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I-PRIME SUBMODULES

ISMAEL AKRAY AND HALGURD S. HUSSEIN

Abstract. We introduce a new generalization of prime submodules called I-prime submodule for I a fixed ideal of a commutative ring R. We study some of its properties and show that the intersection of I-prime submodules is again I-prime. Finally, we proved that if F is a flat module and P an I-prime submodule of a module M then $F \otimes P$ is I-prime submodule of $F \otimes M$.

1. Introduction

Throughout this paper R will be a commutative ring with nonzero identity and I a fixed ideal of R and M a unitary left R-module. Prime ideals play a central role in commutative ring theory. We recall that a prime ideal P of R is a proper ideal with the property that for $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$; or equivalently, for ideals A and B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. The concept of weakly prime ideals was introduced by Anderson and Smith (2003), where a proper ideal P is called weakly prime if, for $a, b \in R$ with $0 \neq ab \in P$, either $a \in P$ or $b \in P$, [7]. Bhatwadekar and Sharma [11] defined the notion of almost prime ideal, i.e., a proper ideal I with the property that if $a, b \in R$, $ab \in I - I^2$, then either $a \in I$ or $b \in I$. Thus a weakly prime ideal is almost prime and any proper idempotent ideal is also almost prime. Moreover, an ideal I of R is almost prime if and only if I/I^2 is a weakly prime ideal of R/I^2. We could restrict where a and/or b lies. A proper ideal Q of R is said to be primary provided that for $a, b \in R, ab \in Q$ implies that either $a \in Q$ or $b \in \sqrt{Q}$. We can generalize the concept of primary ideals by restricting the set where ab lies. A proper ideal Q of R is weakly primary if for $a, b \in R$ with $0 \neq ab \in Q$, either $a \in Q$ or $b \in \sqrt{Q}$. Weakly primary ideals were first introduced and studied by Ebrahimi Atani and Farzalipour in 2005, [12].

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An R-module M is called a multiplication module if every submodule N of M has the form IM for some ideal I of R, see [3]. Note that, since $I \subseteq (N :_R M)$ then $N = IM \subseteq (N :_R M)M \subseteq N$. So that $N = (N :_R M)M$. Let N and K be two submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product of N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [4, Theorem 3.4], the product of N and K is independent of presentations of N and K. An R-module M is called faithful if it has zero annihilator. An R-module M is called a cancellation module of R if, for all ideals I and J of R, $IM = JM$ implies that $I = J$, see [6, 5]. For example, every invertible ideal, free module and finitely generated faithful multiplication module over a ring R is cancellation module of R. It is clear that if N is a submodule of a nitely generated faithful multiplication (and so cancellation) R-module M, then we have $(IN : M) = I(N : M)$ for every ideal I of R.

The class of prime submodules of modules was introduced and studied in 1992 as a generalization of the class of prime ideals of rings. Then, many generalizations of prime submodules were studied such as primary, classical prime, weakly prime and classical primary submodules, see [8, 9, 10, 17] and [3]. A proper ideal P of R is called ϕprime ideal if for all $a, b \in P - \phi(P)$ implies either $a \in P$ or $b \in P$, where $\phi : \tau(R) \longrightarrow \tau(R) \cup \{\phi\}$ is a function defined on the set of ideals $\tau(R)$ of R (see [13] and [19]). Let M be a module and $\tau(M)$ be the set of all submodules of M and let $\phi : \tau(M) \longrightarrow \tau(M) \cup \{\phi\}$ be a function. A proper submodule P is called ϕprime if for all $r \in R, m \in M$ such that $rm \in P\phi(P)$ implies $r \in (P : M)$ or $m \in P$ (see [16] and [20]). In [1], the notion of I-prime ideal was introduced which can be considered as a special case of ϕprime ideals by defining $\phi(P) = IP$.

In this article, we generalize I-prime ideals to submodules and we study several properties of such generalization. We give some characterizations of I-prime submodules. Finally we show that if F is an R-module and P an I-prime submodule of an R-module M, then under a particular condition, $P \otimes F$ will be an I-prime submodule of $M \otimes F$.

