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EINSTEIN EQUATIONS OF G-NATURAL COMPLEX FINSLER METRICS

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Dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday

Abstract. In this paper, we endow the holomorphic tangent bundle with a generalized Sasaki type lift G of the fundamental metric tensor of a complex Finsler space. In order to build the Einstein equations on the holomorphic tangent bundle, we determine the Levi-Civita complex linear connection corresponding to this metric. As an application, we give some solutions of the complex Einstein equations in a weakly gravitational space.

1. Preliminaries

In the papers [3, 4] are studied complex Einstein equations for the weakly gravitational field and for complex version of Schwartzschild metric. This study is based on the idea to write the complex Einstein equations for these metrics relative to the Chern-Finsler connection, which is metrical but with torsion. For such theory to be consistent some restrictions are required, called conservation laws, because the connection used is with torsion.

An alternative to this theory is the one expressed in the following. We extend the metric structure of the weakly gravitational field to one on the holomorphic tangent bundle $T'M$ of a complex manifold M, and we then consider the Levi-Civita connection of this lifted metric, which is metrical and torsion-free. Therefore the complex Einstein equations with respect to the Levi-Civita connection have the classical from. In particular, if the space is vacuum the complex Einstein equations reduce to the vanishing of the complex Ricci tensor. Basically, the idea seems to be simple, but the first problem is how to get such a lift and how they can be general. Then is the issue of writing

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the curvature tensors and Ricci tensors on $T'M$. For this we turn again to the well-known adapted frames of Chern-Finsler connection, and express all in this complex adapted frames. A similar idea has been applied to the real case by M. Anastasiei and H. Shimada, [5]. Finally, we propose to solve these complex Einstein equations, at least in same particular cases of weakly gravitational metric.

Let M be a complex manifold of complex dimension n. We consider $z \in M$, and so $z = (z^1, \dots, z^n)$ are complex coordinate in a local chart. Since $z^k = x^k + \sqrt{-1}x^{k+n}$, $k = 1, \dots, n$, the complex coordinates induce the real coordinates $\{x^1, x^2, \dots, x^{2n}\}$ on M. Let $T_R M$ be the real tangent bundle. Its complexified tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$, under the action of the natural complex structure J on M. The holomorphic tangent bundle $T'M$ is itself a complex manifold, and the coordinates in a local chart will be denoted by $u = (z^k, \eta^a)$, $k, a = 1, \dots, n$. with $\eta^a = y^a + \sqrt{-1}y^{a+n}$, $a = 1, \dots, n$. Trough this paper the indices $i, j, k, \dots$ and $a, b, c, \dots$ run over $\{1, \dots, n\}$, where the second denotes geometric objects in local fibers of the holomorphic tangent bundle. This notation of the indices is important for the clarity of the notions in geometry of $T'M$ manifold.

Consider the sections of the complexified tangent bundle $T_C M$. Let $VT'M \subset T'(T'M)$ be the vertical bundle, locally spanned by $\left\{\frac{\partial}{\partial \eta^a}\right\}_{a=\overline{1,n}}$, and $VT''M$ its conjugate. The idea of the complex non-linear connection, briefly c.n.c. is an instrument in 'linearization' of the geometry of $T'M$ manifold. A c.n.c. is a supplementary subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_u T'M$ is locally spanned by

$$\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^a \frac{\partial}{\partial \eta^a}, \tag{1}$$

where $N_k^a(z, \eta)$ are the coefficients of the c.n.c. The pair $\{\delta_k := \frac{\delta}{\delta z^k},\ \dot{\partial}_a := \frac{\partial}{\partial \eta^a}\}$ will be called the adapted frame of the c.n.c., which obey the transformation laws $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and $\dot{\partial}_a = \delta_a^k \delta_j^b \frac{\partial z'^j}{\partial z^k}\dot{\partial}'_b$, where δ_a^k is the Kronecker symbol. By conjugation everywhere we obtain an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{a}}\}$ on $T''_u(T'M)$. The dual adapted frame are $\{\mathrm{d}z^k, \mathrm{d}\eta^a\}$ and $\{\mathrm{d}\bar{z}^k, \mathrm{d}\bar{\eta}^a\}$ its conjugate, where

$$\delta\eta^a = \mathrm{d}\eta^a + N_k^a(z, \eta)\mathrm{d}z^k. \tag{2}$$

Let N be a c.n.c. on $T'M$. An $h-$metric on $T'M$ is a $d-$tensor field $h\mathcal{G} = g_{j\bar{k}}(z, \eta)\mathrm{d}z^j \otimes \mathrm{d}\bar{z}^k$, with $g_{j\bar{k}}(z, \eta) = \overline{g_{k\bar{j}}(z, \eta)}$, $\det \|g_{j\bar{k}}(z, \eta)\| \neq 0$. A $v-$metric on $T'M$ is a $d-$tensor field $v\mathcal{G} = h_{a\bar{b}}(z, \eta)\delta\eta^a \otimes \delta\bar{\eta}^b$, with $h_{a\bar{b}}(z, \eta) = \overline{h_{b\bar{a}}(z, \eta)}$, $\det \|h_{a\bar{b}}(z, \eta)\| \neq 0$. From here we obtain that an $(h, v)-$metric on $T'M$ is tensor field $\mathcal{G} = h\mathcal{G} + v\mathcal{G}$. So, this metric will be written in the next form:

$$\mathcal{G}(z, \eta) = g_{j\bar{k}}(z, \eta)\mathrm{d}z^j \otimes \mathrm{d}\bar{z}^k + h_{a\bar{b}}(z, \eta)\delta\eta^a \otimes \delta\bar{\eta}^b. \tag{3}$$

A distinguished complex linear connection, briefly d-c.l.c., D on $T'M$ is called compatible with the metric $\mathcal{G}$ if $D\mathcal{G} = 0$.

Definition 1. An $n-$dimensional *complex Finsler space* is a pair (M, F), where $F : T'M \to \mathbb{R}^+$ is a continuous function satisfying the conditions:

a) $L := F^2$ is smooth on $\widetilde{T'M} := T'M \setminus \{0\}$;
b) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;
c) $F(z, \lambda\eta) = |\lambda| F(z, \eta)$ for $\forall \lambda \in \mathbb{C}$;
d) the Hermitian matrix

$$g_{j\bar{k}} = \delta^a_j \overline{\delta^b_k} \frac{\partial^2 L}{\partial \eta^a \partial \bar{\eta}^b}, \tag{4}$$

is positive-definite on $\widetilde{T'M}$.

Then, $g_{j\bar{k}}$ is called the *fundamental metric tensor* of the complex Finsler space. Consequently, from c) we have $\frac{\partial L}{\partial \eta^a}\eta^a = \frac{\partial L}{\partial \bar{\eta}^a}\bar{\eta}^a = L$, $\frac{\partial g_{j\bar{k}}}{\partial \eta^a}\eta^a = \frac{\partial g_{j\bar{k}}}{\partial \bar{\eta}^a}\bar{\eta}^a = 0$, and $L = \delta^j_a \overline{\delta^k_b} g_{j\bar{k}} \eta^a \bar{\eta}^b$.

