Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
30 (2014), 125–131
www.emis.de/journals
ISSN 1786-0091

INTEGRABILITY OF DISTRIBUTIONS ON TWO KINDS OF MANIFOLD

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Abstract. In this paper, we give some sufficient and necessary conditions for integrability of distributions on an almost Hermitian manifold and a quasi Kaehlerian manifold, and generalize Bejancu's and WanYong's research work.

1. Introduction

Let $\overline{M}$ be a real differentiable manifold. An almost complex structure on $\overline{M}$ is a tensor field J of type (1, 1) on $\overline{M}$ such that at every point $x \in \overline{M}$ we have $J^2 = -I$, where I denotes the identify transformation of $T_x\overline{M}$. A manifold $\overline{M}$ endowed with an almost complex structure is called an almost complex manifold.

A Hermitian metric on an almost complex manifold $\overline{M}$ is a Riemannian metric g satisfying

$$g(JX, JY) = g(X, Y), \tag{1.1}$$

for any $X, Y \in \Gamma(T\overline{M})$.

An almost complex manifold endowed with a Hermitian metric is an almost Hermitian manifold. More, we defined the torsion tensor of J or the Nijenhuis tensor of J by

$$[J, J](X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY], \tag{1.2}$$

for any $X, Y \in \Gamma(T\overline{M})$, where $[X, Y]$ is the Lie bracket of vector fields X and Y.

2010 *Mathematics Subject Classification.* 58A30.
Key words and phrases. almost Hermitian manifold, CR-submanifold, distribution, connection, integrability.
Supported by Foundation of Department of Science and Technology of Hunan Province (No. 2010SK3023).

Definition 1.1 ([1])**.** An almost Hermitian manifold $\overline{M}$ with Levi-Civita connection $\overline{\nabla}$ is called a quasi-Kaehlerian manifold if we have

$$(\overline{\nabla}_X J)Y + (\overline{\nabla}_{JX} J)JY = 0, \tag{1.3}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 1.2 ([1])**.** An almost Hermitian manifold $\overline{M}$ with Levi-Civita connection $\overline{\nabla}$ is called a Kaehlerian manifold if we have

$$\overline{\nabla}_X J = 0, \tag{1.4}$$

for any $X \in \Gamma(T\overline{M})$.

Obviously, a Kaehlerian manifold is a quasi-Kaehlerian manifold.

Let M be an m-dimensional Riemannian submanifold of an n-dimensional Riemannian manifold $\overline{M}$. We denote by $TM^{\perp}$ the normal bundle to M and by g both metric on M and $\overline{M}$. Also, we denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$, denote by ∇ the induced connection on M, and denote by $\nabla^{\perp}$ the induced normal connection on M.

Then, for any $X, Y \in \Gamma(TM)$ we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.5}$$

where $h\colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^{\perp})$ is a normal bundle valued symmetric bilinear form on $\Gamma(TM)$. The equation (1.5) is called the Gauss formula and h is called the second fundamental form of M.

Now, for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$ we denote by $-A_V X$ and $\nabla^{\perp}_X V$ the tangent part and normal part of $\overline{\nabla}_X V$ respectively. Then we have

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V. \tag{1.6}$$

Thus, for any $V \in \Gamma(TM^{\perp})$ we have a linear operator, satisfying

$$g(A_V X, Y) = g(X, A_V Y) = g(h(X, Y), V). \tag{1.7}$$

The equation (1.6) is called the Weingarten formula.

An m-dimensional distribution on a manifold $\overline{M}$ is a mapping D defined on $\overline{M}$, which assigns to each point x of $\overline{M}$ an m-dimensional linear subspace D_x of $T_x\overline{M}$. A vector field X on $\overline{M}$ belongs to D if we have $X_x \in D_x$ for each $x \in \overline{M}$. When this happens we write $X \in \Gamma(D)$. The distribution D is said to be differentiable if for any $x \in \overline{M}$ there exist m differentiable linearly independent vector fields $X_i \in \Gamma(D)$ in a neighbourhood of x. From now on, all distributions are supposed to be differentiable of class C^{∞}.

The distribution D is said to be involutive if for all vector fields $X, Y \in \Gamma(D)$ we have $[X, Y] \in \Gamma(D)$. A sub-manifold M of $\overline{M}$ is said to be an integral manifold of D if for every point $x \in M$, D_x coincides with the tangent space to M at x. If there exists no integral manifold of D which contains M, then M is called a maximal integral manifold or a leaf of D. The distribution D is said to be integrable if for every $x \in \overline{M}$ there exists an integral manifold of D containing x.