2. Main results

A proper submodule P of an R-module M is called I-prime submodule of M if $rm \in P - IP$ for all $r \in R$ and $m \in M$ implies that either $m \in P$ or $r \in (P : M)$. It is clear that every prime and weakly prime submodule is I-prime but the converse is not true in general as we see in the following example.

Example 2.1. Consider the ring of integers Z and the Z-module Z_{12}. Take $I = 4Z$ as an ideal of Z and $P = (4)$ be a submodule of Z_{12} generated by 4. Then P is an I-prime submodule of Z_{12} since $P - IP = (4) - 4Z.(4) = (4) - (4) = \phi$. In other side, P is not prime even not weakly prime submodule since $4 = 2.2 \in P$ but not $2 \in P$ nor $2.Z_{12} \subseteq P$.

Note that the similar statements of our results from Theorem 2.2 to Corollary 2.5 are present for $\phi-$prime submodules in [20] and [16] but here new proofs are provided for $I-$prime submodules. We begin with the following evident useful theorem.

Theorem 2.2. *Let P be an I-Prime. Then P is prime if $(P : M)P \nsubseteq IP$.*

Proof. Let $rm \in P$ for $r \in R$ and $m \in M$. If $rm \notin IP$, then P is prime submodule of M. If $rm \in IP$, then we can assume that $rP \subseteq IP$, because for otherwise there exists $x \in P$ such that $rx \notin IP$ so $r(m + x) \notin IP$. As P is I-prime, $r(m + x) \in P - IP$ implies that $r \in (P : M)$ or $m + x \in P$, that is $r \in (P : M)$ or $m \in P$. If $(P : M)m \nsubseteq IP$, then there exists $a \in (P : M)$ such that $am \notin IP$, so $(a + r)m \notin IP$. Thus $(a + r)m \in P - IP$ which imply that $a + r \in (P : M)$ or $m \in P$, that is $r \in (P : M)$ or $m \in P$. Hence we may take $(P : M)m \subseteq IP$. Since given $(P : M)P \nsubseteq IP$, there exists $a \in (P : M)$ and $x \in P$ such that $ax \notin IP$. Therefore $(r + a)(m + x) \in P - IP$ and this implies that $r + a \in (P : M)$ or $m + x \in P$, that is $r \in (P : M)$ or $m \in P$. □

Corollary 2.3. *Let P be an 0-prime submodule of M such that $(P : M)P \neq 0$. Then P is a prime submodule of M.*

Proof. Take $I = 0$ in the Theorem 2.2. □

Corollary 2.4. *Let P be I-prime submodule of M and $IP \subseteq (P : M)^2P$. Then P is J-prime where $J = \cap_{k=1}^{\infty}(P :_R M)^k$.*

Proof. In the case P is prime submodule, then there is nothing to prove. Now, in the case P is not prime submodule, by Theorem 2.2 we have $(P : M)P \subseteq IP$ but given $IP \subseteq (P : M)^2P$, so $IP = (P : M)^2P$ and inductionally, we have $IP = (P : M)^kP$ for all positive integer k. Hence $IP = \cap_{k=1}^{\infty}(P : M)^kP = JP$ and therefore P is J-prime. □

Corollary 2.5. *Let M be a multiplication R-module and P an I-prime submodule of M. If P is not prime, then $P^2 \subseteq IP$.*

Proof. Since M is multiplication R-module, $P = (P : M)M$. By Theorem 2.2 and being P non prime submodule we include that $(P : M)P \subseteq IP$. Therefore $P^2 = (P : M)^2M = (P : M)(P : M)M = (P : M)P \subseteq IP$. □

Recall that if N is a proper submodule of a nonzero R-module M. Then the M-radical of N, denoted by $M - rad(N)$, is defined to be the intersection of all prime submodules of M containing N. If M has no prime submodule containing N, then we say $M - rad(N) = M$. It is shown in [15, Theorem 2.12] that if N is a proper submodule of a multiplication R-module M, then $M - rad(N) = \sqrt{(N :_R M)}M$.