From [1, 7] we know there exists a unique metric Hermitian connection D, of type $(1, 0)-$type, which satisfies in addition $D_{JX}Y = JD_XY$, for all X horizontal vectors, called the *Chern-Finsler connection*, in brief C-F, which have a special meaning in complex Finsler geometry. The C-F connection $D\Gamma = (N^a_j, L^i_{jk}, C^a_{bc}, 0, 0)$ is locally given by the following coefficients:

$$N^a_j = \delta^a_k \delta^l_b g^{\bar{m}k} \frac{\partial g_{l\bar{m}}}{\partial z^j} \eta^b = \delta^a_k \delta^l_b L^k_{lj} \eta^b; \quad L^h_{jk} = g^{\bar{l}h} \delta_k g_{j\bar{l}}; \quad C^a_{bc} = \delta^a_h \delta^j_b g^{\bar{l}h} \dot{\partial}_c g_{j\bar{l}}, \tag{5}$$

where the non-vanishing expressions of the C-F connection are $D_{\delta_k}\delta_j = L^h_{jk}\delta_h$, $D_{\dot{\partial}_b}\dot{\partial}_a = C^d_{ab}\dot{\partial}_d$ and its conjugates.

A particular situation of the $d-$tensor $g_{j\bar{k}}$ from (4) is:

Definition 2. If $g_{j\bar{k}}$ depends only on the variable z, then we say that the space (M, F) is purely Hermitian.

The metric tensor $g_{j\bar{k}}$ from (4) determines a metric structure G_S on $T_C(T'M)$, called the *Sasaki lift* of $g_{j\bar{k}}$, [7], p.96:

$$G_S = g_{j\bar{k}} \mathrm{d}z^j \otimes \mathrm{d}\bar{z}^k + \delta^j_a \overline{\delta^k_b} g_{j\bar{k}}(z, \eta) \delta\eta^a \otimes \delta\bar{\eta}^b. \tag{6}$$

We introduce a generalization of the lift (6), which defines also an $(h, v)-$metric on $T_C(T'M)$:

$$G(z, \eta) = g_{j\bar{k}}(z, \eta) \mathrm{d}z^j \otimes \mathrm{d}\bar{z}^k + h_{a\bar{b}}(z, \eta) \delta\eta^a \otimes \delta\bar{\eta}^b, \tag{7}$$

where $g_{j\bar{k}}$ is the fundamental metric tensor of the Finsler space (M, F), and $h_{a\bar{b}}$ is an arbitrary $d-$tensor of $(0, 2)-$type.

2. The Levi-Civita connection on $T'M$

From the standard definition of a complex linear connection on the manifold $T'M$, extended on the complexified tangent bundle $T_C(T'M)$, a complex linear connection ∇ can be decomposed in the sum $\nabla = \nabla' + \nabla''$, where $\nabla'\colon \Gamma(T_C(T'M)) \to \Gamma(T_C(T'M) \otimes T'(T'^*M))$ and $\nabla''\colon \Gamma(T_C(T'M)) \to \Gamma(T_C(T'M) \otimes T''(T'^*M))$, which can be decomposed in

$$\nabla' = \nabla'^h + \nabla'^v \text{ and } \nabla'' = \nabla''^h + \nabla''^v.$$

So, in the adapted frame of the C-F c.n.c. $\{\delta_k, \dot\partial_a, \delta_{\bar k}, \dot\partial_{\bar a}\}$, ∇ is defined by the following coefficients:

$$\begin{aligned}
\nabla_{\delta_k}\delta_j &= \overset{1}{L}{}^i_{jk}\delta_i + \overset{2}{A}{}^d_{jk}\dot\partial_d + \overset{3}{A}{}^{\bar\imath}_{jk}\delta_{\bar\imath} + \overset{4}{A}{}^{\bar d}_{jk}\dot\partial_{\bar d};\\
\nabla_{\delta_k}\dot\partial_a &= \overset{1}{B}{}^i_{ak}\delta_i + \overset{2}{L}{}^d_{ak}\dot\partial_i + \overset{3}{B}{}^{\bar\imath}_{ak}\delta_{\bar\imath} + \overset{4}{B}{}^{\bar d}_{ak}\dot\partial_{\bar d};\\
\nabla_{\delta_k}\delta_{\bar\jmath} &= \overset{1}{D}{}^i_{\bar\jmath k}\delta_i + \overset{2}{D}{}^d_{\bar\jmath k}\dot\partial_d + \overset{3}{L}{}^{\bar\imath}_{\bar\jmath k}\delta_{\bar\imath} + \overset{4}{D}{}^{\bar d}_{\bar\jmath k}\dot\partial_{\bar d};\\
\nabla_{\delta_k}\dot\partial_{\bar a} &= \overset{1}{E}{}^i_{\bar a k}\delta_i + \overset{2}{E}{}^d_{\bar a k}\dot\partial_d + \overset{3}{E}{}^{\bar\imath}_{\bar a k}\delta_{\bar\imath} + \overset{4}{L}{}^{\bar d}_{\bar a k}\dot\partial_{\bar d};\\
\nabla_{\dot\partial_b}\delta_j &= \overset{1}{C}{}^i_{jb}\delta_i + \overset{2}{F}{}^d_{jb}\dot\partial_d + \overset{3}{F}{}^{\bar\imath}_{jb}\delta_{\bar\imath} + \overset{4}{F}{}^{\bar d}_{jb}\dot\partial_{\bar d};\\
\nabla_{\dot\partial_b}\dot\partial_a &= \overset{1}{G}{}^i_{ab}\delta_i + \overset{2}{C}{}^d_{ab}\dot\partial_d + \overset{3}{G}{}^{\bar\imath}_{ab}\delta_{\bar\imath} + \overset{4}{G}{}^{\bar d}_{ab}\dot\partial_{\bar d};\\
\nabla_{\dot\partial_b}\delta_{\bar\jmath} &= \overset{1}{H}{}^i_{\bar\jmath b}\delta_i + \overset{2}{H}{}^d_{\bar\jmath b}\dot\partial_d + \overset{3}{C}{}^{\bar\imath}_{\bar\jmath b}\delta_{\bar\imath} + \overset{4}{H}{}^{\bar d}_{\bar\jmath b}\dot\partial_{\bar d};\\
\nabla_{\dot\partial_b}\dot\partial_{\bar a} &= \overset{1}{M}{}^i_{\bar a b}\delta_i + \overset{2}{M}{}^d_{\bar a b}\dot\partial_d + \overset{3}{M}{}^{\bar\imath}_{\bar a b}\delta_{\bar\imath} + \overset{4}{C}{}^{\bar d}_{\bar a b}\dot\partial_{\bar d}
\end{aligned} \tag{8}$$

and its conjugates, by $\nabla_{\bar X}\bar Y = \overline{\nabla_X Y}$.

Since $\nabla G = 0$ and ∇ is a symmetric connection, direct calculus leads to

Theorem 1. *The Hermitian manifold $(T'M, G_H)$ admits a unique complex linear connection, which is symmetric and metrical in respect to G, defined by (3). This is called the Levi-Civita connection on $T'M$, and its local coefficients are represented in the local adapted frame $\{\delta_k, \dot\partial_a, \delta_{\bar k}, \dot\partial_{\bar a}\}$ by the following non-zero expressions:*