Definition 1.3 ([1]). Let $\overline{M}$ be a real n-dimensional almost Hermitian manifold with almost complex structure J and with Hermitian metric g. Let M be a real m-dimensional Riemannian manifold isometrically immersed in $\overline{M}$. Then M is called a CR-submanifold of $\overline{M}$ if there exist a differentiable distribution $D\colon x \to D_x \subset T_xM$, on M satisfying the following conditions:

(1) D is holomorphic, that is, $J(D_x) = D_x$, for each $x \in M$,
(2) the complementary orthogonal distribution $D^\perp\colon x \to D_x^\perp \subset T_xM$, is anti-invariant, that is, $J(D_x^\perp) \subset T_xM^\perp$, for each $x \in M$.

Now let M be an arbitrary Riemannian manifold isometrically immersed in an almost Hermitian manifold $\overline{M}$. For each vector field X tangent to M, we put

$$JX = \phi X + \omega X, \tag{1.8}$$

where ϕX and ωX are respectively the tangent part and the normal part of JX. Also, for each vector field V normal to M, we put

$$JV = BV + CV, \tag{1.9}$$

where BV and CV are respectively the tangent part and the normal part of JV.

Denote by P and Q the project morphisms of TM to D and $D^\perp$, then we have

$$\phi X = JPX, \tag{1.10}$$

and

$$\omega X = JQX, \tag{1.11}$$

for any $X \in \Gamma(TM)$.

The covariant derivative of ϕ is defined by

$$(\nabla_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y, \tag{1.12}$$

for any $X, Y \in \Gamma(TM)$. On the other hand the covariant derivative of ω is defined by

$$(\nabla_X \omega)Y = \nabla_X^\perp \omega Y - \omega \nabla_X Y, \tag{1.13}$$

for any $X, Y \in \Gamma(TM)$.

2. Main Results

Lemma 2.1 (Frobenius[1, 3]). *Distribution D on manifold M is integrable if and only if $[X, Y] \in \Gamma(D)$, for all vector fields $X, Y \in \Gamma(D)$.*

Lemma 2.2. *Let M be a CR-sub-manifold of a quasi-Kaehlerian manifold $\overline{M}$. Then we have*

$$\begin{aligned}(\nabla_X \omega)Y = {}& -h(X, \phi Y) + Ch(X, Y) + h(\phi X, Y) + Ch(\phi X, \phi Y) \\ & + \omega \nabla_{\phi X} \phi Y - \omega A_{\omega Y} \phi X + C \nabla_{\phi X}^\perp \omega Y,\end{aligned} \tag{2.1}$$

for any $X \in \Gamma(D)$ *and* $Y \in \Gamma(TM)$.

Proof. For any $X \in \Gamma(D)$ and $Y \in \Gamma(TM)$, from (1.3) we have

$$0 = \overline{\nabla}_X JY - J\overline{\nabla}_X Y - \overline{\nabla}_{JX} Y - J\overline{\nabla}_{JX} JY. \tag{2.2}$$

By using (2.2), (1.5), (1.6) and (1.8) we get

$$\begin{aligned} 0 = {} & \nabla_X \phi Y + h(X, \phi Y) - A_{\omega Y} X + \nabla^{\perp}_X \omega Y - J\nabla_X Y - Jh(X, Y) \\ & -\nabla_{\phi X} Y - h(\phi X, Y) - h(\phi X, Y) - J\nabla_{\phi X} \phi Y \\ & -Jh(\phi X, \phi Y) + JA_{\omega Y} \phi X - J\nabla^{\perp}_{\phi X} \omega Y. \end{aligned} \tag{2.3}$$

Taking account of (2.3), (1.8) and (1.9), we obtain

$$\begin{aligned} 0 = {} & h(X, \phi Y) - Ch(X, Y) - h(\phi X, Y) - Ch(\phi X, \phi Y) + \nabla^{\perp}_X \omega Y \\ & -\omega\nabla_X Y - \omega\nabla_{\phi X} \phi Y + \omega A_{\omega Y} \phi X - C\nabla^{\perp}_{\phi X} \omega Y + \nabla_X \phi Y \\ & -A_{\omega Y} X - \phi\nabla_X Y - Bh(X, Y) - \nabla_{\phi X} Y - \phi\nabla_{\phi X} \phi Y \\ & -Bh(\phi X, \phi Y) + \phi A_{\omega Y} \phi X - B\nabla^{\perp}_{\phi X} \omega Y. \end{aligned} \tag{2.4}$$