Corollary 2.6. *Let M be a multiplication R-module and P an I-prime submodule of M. Then $P \subseteq \sqrt{IP}$ or $\sqrt{IP} \subseteq P$.*

Proof. If P is prime submodule, then $\sqrt{IP} \subseteq \sqrt{P} = P$. Now if P is not prime submodule, then by Corollary 2.5 $P^2 \subseteq IP$, so $P \subseteq \sqrt{IP}$. □

The following two famous theorems are crucial in our investigation because they give several charactrizations of I-prime submodules.

Theorem 2.7. *Let M be R-module and P be a proper submodule of M. Then the following are equivalent.*
(1) P is I-prime submodule of M.
(2) For $r \in R - (P : M), (P : r) = P \cup (IP : r)$.
(3) For $r \in R - (P : M), (P : r) = P$ or $(P : r) = (IP : r)$.

Proof. (1) ⇒ (2) Let P be an I-prime. Take $r \in R-(P : M)$ and $m \in (P :_M r)$. So $rm \in P$. If $rm \notin IP$, then P I-prime gives $m \in P$. If $rm \in IP$, then $m \in (IP : r)$.

(2) ⇒ (3) If a submodule is a union of two submodules, it is equal to one of them.

(3) ⇒ (1) Let $rm \in P - IP$ for $r \in R$ and $m \in M$. If $r \notin (P : M)$, then by hypothesis $(P : r) = P$ or $(P : r) = (IP : r)$. Since $rm \notin IP$, $m \notin (IP : r)$. But $m \in (P : r)$ which means that $(P : r) \neq (IP : r)$. Hence $(P : r) = P$ and so $m \in P$. Therefore P is I-prime submodule of M. □

Theorem 2.8. *Let P be a proper submodule of an R-module M. Then P is I-prime submodule in M if and only if P/IP is weakly prime in M/IP.*

Proof. (⇒) Let P be I-prime in M. Let $r \in R$ and $m \in M$ with $0 \neq r(m + IP) \in P/IP$ in M/IP. Then $rm \in P - IP$ implies $r \in (P : M)$ or $m \in P$, hence $r \in (P : M) = (P/IP : M/IP)$ or $m + IP \in P/IP$. So P/IP is weakly prime submodule in M/IP.

(⇐) Suppose that P/IP is weakly prime in M/IP and take $r \in R, m \in M$ such that $rm \in P - IP$. Then $0 \neq rm + IP = r(m + IP) \in P/IP$ so $m + IP \in P/IP$ or $r \in (P/IP : M/IP) = (P : M)$. Therefore $m \in P$ or $r \in (P : M)$. Thus P is I-prime. □

Lemma 2.9. *Let M be multiplication R-module, P an I-prime of M and $(P : M) \subseteq I$. Then $\sqrt{(IP : M)}P = IP$.*

Proof. Let $r \in \sqrt{(IP : M)}$. If $r \in I$, then $rP \subseteq IP$. For $r \notin I$, if $r \notin (P : M)$, then $(P : r) = P$ or $(P : r) = (IP : r)$ by Theorem 2.7. If $(P : r) = (IP : r)$, then $rP \subseteq r(P : r) \subseteq r(IP : r) \subseteq IP$. For the case $(P : r) = P$, let n be the smallest positive integer such that $r^n \in (IP : M)$.

Then as clearly as $(P : r) = P$, $r(r^k)M \subseteq P$ implies $r^k M \subseteq P$, hence as clearly $n \geq 2$ and $IP \subseteq P$, we conclude $rM \subseteq P$ contradicting $r \notin (P : M)$. The case $r \notin I, r \notin (P : M)$ is impossible as by assumtion, $(P : M) \subseteq I$. Hence $\sqrt{(IP : M)}P \subseteq IP$. For the reverse inclusion, since $IP = (IP : M)M \subseteq \sqrt{(IP : M)}M$, the result follows □

The next Theorem is an I-prime version of [3, Proposition 13]. First, we need the following lemma from [2].

Lemma 2.10. *Let P be a submodule of a faithful multiplication R-module M and J a finitely generated faithful multiplication ideal of R. Then,*

(1) $P = (JP : J)$.

(2) If $P \subseteq JM$, then $(KP : J) = K(P : J)$ for any ideal K of R.