(9)

$$\overset{1}{L}{}^i_{jk} = \frac{1}{2}g^{\bar l i}(\delta_k g_{j\bar l} + \delta_j g_{k\bar l}); \qquad \overset{2}{D}{}^c_{\bar\jmath k} = -\overset{4}{D}{}^c_{k\bar\jmath} = \frac{1}{2}[\delta_{\bar\jmath}N^c_k - h^{\bar d c}(\dot\partial_{\bar d} g_{k\bar\jmath})];$$

$$\overset{2}{L}{}^c_{ak} = \frac{1}{2}[h^{\bar d c}(\delta_k h_{a\bar d}) + \dot\partial_a N^c_k]; \qquad \overset{2}{E}{}^c_{\bar a k} = \frac{1}{2}h^{\bar d c}[(\dot\partial_{\bar a}N^e_k)h_{e\bar d} - (\dot\partial_{\bar d}N^e_k)h_{e\bar a}];$$

$$\overset{3}{L}{}^i_{j\bar k} = \overset{1}{D}{}^i_{\bar k j} = \frac{1}{2}g^{\bar l i}(\delta_{\bar k} g_{j\bar l} - \delta_{\bar l} g_{j\bar k}); \qquad \overset{2}{F}{}^c_{jb} = \frac{1}{2}[h^{\bar d c}(\delta_j h_{a\bar d}) - \dot\partial_a N^c_j];$$

$$\overset{4}{L}{}^{c}_{a\bar{k}} = \overset{2}{H}{}^{c}_{\bar{k}a} = \frac{1}{2}h^{\bar{d}c}[\delta_{\bar{k}}h_{a\bar{d}} - (\dot{\partial}_{\bar{d}}N^{\bar{e}}_{\bar{k}})h_{a\bar{e}}]; \quad \overset{1}{G}{}^{i}_{ab} = \frac{1}{2}g^{\bar{l}i}[(\dot{\partial}_b N^{\bar{d}}_{\bar{l}})h_{a\bar{d}} + (\dot{\partial}_a N^{\bar{d}}_{\bar{l}})h_{b\bar{d}}];$$
$$\overset{1}{C}{}^{i}_{ka} = \overset{1}{B}{}^{i}_{ak} = \frac{1}{2}g^{\bar{l}i}[\dot{\partial}_a g_{k\bar{l}} + (\delta_k N^{\bar{d}}_{\bar{l}})h_{a\bar{d}}]; \quad \overset{4}{H}{}^{c}_{j\bar{b}} = -\frac{1}{2}h^{\bar{d}c}[(\dot{\partial}_{\bar{d}}N^{e}_{j})h_{e\bar{b}} + (\dot{\partial}_{\bar{b}}N^{e}_{j})h_{e\bar{d}}];$$
$$\overset{2}{C}{}^{c}_{ab} = \frac{1}{2}h^{\bar{d}c}(\dot{\partial}_b h_{a\bar{d}} + \dot{\partial}_a h_{b\bar{d}}); \quad \overset{1}{M}{}^{i}_{\bar{a}b} = \overset{3}{M}{}^{i}_{b\bar{a}} = \frac{1}{2}g^{\bar{l}i}[(\dot{\partial}_{\bar{a}}N^{\bar{d}}_{\bar{l}})h_{b\bar{d}} - \delta_{\bar{l}}h_{b\bar{a}}];$$
$$\overset{3}{C}{}^{i}_{j\bar{b}} = \overset{1}{E}{}^{i}_{\bar{a}j} = \frac{1}{2}g^{\bar{l}i}[\dot{\partial}_{\bar{b}}g_{j\bar{l}} - (\delta_{\bar{l}}N^{d}_{j})h_{d\bar{b}}]; \quad \overset{4}{C}{}^{c}_{a\bar{b}} = \overset{2}{M}{}^{c}_{\bar{b}a} = \frac{1}{2}h^{\bar{d}c}(\dot{\partial}_{\bar{b}}h_{a\bar{d}} - \dot{\partial}_{\bar{d}}h_{a\bar{b}}),$$

and its conjugates.

This connection is not h- or v-metrical.

To study the Levi-Civita connection, we may consider a similar connection, which help us to express easier the different properties of the Levi-Civita connection. Indeed let $\widetilde{D}$ be a d-c.l.c. on $T_C(T'M)$:

$$\widetilde{D}_{\delta_k}\delta_j = \overset{1}{L}{}^{i}_{jk}\delta_i;\ \widetilde{D}_{\delta_k}\dot{\partial}_a = \overset{2}{L}{}^{d}_{ak}\dot{\partial}_d\ ;\ \widetilde{D}_{\delta_k}\delta_{\bar{j}} = \overset{3}{L}{}^{\bar{i}}_{\bar{j}k}\delta_{\bar{i}}\ ;\ \widetilde{D}_{\delta_k}\dot{\partial}_{\bar{a}} = \overset{4}{L}{}^{\bar{d}}_{\bar{a}k}\dot{\partial}_{\bar{d}}; \tag{10}$$
$$\widetilde{D}_{\dot{\partial}_a}\delta_j = \overset{1}{C}{}^{i}_{ja}\delta_i;\ \widetilde{D}_{\dot{\partial}_b}\dot{\partial}_a = \overset{2}{C}{}^{d}_{ab}\dot{\partial}_d\ ;\ \widetilde{D}_{\dot{\partial}_a}\delta_{\bar{j}} = \overset{3}{C}{}^{\bar{i}}_{\bar{j}a}\delta_{\bar{i}}\ ;\ \widetilde{D}_{\dot{\partial}_b}\dot{\partial}_{\bar{a}} = \overset{4}{C}{}^{\bar{d}}_{\bar{a}b}\dot{\partial}_{\bar{d}}$$

and its conjugates, where the local coefficients are expressed in (9).

This d-c.l.c. is metrical with respect to G, i.e.

$$g_{j\bar{k}|m} = g_{j\bar{k}}|_d = g_{j\bar{k}|\bar{m}} = g_{j\bar{k}}|_{\bar{d}} = h_{a\bar{b}|m} = h_{a\bar{b}}|_d = h_{a\bar{b}|\bar{m}} = h_{a\bar{b}}|_{\bar{d}} = 0, \tag{11}$$

where with "$_|$", "$|$", "$_{\bar{|}}$", "$\bar{|}$" are notated the $h-$, $v-$, $\bar{h}-$ and $\bar{v}-$covariant derivatives with respect to $\widetilde{D}$.

Proposition 1. *The non-zero components of the torsion of the d-c.l.c. $\widetilde{D}$ are*
(12)
$$h\widetilde{\mathbb{T}}(\delta_{\bar{k}},\delta_j) = \widetilde{\tau}^{i}_{j\bar{k}}\delta_i; \quad v\widetilde{\mathbb{T}}(\delta_{\bar{k}},\delta_j) = \widetilde{\Theta}^{d}_{j\bar{k}}\dot{\partial}_d; \quad h\widetilde{\mathbb{T}}(\dot{\partial}_{\bar{a}},\delta_j) = \widetilde{\Upsilon}^{i}_{j\bar{a}}\delta_i; \quad h\widetilde{\mathbb{T}}(\dot{\partial}_a,\delta_j) = \widetilde{Q}^{i}_{ja}\delta_i;$$
$$v\widetilde{\mathbb{T}}(\dot{\partial}_{\bar{b}},\dot{\partial}_a) = \widetilde{\chi}^{d}_{a\bar{b}}\dot{\partial}_d; \quad v\widetilde{\mathbb{T}}(\dot{\partial}_{\bar{a}},\delta_j) = \widetilde{\rho}^{d}_{j\bar{a}}\dot{\partial}_d; \quad v\widetilde{\mathbb{T}}(\delta_{\bar{k}},\dot{\partial}_a) = \widetilde{\Sigma}^{d}_{a\bar{k}}\dot{\partial}_d; \quad v\widetilde{\mathbb{T}}(\dot{\partial}_a,\delta_j) = \widetilde{P}^{d}_{ja}\dot{\partial}_d;$$
and their conjugates.