By comparing to the tangent part and the normal part in (2.4), we get

$$\begin{aligned} 0 = {} & \nabla_X \phi Y - A_{\omega Y} X - \phi\nabla_X Y - Bh(X, Y) - \nabla_{\phi X} Y - \phi\nabla_{\phi X} \phi Y \\ & -Bh(\phi X, \phi Y) + \phi A_{\omega Y} \phi X - B\nabla^{\perp}_{\phi X} \omega Y \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} 0 = {} & h(X, \phi Y) - Ch(X, Y) - h(\phi X, Y) - Ch(\phi X, \phi Y) + \nabla^{\perp}_X \omega Y \\ & -\omega\nabla_X Y - \omega\nabla_{\phi X} \phi Y + \omega A_{\omega Y} \phi X - C\nabla^{\perp}_{\phi X} \omega Y. \end{aligned} \tag{2.6}$$

Thus (2.1) follows from (2.6) and (1.13). □

Theorem 2.1. *Let* M *be a CR-sub-manifold of a quasi-Kaehlerian manifold* $\overline{M}$. *Then the distribution* D *is integrable if and only if*

$$\omega[\phi Y, \phi X] + 2h(X, \phi Y) - 2h(\phi X, Y) = 0, \tag{2.7}$$

for any $X, Y \in \Gamma(D)$.

Proof. For any $X, Y \in \Gamma(D)$. From (1.5) and (1.8) we have

$$\omega[X, Y] = \omega(\nabla_X Y - \nabla_Y X) = \nabla^{\perp}_Y \omega X - \omega\nabla_Y X - \nabla^{\perp}_X \omega Y + \omega\nabla_X Y. \tag{2.8}$$

By using (2.8) and (1.13) we obtain

$$\omega[X, Y] = (\nabla_Y \omega)X - (\nabla_X \omega)Y. \tag{2.9}$$

Taking account of (2.9) and (2.1) we get

$$\omega[X, Y] = \omega[\phi Y, \phi X] + 2h(X, \phi Y) - 2h(\phi X, Y). \tag{2.10}$$

According to Frobenius's Theorem, we know that the distribution D is integrable if and only if $\omega[X, Y] = 0$, for any $X, Y \in \Gamma(D)$. Taking into account (2.10), we see that the distribution D is integrable if and only if (2.7) is satisfied. □

Lemma 2.3 ([4, 5])**.** *Let $\overline{M}$ be a quasi-Karhlerian manifold. Then we have*

$$(\overline{\nabla}_X J)Y - (\overline{\nabla}_Y J)X = \frac{1}{2}J[J,J](X,Y), \tag{2.11}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Lemma 2.4 ([6, 7])**.** *Let M be a CR-sub-manifold of a quasi-Kaehlerian manifold $\overline{M}$. Then the distribution D is integrable if and only if*

$$h(X, JY) = h(JX, Y) \tag{2.12}$$

and

$$[J,J](X,Y) \in \Gamma(D), \tag{2.13}$$

for any $X, Y \in \Gamma(D)$.

Theorem 2.2. *Let M be a CR-sub-manifold of a quasi-Kaehlerian manifold $\overline{M}$. Then the distribution D is integrable if and only if*

$$h(X, JY) = h(JX, Y) \tag{2.14}$$

and

$$4g((\overline{\nabla}_U J)Y, JX) = g([J,J](X,U), Y) - g([J,J](Y,U), X), \tag{2.15}$$

for any $X, Y \in \Gamma(D), U \in \Gamma(D^{\perp})$.

Proof. For any $X, Y \in \Gamma(D), U \in \Gamma(D^{\perp})$. From (2.11) and (1.1) we have

$$\frac{1}{2}g([J,J](X,U), Y) = g((\overline{\nabla}_X J)U - (\overline{\nabla}_U J)X, JY). \tag{2.16}$$

From (2.16), (1.1) and (1.5) we get

$$\begin{aligned} &\frac{1}{2}g([J,J](X,U), Y) \\ &\qquad = -g(JU, h(X, JY)) + g(U, \overline{\nabla}_X Y) - g((\overline{\nabla}_U J)X, JY). \end{aligned} \tag{2.17}$$

Exchanging X with Y in (2.17) we obtain

$$\begin{aligned} &\frac{1}{2}g([J,J](Y,U), X) \\ &\qquad = -g(JU, h(Y, JX)) + g(U, \overline{\nabla}_Y X) - g((\overline{\nabla}_U J)Y, JX). \end{aligned} \tag{2.18}$$

On the other hand, by a direct computation we achieve

$$g((\overline{\nabla}_U J)X, JY) = -g((\overline{\nabla}_U J)Y, JX). \tag{2.19}$$