Theorem 2.11. *Let P be a submodule of a faithful multiplication R-module M and J a finitely generated faithful multiplication ideal of R. Then P is I-prime submodule of JM if and only if $(P : J)$ is I-prime in M.*

Proof. Suppose that P is I-prime in JM. Let $r \in R$ and $m \in M$ such that $rm \in (P : J) - I(P : J)$. Then $rJm \subseteq P - IP$ because, if $rJm \subseteq IP$ then by Lemma 2.10 $rm \in (IP : J) = I(P : J)$ which is a contradiction. If $r \notin (P : JM)$ we may apply Theorem 2.7 (3) and weinfer $(P :_{JM} r) = P$, $m \in P$. Now, suppose $r \in (P : JM)$, so that $rJM \subseteq P$ and then again by Lemma 2.10 $rM = r(JM : J) \subseteq (rJM : J) \subseteq (P : M)$ and so $r \in ((P : J) : M)$. Therefore $(P : J)$ is $I-$prime in M. Conversely, suppose that $(P : J)$ is I-prime in M. Let K be an ideal of R and N a submodule of JM such that $KN \subseteq P - IP$. Then taking Lemma 2.10 in mind we have $K(N : J) \subseteq (KN : J) \subseteq (P : J)$. Moreover, if $K(N : J) \subseteq I(P : J)$, then $KN = K(JN : J) = JK(N : J) \subseteq IJ(P : J) = IP$ a contradiction. Hence $K(N : J) \subseteq (P : J) - I(P : J)$. By [20, Theorem 2.11] $(P : J)$ I-prime in M implies either $K \subseteq ((P : J) : M) = (P : JM)$ or $(N : J) \subseteq (P : J)$, which implies that $N = J(N : J) \subseteq J(P : J) = P$. Hence P is I-prime submodule in JM. □

Now we give other charactrizations of I-prime submodules which connect between the I-primeness of a submodule P of an R-module M and the ideal $(P : M)$ of R.

Theorem 2.12. *Let M be a finitely generated faithful multiplication module and P be a proper subset of M. Then the following are equivalent:*

(i) P is I-prime submodule in M.

(ii) $(P : M)$ is I-prime ideal in R.

(iii) $P = JM$ for some I-prime ideal J of R.

Proof. We may apply Theorem 2.7 and Lemma 2.10, hence we have P is I-prime in M if and only if for any $r \in R - (P : M)$,

(*) $$(P : r) = P \text{ or } (P : r) = (IP : r) \ldots$$

$(P : M)$ is I-prime in R if and only if for any $r \in R - (P : M)$,

(**) $$((P : M) : r) = (P : M) \text{ or } ((P : M) : r) = ((IP : M) : r) \ldots$$

(i) ⇒ (ii) Let $a, b \in R$ with $ab \in (P : M) - I(P : M)$. If $abM \subseteq IP$, then $ab \in (IP : M) = I(P : M)$ which is a contradiction. So $abM \nsubseteq IP$. Assuming

$a \notin (P : M)$ by condition (*) we infer $(P : M) = P$. Thus $a \in (P : M)$ or $bM \subseteq P$, that is $a \in (P : M)$ or $b \in (P : M)$. Hence $(P : M)$ is I-prime ideal in R.

(ii) $\Rightarrow$ (i) Let $rm \in P - IP$. Assuming $r \notin (P : M)$. By condition (**) we infer $((P : M) : r) = (P : M)$ and $(Rm : M) \subseteq (P : M)$. Apply [4, Theorem 3.2] and the result obtained.

(ii) $\Rightarrow$ (iii) Take $J = (P : M)$ and as M is multiplication, then $P = (P : M)M = JM$.

(iii) $\Rightarrow$ (ii) Let $P = JM$, for some I-prime ideal J of R. Then as M is multiplication module, we have $P = (P : M)M$. Hence $(P : M)M = JM$ and as M is cancelation module, $(P : M) = J$ and so $(P : M)$ is I-prime ideal in R. □

Applying [4, Theorem 3.2] we see that in this particular case the ideal lattice of R and the submodule lattice of M are isomorphic, this way we may prove the analogue of the charactrization of [20, Theorem 2.11 (iv)].