After a straightforward computation we obtain the expressions for (12)

$$\widetilde{\tau}^{i}_{j\bar{k}} = \overset{3}{L}{}^{i}_{j\bar{k}};\ \widetilde{\Theta}^{d}_{j\bar{k}} = \delta_{\bar{k}}N^{d}_{j};\ \ \widetilde{\Upsilon}^{i}_{j\bar{a}} = \overset{3}{C}{}^{i}_{j\bar{a}};\ \ \widetilde{Q}^{i}_{ja} = \overset{1}{C}{}^{i}_{ja}; \tag{13}$$
$$\widetilde{\chi}^{d}_{a\bar{b}} = \overset{4}{C}{}^{d}_{a\bar{b}};\ \widetilde{\rho}^{d}_{j\bar{a}} = \dot{\partial}_{\bar{a}}N^{d}_{j};\ \ \widetilde{\Sigma}^{d}_{b\bar{j}} = \overset{4}{L}{}^{d}_{b\bar{j}};\ \ \widetilde{P}^{d}_{ja} = \dot{\partial}_b N^{d}_{j} - \overset{2}{L}{}^{d}_{jb}.$$

The curvature of $\widetilde{D}$ has twenty components in the form (see p. 44 of [7]):

$$\widetilde{R}^{i}_{jkh} = \mathcal{A}_{hk}\{\delta_h\overset{1}{L}{}^{i}_{jk} + \overset{1}{L}{}^{m}_{jk}\overset{1}{L}{}^{i}_{mh}\}; \tag{14}$$
$$\widetilde{R}^{\bar{\imath}}_{\bar{j}kh} = \mathcal{A}_{hk}\{\delta_h\overset{3}{L}{}^{\bar{\imath}}_{\bar{j}k} + \overset{3}{L}{}^{\bar{m}}_{\bar{j}k}\overset{3}{L}{}^{\bar{\imath}}_{\bar{m}h}\};$$

$$\widetilde{R}^i_{j\bar{k}h} = \delta_h \overset{3}{L}{}^i_{j\bar{k}} - \delta_{\bar{k}} \overset{1}{L}{}^i_{jh} + \overset{3}{L}{}^m_{j\bar{k}} \overset{1}{L}{}^i_{mh} - \overset{3}{L}{}^i_{m\bar{k}} \overset{1}{L}{}^m_{jh} + (\delta_h N^{\bar{e}}_{\bar{k}}) \overset{3}{C}{}^i_{j\bar{e}} - (\delta_{\bar{k}} N^e_h) \overset{1}{C}{}^i_{je};$$

$$\widetilde{\Omega}^d_{akh} = \mathcal{A}_{hk}\{\delta_h \overset{2}{L}{}^i_{ak} + \overset{2}{L}{}^e_{ak} \overset{2}{L}{}^d_{eh}\};$$

$$\widetilde{\Omega}^{\bar{d}}_{\bar{a}kh} = \mathcal{A}_{hk}\{\delta_h \overset{4}{L}{}^{\bar{d}}_{\bar{a}k} + \overset{4}{L}{}^{\bar{e}}_{\bar{a}k} \overset{4}{L}{}^{\bar{d}}_{\bar{e}h}\};$$

$$\widetilde{\Omega}^d_{a\bar{k}h} = \delta_h \overset{4}{L}{}^d_{a\bar{k}} - \delta_{\bar{k}} \overset{2}{L}{}^d_{ah} + \overset{4}{L}{}^e_{a\bar{k}} \overset{2}{L}{}^d_{eh} - \overset{4}{L}{}^d_{e\bar{k}} \overset{2}{L}{}^e_{ah} + (\delta_h N^{\bar{e}}_{\bar{k}}) \overset{4}{C}{}^d_{a\bar{e}} - (\delta_{\bar{k}} N^e_h) \overset{2}{C}{}^d_{ae};$$

$$\widetilde{\Pi}^i_{jkc} = \dot{\partial}_c \overset{1}{L}{}^i_{jk} - \delta_k \overset{1}{C}{}^i_{jc} + \overset{1}{L}{}^m_{jk} \overset{1}{C}{}^i_{mc} - \overset{1}{L}{}^i_{mk} \overset{1}{C}{}^i_{jc} + (\dot{\partial}_c N^e_k) \overset{1}{C}{}^i_{ej};$$

$$\widetilde{\Pi}^{\bar{i}}_{\bar{j}kc} = \dot{\partial}_c \overset{3}{L}{}^{\bar{i}}_{\bar{j}k} - \delta_k \overset{3}{C}{}^{\bar{i}}_{\bar{j}c} + \overset{3}{L}{}^m_{\bar{j}k} \overset{3}{C}{}^{\bar{i}}_{mc} - \overset{3}{L}{}^{\bar{i}}_{mk} \overset{3}{C}{}^{\bar{i}}_{\bar{j}c} + (\dot{\partial}_c N^e_k) \overset{3}{C}{}^i_{e\bar{j}};$$

$$\widetilde{\Pi}^i_{j\bar{k}c} = \dot{\partial}_c \overset{3}{L}{}^i_{j\bar{k}} - \delta_{\bar{k}} \overset{1}{C}{}^i_{jc} + \overset{3}{L}{}^m_{j\bar{k}} \overset{1}{C}{}^i_{mc} - \overset{3}{L}{}^i_{m\bar{k}} \overset{1}{C}{}^m_{jc} + (\dot{\partial}_c N^{\bar{e}}_{\bar{k}}) \overset{1}{C}{}^i_{j\bar{e}};$$

$$\widetilde{P}^d_{akc} = \dot{\partial}_c \overset{2}{L}{}^d_{ak} - \delta_k \overset{2}{C}{}^d_{ac} + \overset{2}{L}{}^e_{ak} \overset{2}{C}{}^d_{ec} - \overset{2}{L}{}^d_{ek} \overset{2}{C}{}^e_{ac} + (\dot{\partial}_c N^e_k) \overset{2}{C}{}^d_{ae};$$

$$\widetilde{P}^{\bar{d}}_{\bar{a}kc} = \dot{\partial}_c \overset{4}{L}{}^d_{\bar{a}k} - \delta_k \overset{4}{C}{}^d_{ac} + \overset{4}{L}{}^{\bar{e}}_{\bar{a}k} \overset{4}{C}{}^{\bar{d}}_{\bar{e}c} - \overset{4}{L}{}^{\bar{d}}_{\bar{e}k} \overset{4}{C}{}^{\bar{e}}_{\bar{a}c} + (\dot{\partial}_c N^e_k) \overset{4}{C}{}^{\bar{d}}_{\bar{a}e};$$

$$\widetilde{P}^d_{a\bar{k}c} = \dot{\partial}_c \overset{4}{L}{}^d_{a\bar{k}} - \delta_{\bar{k}} \overset{2}{C}{}^d_{ac} + \overset{4}{L}{}^e_{a\bar{k}} \overset{2}{C}{}^d_{ec} - \overset{4}{L}{}^d_{e\bar{k}} \overset{2}{C}{}^e_{ac} + (\dot{\partial}_c N^{\bar{e}}_{\bar{k}}) \overset{4}{C}{}^d_{a\bar{e}};$$

$$\widetilde{\Theta}^i_{j\bar{b}h} = \delta_h \overset{3}{C}{}^i_{j\bar{b}} - \dot{\partial}_{\bar{b}} \overset{1}{L}{}^i_{jh} + \overset{3}{C}{}^m_{j\bar{b}} \overset{1}{L}{}^i_{mh} - \overset{3}{C}{}^i_{m\bar{b}} \overset{1}{L}{}^m_{jh} - (\dot{\partial}_{\bar{b}} N^e_h) \overset{1}{C}{}^i_{je};$$