From (2.17) and (2.19) we find

$$\begin{aligned} &\frac{1}{2}g([J,J](X,U), Y) \\ &\qquad = -g(JU, h(X, JY)) + g(U, \overline{\nabla}_X Y) + g((\overline{\nabla}_U J)Y, JX). \end{aligned} \tag{2.20}$$

(2.20)−(2.18) follows

$$(2.21)\quad \frac{1}{2}g([J,J](X,U),Y)-\frac{1}{2}g([J,J](Y,U),X) = g(JU,h(Y,JX)-h(X,JY))+g(U,[X,Y])+2g((\overline{\nabla}_U J)Y,JX).$$

(2.21) can be become

$$(2.22)\quad g([X,Y],U)=\frac{1}{2}g([J,J](X,U),Y)-\frac{1}{2}g([J,J](Y,U),X) +g(JU,h(X,JY)-h(Y,JX))-2g((\overline{\nabla}_U J)Y,JX).$$

Suppose D is integrable. Then from Lemma 2.4 and (2.22) we have

$$h(X,JY)=h(Y,JX)$$

and

$$0=\frac{1}{2}g([J,J](X,U),Y)-\frac{1}{2}g([J,J](Y,U),X)-2g((\overline{\nabla}_U J)Y,JX)$$

for any $X,Y\in\Gamma(D),U\in\Gamma(D^{\perp})$, which is equivalent to (2.15).

Conversely, suppose (2.14) and (2.15) are satisfied. From (2.22), (2.14) and (2.15) we have $[X,Y]\in\Gamma(D)$ for any $X,Y\in\Gamma(D)$. By Frobenius's Theorem, we know that the distribution D is integrable. □

Lemma 2.5. *Let M be a CR-sub-manifold of an almost Hermitian manifold $\overline{M}$. Then we have*

$$(2.23)\quad (\nabla_X\phi)Y=A_{\omega Y}X+Bh(X,Y)+((\overline{\nabla}_X J)Y)^{\top},$$

for any $X,Y\in\Gamma(TM)$.

Proof. For any $X,Y\in\Gamma(TM)$. From (1.5) and (1.8) we have

$$(2.24)\quad (\overline{\nabla}_X J)Y=\overline{\nabla}_X(\phi Y+\omega Y)-J(\nabla_X Y+h(X,Y)).$$

By using (2.24), (1.5), (1.6), (1.8) and (1.9) we get

$$(2.25)\quad (\overline{\nabla}_X J)Y=\nabla_X\phi Y+h(X,\phi Y)-A_{\omega Y}X+\nabla^{\perp}_X\omega Y -\phi\nabla_X Y-\omega\nabla_X Y-Bh(X,Y)-Ch(X,Y).$$

By comparying to the tangent part and the normal part in (2.25), we obtain

$$(2.26)\quad ((\overline{\nabla}_X J)Y)^{\top}=\nabla_X\phi Y-A_{\omega Y}X-\phi\nabla_X Y-Bh(X,Y),$$

and

$$(2.27)\quad ((\overline{\nabla}_X J)Y)^{\perp}=h(X,\phi Y)+\nabla^{\perp}_X\omega Y-\omega\nabla_X Y-Ch(X,Y).$$

Taking account of (2.26) and (1.12), (2.23) is satisfied. □

Theorem 2.3. *Let M be a CR-sub-manifold of an almost Hermitian manifold $\overline{M}$. Then the distribution $D^{\perp}$ is integrable if and only if*

$$(2.28)\quad A_{\omega U}V-A_{\omega V}U+((\overline{\nabla}_V J)U)^{\top}-((\overline{\nabla}_U J)V)^{\top}=0,$$

for any $U,V\in\Gamma(D^{\perp})$.

Proof. For any $U, V \in \Gamma(D^\perp)$. From (1.5) and (1.8) we have

$$\phi[U,V] = \phi(\nabla_U V - \nabla_V U) = -\nabla_U \phi V + \phi \nabla_U V + \nabla_V \phi U - \phi \nabla_V U. \tag{2.29}$$

By using (2.29) and (1.12) we obtain

$$\phi[U,V] = (\nabla_V \phi)U - (\nabla_U \phi)V. \tag{2.30}$$

Taking account of (2.23) and (2.30) we get

$$\phi[U,V] = A_{\omega U} V + ((\overline{\nabla}_V J)U)^\top - A_{\omega V} U - ((\overline{\nabla}_U J)V)^\top. \tag{2.31}$$

According to Frobenius's Theorem, we know that the distribution $D^\perp$ is integrable if and only if $\phi[U,V] = 0$, for any $U, V \in \Gamma(D^\perp)$. Taking into account (2.31), we see that the distribution $D^\perp$ is integrable if and only if (2.28) is satisfied. □

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Received March 29, 2013.

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