Theorem 2.13. *Let M be a finitely generated multiplication R-module and P a proper submodule of M such that $I(P : M) = (IP : M)$. Then P is I-prime submodule in M if and only if for any two submodules A and B of M with $A.B \subseteq P$ and $A.B \nsubseteq IP$ implies either $A \subseteq P$ or $B \subseteq P$.*

Proof. Let P be an I-prime submodule of M and A, B be any two submodules of M with $A.B \subseteq P$, $A.B \nsubseteq IP$ with $A \nsubseteq P$ and $B \nsubseteq P$. As M is multiplication R-module, $A = (A : M)M$ and $B = (B : M)M$ and so $A.B = (A : M)(B : M)M$. Thus $(A : M) \nsubseteq (P : M)$ and $(B : M) \nsubseteq (P : M)$. By Theorem 2.12 $(P : M)$ is I-prime ideal in R and by [1, Theorem 2.12] we have either $(A : M)(B : M) \nsubseteq (P : M)$ or $(A : M)(B : M) \subseteq I(P : M)$. In the first case, we have $AB = (A : M)(B : M)M \nsubseteq (P : M)M = P$ and in the second case, we have $AB = (A : M)(B : M)M \subseteq I(P : M)M = IP$ and both contradict our hypothesis. Hence either $A \subseteq P$ or $B \subseteq P$. For the converse, it is enough by Theoprem 2.12 to prove that $(P : M)$ is I-prime ideal in R. Let $a, b \in R$ such that $ab \in (P : M) - I(P : M)$ with $a \notin (P : M)$ and $b \notin (P : M)$. Take $A = aM, B = bM$. Then $AB = abM \subseteq (P : M)M = P$. If $AB = abM \subseteq IP$ then $ab \in (IP : M) = I(P : M)$ which is a contradiction. Hence $AB \subseteq P - IP$ and by the hypothesis we have either $A = aM \subseteq P$ or $B = bM \subseteq P$ which means that $a \in (P : M)$ or $b \in (P : M)$. Therefore $(P : M)$ is I-prime ideal of R. □

Suppose M is a multiplication module and $x, y \in M$. Then we can define the product of x and y as $xy = Rx.Ry = (Rx : M)(Ry : M)M$. Thus we have the following corollary.

Corollary 2.14. *Let P be a proper submodule of finitely generated multiplication R-module such that $I(P : M) = (IP : M)$. Then P is I-prime submodule of M if and only if whenever $x, y \in M$ with $xy \in P - IP$ implies $x \in P$ or $y \in P$*

Let M and F be R-modules and $r \in R$. Then it is clear that for any submodule P of M, $F \otimes (P : r) \subseteq (F \otimes P : r)$. In the following lemma we give a condition under which the equality holds.

Lemma 2.15. *Let $r \in R$ and P a submodule of M. Then for any flat R-module F, we have $F \otimes (P : r) = (F \otimes P : r)$.*

Proof. Consider the exact sequence $0 \longrightarrow (P : r) \longrightarrow M \xrightarrow{f_r} \frac{M}{P}$ where $f_r(m) = rm + P$. As F is flat , the exactness of the sequence $0 \longrightarrow P \longrightarrow M \longrightarrow \frac{M}{P} \longrightarrow 0$ implies to the exactness of the sequence $0 \longrightarrow F \otimes P \longrightarrow F \otimes M \longrightarrow F \otimes \frac{M}{P} \longrightarrow 0$ which gives the isomorphism, $F \otimes \frac{M}{P} \cong \frac{F \otimes M}{F \otimes P}$. So the exactness of the sequence $0 \longrightarrow (P : r) \longrightarrow M \longrightarrow \frac{M}{P}$ imply the exactness of the sequence $0 \longrightarrow F \otimes (P : r) \longrightarrow F \otimes M \xrightarrow{1 \otimes \acute{f}_r} \frac{F \otimes M}{F \otimes P}$ where $(1 \otimes \acute{f}_r)(n \otimes m) = r.(n \otimes m) + F \otimes P$ for $n \in F$. Therefore $F \otimes (P : r) = ker(1 \otimes \acute{f}_r) = (F \otimes P :_{F \otimes M} r)$. □

The next two assersions are closely related to Theorem 2.18 in [18].