$$\widetilde{Q}^d_{a\bar{b}h} = \delta_h \overset{4}{C}{}^d_{a\bar{b}} - \dot{\partial}_{\bar{b}} \overset{2}{L}{}^d_{ah} + \overset{4}{C}{}^e_{a\bar{b}} \overset{2}{L}{}^d_{eh} - \overset{4}{C}{}^d_{e\bar{b}} \overset{2}{L}{}^e_{ah} - (\dot{\partial}_{\bar{b}} N^e_h) \overset{2}{C}{}^d_{je};$$

$$\widetilde{\Xi}^i_{jbc} = \mathcal{A}_{cb}\{\dot{\partial}_c \overset{1}{C}{}^i_{jb} + \overset{1}{C}{}^m_{jb} \overset{1}{C}{}^i_{mc}\};$$

$$\widetilde{\Xi}^{\bar{i}}_{\bar{j}bc} = \mathcal{A}_{cb}\{\dot{\partial}_c \overset{3}{C}{}^{\bar{i}}_{\bar{j}b} + \overset{3}{C}{}^{\bar{m}}_{\bar{j}b} \overset{3}{C}{}^{\bar{i}}_{\bar{m}c}\};$$

$$\widetilde{\Xi}^i_{j\bar{b}c} = \dot{\partial}_c \overset{3}{C}{}^i_{j\bar{b}} - \dot{\partial}_{\bar{b}} \overset{1}{C}{}^i_{jc} + \overset{3}{C}{}^m_{j\bar{b}} \overset{1}{C}{}^i_{mc} - \overset{3}{C}{}^i_{m\bar{b}} \overset{1}{C}{}^m_{jc};$$

$$\widetilde{S}^d_{abc} = \mathcal{A}_{cb}\{\dot{\partial}_c \overset{2}{C}{}^d_{ab} + \overset{2}{C}{}^e_{ab} \overset{2}{C}{}^d_{ec}\};$$

$$\widetilde{S}^{\bar{d}}_{\bar{a}bc} = \mathcal{A}_{cb}\{\dot{\partial}_c \overset{4}{C}{}^{\bar{d}}_{\bar{a}b} + \overset{4}{C}{}^{\bar{e}}_{\bar{a}b} \overset{4}{C}{}^{\bar{d}}_{\bar{e}c}\};$$

$$\widetilde{S}^d_{a\bar{b}c} = \dot{\partial}_c \overset{4}{C}{}^d_{a\bar{b}} - \dot{\partial}_{\bar{b}} \overset{2}{C}{}^d_{ac} + \overset{4}{C}{}^e_{a\bar{b}} \overset{2}{C}{}^d_{ec} - \overset{4}{C}{}^d_{e\bar{b}} \overset{2}{C}{}^e_{ac},$$

where $\mathcal{A}_{hk}$ means the difference between the terms in the brackets and the terms obtained by replacing k with h.

3. Pure Hermitian metric

The geometrical objects associated with G are generally complicated. Some simplifications appear for particular choices for $g_{j\bar{k}}$ and $h_{a\bar{b}}$. We studied in a previous paper, [9], the case $\delta^a_j \delta^{\bar{b}}_{\bar{k}} h_{a\bar{b}} = \frac{1}{a(F^2)} g_{j\bar{k}}$, and G. Munteanu and N.

Aldea studied the case $\delta^a_j\delta^{\bar{b}}_{\bar{k}}h_{a\bar{b}} = g_{j\bar{k}}$, [7, 2]. Here we resume for a detailed analysis the following particular case of the Sasaki type metric:

$$G_H(z,\eta) = g_{j\bar{k}}(z)\mathrm{d}z^j \otimes \mathrm{d}\bar{z}^k + h_{a\bar{b}}(z)\delta\eta^a \otimes \delta\bar{\eta}^b. \tag{15}$$

The Chern-Finsler c.l.c. of the complex Finsler space (M,F) is reduced to $\overset{CF}{D\Gamma} = \left(N^a_j = \delta^a_i\delta^l_b g^{\bar{m}i}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^b,\ L^i_{jk} = g^{\bar{m}i}\frac{\partial g_{k\bar{m}}}{\partial z^j}\ ,0,0,0\right)$. The $v-$ and $\bar{v}-$covariant derivatives coincides with the partial derivatives with respect to η^a and $\bar{\eta}^a$, respectively. By direct calculation we prove:

Proposition 2. *The Levi-Civita connection of the metric (15) is given by the following non-zero coefficients*

$$\begin{aligned}
\overset{1}{L}{}^i_{jk} &= \frac{1}{2}g^{\bar{l}i}(\partial_k g_{j\bar{l}} + \partial_j g_{k\bar{l}});\\
\overset{2}{L}{}^c_{ak} &= \frac{1}{2}[h^{\bar{d}c}(\partial_k h_{a\bar{d}}) + \dot{\partial}_a N^c_k];\\
\overset{3}{L}{}^i_{j\bar{k}} &= \overset{1}{D}{}^i_{\bar{k}j} = \frac{1}{2}g^{\bar{l}i}(\partial_{\bar{k}} g_{j\bar{l}} - \partial_{\bar{l}} g_{j\bar{k}});\\
\overset{4}{L}{}^c_{a\bar{k}} &= \overset{2}{H}{}^c_{\bar{k}a} = \frac{1}{2}h^{\bar{d}c}[\partial_{\bar{k}} h_{a\bar{d}} - (\dot{\partial}_{\bar{d}} N^{\bar{e}}_{\bar{k}})h_{a\bar{e}}];\\
\overset{2}{F}{}^c_{jb} &= \frac{1}{2}[h^{\bar{d}c}(\partial_j h_{a\bar{d}}) - \dot{\partial}_a N^c_j];\\
\overset{1}{M}{}^i_{\bar{a}b} &= \overset{3}{M}{}^i_{b\bar{a}} = \frac{1}{2}g^{\bar{l}i}[(\dot{\partial}_{\bar{a}} N^{\bar{d}}_{\bar{l}})h_{b\bar{d}} - \partial_{\bar{l}} h_{b\bar{a}}],
\end{aligned} \tag{16}$$

where $\partial_k = \frac{\partial}{\partial z^k}$.

The curvature of the d-c.l.c. $\widetilde{D}$ from (15) is reduced to

$$\begin{aligned}
\widetilde{R}^i_{jkh} &= \mathcal{A}_{hk}\{\partial_h \overset{1}{L}{}^i_{jk} + \overset{1}{L}{}^m_{jk}\overset{1}{L}{}^i_{mh}\};\\
\widetilde{R}^{\bar{i}}_{\bar{j}kh} &= \mathcal{A}_{hk}\{\partial_h \overset{3}{L}{}^{\bar{i}}_{\bar{j}k} + \overset{3}{L}{}^{\bar{m}}_{\bar{j}k}\overset{3}{L}{}^{\bar{i}}_{\bar{m}h}\};\\
\widetilde{R}^i_{j\bar{k}h} &= \partial_h \overset{3}{L}{}^i_{j\bar{k}} - \partial_{\bar{k}}\overset{1}{L}{}^i_{jh} + \overset{3}{L}{}^m_{j\bar{k}}\overset{1}{L}{}^i_{mh} - \overset{3}{L}{}^i_{m\bar{k}}\overset{1}{L}{}^m_{jh};\\
\widetilde{\Omega}^d_{akh} &= \mathcal{A}_{hk}\{\partial_h \overset{2}{L}{}^i_{ak} + \overset{2}{L}{}^e_{ak}\overset{2}{L}{}^d_{eh}\};\\
\widetilde{\Omega}^{\bar{d}}_{\bar{a}kh} &= \mathcal{A}_{hk}\{\partial_h \overset{4}{L}{}^{\bar{d}}_{\bar{a}k} + \overset{4}{L}{}^{\bar{e}}_{\bar{a}k}\overset{4}{L}{}^{\bar{d}}_{\bar{e}h}\};\\
\widetilde{\Omega}^d_{a\bar{k}h} &= \partial_h \overset{4}{L}{}^d_{a\bar{k}} - \partial_{\bar{k}}\overset{2}{L}{}^d_{ah} + \overset{4}{L}{}^e_{a\bar{k}}\overset{2}{L}{}^d_{eh} - \overset{4}{L}{}^d_{e\bar{k}}\overset{2}{L}{}^e_{ah}.
\end{aligned} \tag{17}$$

