[\prod_{i=1}^{l} (A_i(x) - A_i(y)) \right] K(x,y) f(y) dy.$$

Then it is easily to see that T satisfies the conditions in Theorem 1 and 2.

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College of Mathematics,
Changsha University of Science and Technology,
Changsha 410077,
P. R. of China
E-mail address: lanzheliu@163.com