Theorem 2.16. *Let P be I-prime submodule of an R-module M and F a flat R-module with $F \otimes P \neq F \otimes M$. Then $F \otimes P$ is I-prime submodule of $F \otimes M$.*

Proof. Suppose that P is I-prime and $r \in R - (P : M)$. Then by Theorem 2.7 $(P : r) = P$ or $(P : r) = (IP : r)$. Now Lemma 2.15 gives us $(F \otimes P : r) = F \otimes (P : r) = F \otimes P$ or $(F \otimes P : r) = F \otimes (P : r) = F \otimes (IP : r) = (F \otimes IP : r) = (I(F \otimes P) : r)$ and consequently $F \otimes P$ is I-prime submodule of $F \otimes M$. □

An R-module F is called faithfully flat if for any two R-modules A and B, the sequence $0 \longrightarrow A \longrightarrow B$ is exact if and only if the sequence $0 \longrightarrow F \otimes A \longrightarrow F \otimes B$ is exact. By using this definition we are thus led to the following strengthening of the Theorem 2.16.

Proposition 2.17. *Let F be a faithfully flat R-module. Then a submodule P of an R-module M is I-prime if and only if $F \otimes P$ is I-prime submodule of $F \otimes M$.*

Proof. Suppose that P is I-prime submodule of an R-module M and F a faithfully flat R-module. If $F \otimes P = F \otimes M$, then the exactness of the sequence $0 \longrightarrow F \otimes P \longrightarrow F \otimes M \longrightarrow 0$ imply the exactness of $0 \longrightarrow P \longrightarrow M \longrightarrow 0$ and hence $P = M$ which is a contradiction. So $F \otimes P \neq F \otimes M$ and by Theorem 2.16 $F \otimes P$ is an I-prime submodule of $F \otimes M$. Conversely, let $F \otimes P$ be an I-prime submodule of $F \otimes M$. Hence $F \otimes P \neq F \otimes M$ and so $P \neq M$. Now for every $r \in R - (P : M)$ we have $r \in R - (F \otimes P : F \otimes M)$ and so by Lemma 2.15, $F \otimes (P : r) = (F \otimes P : r) = F \otimes P$ or $F \otimes (P : r) = (F \otimes P : r) = (I(F \otimes P) : r) = (F \otimes IP : r) = F \otimes (IP : r)$. Assume $F \otimes (P : r) = F \otimes P$. Then $0 \longrightarrow F \otimes (P : r) \longrightarrow F \otimes P \longrightarrow 0$ is an exact sequence and as F is faithfully flat, $0 \longrightarrow (P : r) \longrightarrow P \longrightarrow 0$ is exact sequence and consequently

$(P : r) = P$. The other case can be proved similarly. Thus by Theorem 2.7 P is I-prime submodule of M. □

It is known from Proposition 6.1 in [14] that $J \otimes F \cong JF$ for any ideal J of R and flat R-module F. Thus according to Theorem 2.16 and Corollary 2.17 we conclude the following.

Corollary 2.18. *Let F be a flat R-module and J an I-prime ideal of R with $JF \neq F$. Then JF is an I-prime submodule of F. In the case F is faithfully flat, the converse is also true.*

We know that every polynomial ring $R[x]$ is flat over R and that $R[x] \otimes M \cong M[x]$. Hence as an immediate consequence of the Theorem 2.16 we give the following corollary.

Corollary 2.19. *Let M be an R-module and x an indeterminate. If P is I-prime submodule of M, then $P[x]$ is I-prime submodule of $M[x]$.*

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Ismael Akray,
Department of Mathematics,
Soran University, Erbil city, Kurdistan region,
Iraq
E-mail address: ismael.akray@soran.edu.iq

Halgurd S. Hussein,
Department of Mathematics,
Soran University, Erbil city, Kurdistan region,
Iraq
E-mail address: rasty.rosty1@gmail.com