Let $\mathbb{K}$ be the curvature tensor field of the Levi-Civita connection ∇. We shall denote its components by the same letters as N. Aldea in [2], indexed with two types of indices with the understanding that different indices means different components. From the sixty different curvature components, where will be 21 non-zero:

Proposition 3. *Let $\widetilde{D}$ be a d-c.l.c. on $T_C(T'M)$ which local coefficients are expressed in (17). With respect to the local adapted frame relative to the Chern-Finsler c.n.c., the curvature local coefficients of the Levi-Civita d-c.l.c. on $(T'M, G_H)$ are*

$$R^{i}_{jkh} = \widetilde{R}^{i}_{jkh}; \tag{18}$$

$$R^{i}_{\bar{j}kh} = \mathcal{A}_{hk}\{\overset{3}{L}{}^{i}_{k\bar{j}|h} + \overset{1}{L}{}^{m}_{kh}\overset{3}{L}{}^{i}_{m\bar{j}}\};$$

$$R^{\bar{i}}_{\bar{j}kh} = \widetilde{R}^{\bar{i}}_{\bar{j}kh};$$

$$R^{i}_{j\bar{k}h} = \widetilde{R}^{i}_{j\bar{k}h} + \overset{3}{L}{}^{m}_{j\bar{k}}\overset{1}{L}{}^{i}_{mh} - \overset{1}{L}{}^{m}_{jh}\overset{3}{L}{}^{i}_{m\bar{k}} + \overset{3}{L}{}^{\bar{m}}_{\bar{k}j}\overset{3}{L}{}^{i}_{h\bar{m}};$$

$$R^{\bar{i}}_{j\bar{k}h} = \overset{3}{L}{}^{\bar{i}}_{\bar{k}j|h} + \overset{3}{L}{}^{\bar{i}}_{m\bar{k}}\overset{1}{L}{}^{m}_{jh};$$

$$\Omega^{d}_{akh} = \widetilde{\Omega}^{d}_{akh}; \tag{19}$$

$$\Omega^{d}_{a\bar{k}h} = \widetilde{\Omega}^{d}_{a\bar{k}h};$$

$$\Pi^{d}_{jkc} = \overset{2}{F}{}^{d}_{jc|h} - \overset{2}{F}{}^{d}_{jc}\overset{2}{L}{}^{e}_{ck} + (\dot{\partial}_c N^{e}_{k})\overset{2}{F}{}^{d}_{je}; \tag{20}$$

$$\Pi^{d}_{\bar{j}kc} = -\overset{4}{L}{}^{d}_{\bar{j}c|k} + \overset{3}{L}{}^{m}_{k\bar{j}}\overset{2}{F}{}^{d}_{mc} - \overset{4}{L}{}^{e}_{c\bar{j}}\overset{2}{L}{}^{d}_{ek} + (\dot{\partial}_c N^{e}_{k})\overset{4}{L}{}^{d}_{\bar{j}e};$$

$$\Pi^{d}_{j\bar{k}c} = -\overset{2}{F}{}^{d}_{jc|\bar{k}} - \overset{2}{F}{}^{d}_{je}\overset{4}{L}{}^{e}_{c\bar{k}} + \overset{3}{L}{}^{\bar{m}}_{\bar{k}j}\overset{4}{L}{}^{d}_{c\bar{m}};$$

$$P^{i}_{\bar{a}kc} = -\overset{1}{M}{}^{i}_{\bar{a}c|k} - \overset{2}{L}{}^{e}_{ck}\overset{1}{M}{}^{i}_{\bar{a}e} - \overset{1}{M}{}^{\bar{m}}_{c\bar{a}}\overset{3}{L}{}^{i}_{k\bar{m}} + (\dot{\partial}_c N^{e}_{k})\overset{1}{M}{}^{i}_{\bar{a}e}; \tag{21}$$

$$P^{\bar{i}}_{\bar{a}kc} = -\overset{1}{M}{}^{\bar{i}}_{c\bar{a}|k} - \overset{2}{L}{}^{e}_{ck}\overset{1}{M}{}^{\bar{i}}_{e\bar{a}} + (\dot{\partial}_c N^{e}_{k})\overset{1}{M}{}^{\bar{i}}_{e\bar{a}};$$

$$\Theta^{d}_{j\bar{b}h} = -\overset{4}{L}{}^{\bar{e}}_{\bar{b}j}\overset{2}{E}{}^{d}_{\bar{e}h} - (\dot{\partial}_{\bar{b}} N^{e}_{h})\overset{2}{F}{}^{d}_{je} \tag{22}$$

$$Q^{i}_{a\bar{b}h} = \overset{1}{M}{}^{i}_{\bar{b}a|h} + \overset{4}{L}{}^{\bar{e}}_{\bar{b}h}\overset{1}{M}{}^{i}_{\bar{e}a} + \overset{1}{M}{}^{\bar{m}}_{a\bar{b}}\overset{3}{L}{}^{i}_{h\bar{m}}; \tag{23}$$

$$Q^{\bar{i}}_{a\bar{b}h} = \overset{1}{M}{}^{\bar{i}}_{a\bar{b}|h} + \overset{4}{L}{}^{\bar{e}}_{\bar{b}h}\overset{1}{M}{}^{\bar{i}}_{a\bar{e}};$$

$$\Xi^{i}_{\bar{j}bc} = \mathcal{A}_{cb}\{\overset{4}{L}{}^{\bar{e}}_{b\bar{j}}\overset{1}{M}{}^{i}_{\bar{e}c}\}; \tag{24}$$

$$\Xi^{i}_{j\bar{b}c} = \overset{4}{L}{}^{\bar{e}}_{b\bar{j}}\overset{1}{M}{}^{i}_{\bar{e}c} - \overset{2}{F}{}^{e}_{jc}\overset{1}{M}{}^{i}_{\bar{b}e};$$

$$\Xi^{d}_{j\bar{b}c} = \overset{4}{L}{}^{\bar{e}}_{\bar{b}j}\overset{1}{M}{}^{d}_{c\bar{e}};$$

$$S^{d}_{\bar{a}bc} = \mathcal{A}_{cb}\{\overset{1}{M}{}^{m}_{\bar{a}b}\overset{2}{F}{}^{d}_{mc} + \overset{1}{M}{}^{\bar{m}}_{b\bar{a}}\overset{4}{L}{}^{d}_{c\bar{m}}\}; \tag{25}$$

$$S^d_{a\bar{b}c} = \overset{1}{M}{}^m_{\bar{b}a}\overset{2}{F}{}^d_{mc} + \overset{1}{M}{}^{\bar{m}}_{a\bar{b}}\overset{4}{L}{}^d_{c\bar{m}};$$

and the rest are zero.

The Ricci curvature tensors are $\overset{H}{R}_{jk} := R^i_{jki}$; $\overset{H}{R}_{\bar{j}k} := R^i_{\bar{j}ki}$; $\overset{\bar{H}}{R}_{jk} := R^{\bar{i}}_{j\bar{i}k}$; $\overset{V}{\Pi}_{\bar{j}k} := \Pi^d_{\bar{j}kd}$; $\overset{H}{P}_{\bar{a}b} := P^i_{\bar{a}ib}$; $\overset{V}{S}_{\bar{a}b} := S^d_{\bar{a}bd}$. From which we obtain the following Ricci scalars $r := g^{\bar{j}k}\overset{H}{R}_{\bar{j}k}$; $\pi := g^{\bar{j}k}\overset{V}{\Pi}_{\bar{j}k}$; $p := h^{\bar{a}b}\overset{H}{P}_{\bar{a}b}$; $s := h^{\bar{a}b}\overset{V}{S}_{\bar{a}b}$.

Using some idea from the real case ([6]), a generalization of the classical Einstein equations for an $n-$dimensional complex Finsler space is

$$\mathbf{R}_{\bar{\alpha}\beta} - \frac{1}{2}\rho \cdot \mathbf{G}_{\beta\bar{\alpha}} = \chi \mathbf{T}_{\bar{\alpha}\beta}, \tag{26}$$

where to standardize the notation we use the Greek letters $\alpha, \beta = 1, \dots, n$ for the two types of indices; $\mathbf{R}_{\bar{\alpha}\beta}$ denotes the components of the Ricci tensors; ρ denotes the the Ricci scalar curvature; $\mathbf{G}_{\beta\bar{\alpha}}$ represents the metric tensors $g_{j\bar{k}}$ and $h_{a\bar{b}}$, respectively; χ is the universal constant; $\mathbf{T}_{\bar{\alpha}\beta}$ are the energy-momentum tensors ([3]). Since the Levi-Civita connection ∇ is without torsion, the conservation laws to the Einstein equations (26) are satisfied, i.e. $\nabla_\alpha(\mathbf{R}^\alpha_\beta - \frac{1}{2}\rho\delta^\alpha_\beta) = 0$, or equivalently $\nabla_\alpha \mathbf{T}^\alpha_\beta = 0$.

By this, as in classical theory, in vacuum we have

Proposition 4. *In vacuum the Ricci tensors of the Levi-Civita connection on $T'M$ are vanishing.*

4. Solutions for the Complex Einstein Equation in Vacuum for a weakly gravitational metric

In this section our goal is to solve the complex Einstein equations for the particular case of a 2−dimensional complex Finsler space in vacuum, when the fundamental metric tensor has a form of a weakly gravitational metric [3]:

$$g_{j\bar{k}}(z, \eta) = \eta_{j\bar{k}} + p_{j\bar{k}}, \tag{27}$$

where $(\eta_{j\bar{k}}) := \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$, is the Minkowski metric and

$$(p_{j\bar{k}}) := \begin{pmatrix} \frac{2\Phi}{c^2} & i\frac{2\Phi}{c^2} \\ -i\frac{2\Phi}{c^2} & \frac{2\Phi}{c^2} \end{pmatrix}$$

is a small perturbation of $\eta_{j\bar{k}}$, and Φ has the meaning of a gravitational potential. Here Φ is a real valued smooth function in $T'M$, $\Phi \neq \frac{c^2}{2}$, and $c \in \mathbb{R}$, $c \neq 0$.

In [3], Proposition 4.1 is proved, that the metric (27) is Finsler if the following statements are satisfied:

i) $\Phi > \frac{c^2}{2}$;
ii) Φ is homogeneous function respect to η;
iii) $i\Phi_{\cdot 2} = \Phi_{\cdot 1}$, where $\Phi_{\cdot a} = \frac{\partial \Phi}{\partial \eta^a}$, $a = 1, 2$,

and it can be used in the study of the weakly gravitational fields in the complex Finsler space (M, F), with

$$(28)\quad L = \left(1 + \frac{2\Phi}{c^2}\right)|\eta^1|^2 - i\left(1 - \frac{2\Phi}{c^2}\right)\eta^1\bar{\eta}^2 + i\left(1 - \frac{2\Phi}{c^2}\right)\eta^2\bar{\eta}^1 - \left(1 - \frac{2\Phi}{c^2}\right)|\eta^2|^2.$$

Remark 1. In the following, we will use the letters $j, k, l, \ldots = 1, \ldots, n$ for both the horizontal and vertical indices.

Using some elementary calculus, it is obtained the local expressions of the Chern-Finsler c.l.c.:

$$(29)\quad \begin{aligned} N_k^1 &= 0; \quad N_k^2 = \frac{-2i}{c^2\left(1 - \frac{2\Phi}{c^2}\right)}(\eta^1 - i\eta^2)\Phi_k; \\ L_{jk}^1 &= 0; \quad L_{1k}^2 = -\frac{2i}{c^2\left(1 - \frac{2\Phi}{c^2}\right)}\Phi_k = iL_{2k}^2; \\ C_{jk}^1 &= 0; \quad C_{jk}^2 = 0, \text{ for } j, k = 1, 2, \end{aligned}$$

where $\Phi_k := \frac{\partial\Phi}{\partial z^k}$, $k = 1, 2$.

The above requirements *ii*) and *iii*) imply $\Phi_{\cdot 1}(\eta^1 - i\eta^2) = i\Phi_{\cdot 2}(\eta^1 - i\eta^2) = 0$, and their conjugates, which implies the following

Theorem 2 ([4, Theorem 3.1]). *Let (M, F) be a complex Finsler space, where $L = F^2$ is the metric (28). Then (M, F) is either a purely Hermitian space or a locally Minkowski space with $\eta^1 = i\eta^2$.*

In the following we assume that (M, F) is a purely Hermitian space, i.e. $g_{j\bar{k}}(z, \eta) = g_{j\bar{k}}(z)$, which implies that form (27) the function $\Phi = \Phi(z)$. We consider the corresponding Hermitian metric G_H given by (15) customized for the weakly gravitational metric (27)

$$(30)\quad G_{wg}(z, \eta) = \eta_{j\bar{k}} dz^j \otimes d\bar{z}^k + g_{j\bar{k}}\delta\eta^j \otimes \delta\bar{\eta}^k,$$

where $(\eta_{j\bar{k}}) := \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$, with the inverse matrix $(\eta^{\bar{k}j}) := \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & -\frac{1}{2} \end{pmatrix}$, and

$$(31)\quad (g_{j\bar{k}}(z, \eta))_{j\bar{k}=1,2} = \begin{pmatrix} 1 + \frac{2\Phi}{c^2} & -i\left(1 - \frac{2\Phi}{c^2}\right) \\ i\left(1 - \frac{2\Phi}{c^2}\right) & -\left(1 - \frac{2\Phi}{c^2}\right) \end{pmatrix}, \text{where } i := \sqrt{-1};$$

with its inverse

$$(32)\quad g^{\bar{k}j}(z, \eta))_{j\bar{k}=1,2} = \begin{pmatrix} \frac{1}{2} & \frac{-i}{2} \\ \frac{i}{2} & -\frac{1 + \frac{2\Phi}{c^2}}{2\left(1 - \frac{2\Phi}{c^2}\right)} \end{pmatrix}.$$

In order to solve the Einstein equations for this particular case of Finsler space, we can express the coefficients of the Levi-Civita connection corresponding to the metric (30). The non-vanishing local expressions of the above mentioned connection are:

$$
\begin{aligned}
\overset{2}{L}{}^{2}_{jk} &= \frac{-i^k}{c^2\left(1-\frac{2\Phi}{c^2}\right)}(\Phi+(-1)^k\Phi_k); \\
\overset{2}{F}{}^{2}_{jk} &= \frac{-i^k}{c^2\left(1-\frac{2\Phi}{c^2}\right)}(\Phi+\Phi_k); \\
\overset{1}{M}{}^{1}_{\bar{j}2} &= \frac{-i^j}{c^2}(\Phi_{\bar{1}}+i\Phi_{\bar{2}}); \\
\overset{1}{M}{}^{2}_{\bar{j}2} &= \frac{i^j}{c^2}(i\Phi_{\bar{1}}+\Phi_{\bar{2}}), \quad j,k=1,2.
\end{aligned}
\tag{33}
$$

The non-zero local expression of the Ricci tensors with respect to the Levi-Civita connection (33) are as follows:

$$
\begin{aligned}
\overset{H}{P}_{\bar{j}1} &= \frac{2i}{c^4\left(1-\frac{2\Phi}{c^2}\right)} i^j\left(\Phi_1\Phi_{\bar{1}}+i\Phi_1\Phi_{\bar{2}}-i\Phi_2\Phi_{\bar{1}}-\Phi_2\Phi_{\bar{2}}\right); \\
\overset{H}{P}_{\bar{j}2} &= \frac{i^j}{c^2}(\Phi_{1\bar{1}}+i\Phi_{1\bar{2}}-i\Phi_{2\bar{1}}-\Phi_{2\bar{2}})+ \\
&\quad+\frac{2i}{c^4\left(1-\frac{2\Phi}{c^2}\right)}(1-i^{2-j})\left(\Phi_1\Phi_{\bar{1}}+i\Phi_1\Phi_{\bar{2}}-i\Phi_2\Phi_{\bar{1}}-\Phi_2\Phi_{\bar{2}}\right); \\
\overset{V}{S}_{\bar{j}2} &= \frac{i^j}{c^4\left(1-\frac{2\Phi}{c^2}\right)}(1-i)(\Phi+\Phi_1)\left(\Phi_{\bar{1}}-\Phi_{\bar{2}}\right), \quad j=1,2.
\end{aligned}
\tag{34}
$$

Now we are able to write the complex Einstein equations of this space.

Theorem 3. *The complex Einstein equations in vacuum corresponding to a two-dimensional complex Finlser space (M,F), with the complex Finsler metric (28) and with the Levi-Civita connection (33) are*

$$
\begin{aligned}
&-\frac{2}{c^4\left(1-\frac{2\Phi}{c^2}\right)}\left(\Phi_1\Phi_{\bar{1}}+i\Phi_1\Phi_{\bar{2}}-i\Phi_2\Phi_{\bar{1}}-\Phi_2\Phi_{\bar{2}}\right)+\left(1-\frac{2\Phi}{c^2}\right)\rho=0; \\
&\frac{2i}{c^4\left(1-\frac{2\Phi}{c^2}\right)}\left(\Phi_1\Phi_{\bar{1}}+i\Phi_1\Phi_{\bar{2}}-i\Phi_2\Phi_{\bar{1}}-\Phi_2\Phi_{\bar{2}}\right)+i\left(1-\frac{2\Phi}{c^2}\right)\rho=0; \\
&\frac{i^{2-j}}{c^2}(\Phi_{1\bar{1}}+i\Phi_{1\bar{2}}-i\Phi_{2\bar{1}}-\Phi_{2\bar{2}})+\frac{2i}{c^4\left(1-\frac{2\Phi}{c^2}\right)}(1-i^{2-j})\left(\Phi_1\Phi_{\bar{1}}+\right. \\
&\qquad\qquad \left.+i\Phi_1\Phi_{\bar{2}}-i\Phi_2\Phi_{\bar{1}}-\Phi_2\Phi_{\bar{2}}\right)+(-i)^j\left(1-\frac{2\Phi}{c^2}\right)\rho=0; \\
&\frac{i^{2-j}}{c^4\left(1-\frac{2\Phi}{c^2}\right)}(1-i)(\Phi+\Phi_1)\left(\Phi_{\bar{1}}-\Phi_{\bar{2}}\right)+(-i)^j\left(1-\frac{2\Phi}{c^2}\right)\rho=0,
\end{aligned}
\tag{35}
$$

$(j=1,2)$, and their conjugates, where ρ represents the scalar curvature.

Instead to solve the system of PDE's (35) we can use the Proposition 3.3 to simplify this. According to the mentioned Proposition, the Ricci tensors in vacuum are zero, which implies that the Ricci scalars are vanishing too in the same conditions. Then the system (35) is equivalent with

$$\begin{aligned} \Phi_1\Phi_{\bar{1}} + i\Phi_1\Phi_{\bar{2}} - i\Phi_2\Phi_{\bar{1}} - \Phi_2\Phi_{\bar{2}} &= 0; \\ \Phi_{1\bar{1}} + i\Phi_{1\bar{2}} - i\Phi_{2\bar{1}} - \Phi_{2\bar{2}} &= 0; \\ (\Phi + \Phi_1)(\Phi_{\bar{1}} - \Phi_{\bar{2}}) &= 0. \end{aligned} \tag{36}$$

Proposition 5. *The real valued smooth function on $T'M$, $\Phi(z) = \Phi(z^1, \bar{z}^1, z^2, \bar{z}^2)$, is a solution for the system of equations (36), when it satisfies the following conditions:*
a) $\Phi_1 = \Phi_2$;
b) $\Phi_{\bar{1}} = \Phi_{\bar{2}}$.

Example 1. We consider the function $\Phi(z) = \frac{c^2}{2}e^{i(z^1-\bar{z}^1)+i(z^2-\bar{z}^2)}$ on $\mathbb{C}^2$. Requiring $\Phi > \frac{c^2}{2}$, by (28), it induce on $D := \{z \in \mathbb{C}^2 | \ Im(z^1 + z^2) > 0\}$ the purely Hermitian complex Finsler metric

$$L = \left(1 + e^Z\right)|\eta^1|^2 - i\left(1 - e^Z\right)\eta^1\bar{\eta}^2 + i\left(1 - e^Z\right)\eta^2\bar{\eta}^1 - \left(1 - e^Z\right)|\eta^2|^2, \tag{37}$$

where $Z = i(z^1 - \bar{z}^1) + i(z^2 - \bar{z}^2)$. Since Φ satisfies the conditions a) and b) from Proposition 4.1, the Sasaki type metric defined by (30) is a solution for the complex Einstein equations in vacuum.

We notice that besides the solutions provided by Proposition 4.1, there exists also different type of solutions:

Example 2. Let $\Phi(z) = \frac{c^2}{2}e^{-(z^1+\bar{z}^1)+z^2+\bar{z}^2}$ a real valued function on $T'M$. Requiring $\Phi > \frac{c^2}{2}$, by (28), the purely Hermitian complex Finsler metric induced on $D := \{z \in \mathbb{C}^2 | \ Re(z^2 - z^1) > 0\}$ is

$$L = \left(1 + e^Z\right)|\eta^1|^2 - i\left(1 - e^Z\right)\eta^1\bar{\eta}^2 + i\left(1 - e^Z\right)\eta^2\bar{\eta}^1 - \left(1 - e^Z\right)|\eta^2|^2, \tag{38}$$

where $Z = -(z^1 + \bar{z}^1) + z^2 + \bar{z}^2$. Because Φ is a solution for the system of equations (36), a solution for the complex Einstein equations in vacuum is the Sasaki type metric defined by (30).

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