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LACUNARY SEQUENCE SPACES OF ORLICZ FUNCTION AND INFINITE MATRIX

KULDIP RAJ AND ANU CHOUDHARY

Abstract. In this paper we introduce and study sequence spaces

$$[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_0, \qquad [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_1,$$
$$[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_\infty$$

over n-normed spaces defined by an infinite matrix $A = (a_{jk})$ and a sequence of Orlicz functions. We make an effort to study some algebraic and topological properties of these sequence spaces. Some inclusion relations between these sequence spaces are also studied in the paper.

1. Introduction and Preliminaries

The notion of difference sequence spaces was introduced by Kizmaz [7], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [1] by introducing the spaces $l_\infty(\Delta^e)$, $c(\Delta^e)$ and $c_0(\Delta^e)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let e, f be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_f^e) = \{x = (x_k) \in w : (\Delta_f^e x_k) \in Z\},$$

where $\Delta_f^e x = (\Delta_f^e x_k) = (\Delta_f^{e-1} x_k - \Delta_f^{e-1} x_{k+1})$ and $\Delta_f^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_f^e x_k = \sum_{v=0}^{e} (-1)^v \binom{e}{v} x_{k+fv}.$$

Taking $f = 1$, we get the spaces which were studied by Et and Colak [1]. Taking $e = f = 1$, we get the spaces which were introduced and studied by Kizmaz [7]. For more details about sequence spaces see [11], [12], [13] and references therein.

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Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$,
(2) $p(-x) = p(x)$, for all $x \in X$,
(3) $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called a total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [10], Theorem 10.4.2, P-183).

An Orlicz function M is a function, which is continuous, non decreasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Lindenstrauss and Tzafriri [8] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$l_M = \{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\}$$

which is called an Orlicz sequence space. The space l_M is a Banach space with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}.$$

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function. A Musielak-Orlicz function $\mathcal{M} = (M_k)$ is said to satisfy Δ_2-*condition* if there exist constants a, $K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in l_+^1$ (the positive cone of l^1) such that the inequality

$$M_k(2u) \leq K M_k(u) + c_k$$

holds for all $k \in \mathbb{N}$ and $u \in \mathbb{R}^+$, whenever $M_k(u) \leq a$.

By l_∞, c, and c_0 we denote the classes of all bounded, convergent and null sequence spaces, respectively.

The concept of 2-normed spaces was initially developed by Gähler [3] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [9]. Since then, many others have studied this concept and obtained various results, see Gunawan [4], [5] and Gunawan and Mashadi [6]. Let $n \in \mathbb{N}$ and X be a linear space over the field of real numbers $\mathbb{R}$ of dimension d, where $d \geq n \geq 2$. A real valued function $\|\cdot, \cdots, \cdot\|$ on X^n satisfying the following four conditions:

(1) $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X,
(2) $\|x_1, x_2, \cdots, x_n\|$ is invariant under permutation,
(3) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \ \|x_1, x_2, \cdots, x_n\|$ for any $\alpha \in \mathbb{R}$, and

(4) $\|x+x', x_2, \cdots, x_n\| \leq \|x, x_2, \cdots, x_n\| + \|x', x_2, \cdots, x_n\|$

is called an n-norm on X and the pair $(X, \|\cdot, \cdots, \cdot\|)$ is called an n-normed space over the field $\mathbb{R}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, \|\cdot, \cdots, \cdot\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $\|\cdot, \cdots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \cdots, x_{n-1}\|_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in an n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} \|x_k - x_p, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be an n-Banach space.

Throughout the paper E will represent a seminormed space, seminormed by q. We define $w(E)$ to be the vector space of all E-valued sequences.

A sequence of positive integers $\theta = (k_r)$ is called lacunary if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ is defined by Freedman et al. [2] as follows:

$$N_\theta = \Big\{x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - le| = 0, \text{ for some } l\Big\}.$$

There is a strong connection between N_θ and the space s of strongly Cesàro summable sequences which is defined by

$$s = \Big\{x = (x_k) : \lim_n \frac{1}{n} \sum_{k=0}^{n} |x_k - le| = 0, \text{ for some } l\Big\}.$$

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, θ be a lacunary sequence, $p = (p_k)$ be a bounded sequence of strictly positive real numbers and $A = (a_{jk})$

be a non-negative matrix. In the present paper we define the following sequence spaces:

$$[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_0 = \Big\{x \in w(E) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \Big[M_k\Big(\Big\|\frac{q(\Delta_f^e x_k)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k}\Big] = 0, \text{ uniformly in } j\Big\},$$

$$[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_1 = \Big\{x \in w(E) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \Big[M_k\Big(\Big\|\frac{q(\Delta_f^e x_k - L)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k}\Big] = 0, \text{ uniformly in } j \text{ for some } L\Big\},$$

and

$$[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_\infty = \Big\{x \in w(E) : \sup_j \sup_r \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \Big[M_k\Big(\Big\|\frac{q(\Delta_f^e x_k)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k}\Big] < \infty\Big\}.$$

For $M_k(x) = x$

$$[V_\theta^E, A, \Delta_f^e, \|\cdot, \cdots, \cdot\|, p]_0 = \Big\{x \in w(E) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \Big[\Big(\|q(\Delta_f^e x_k), z_1, \cdots, z_{n-1}\|\Big)^{p_k}\Big] = 0, \text{ uniformly in } j\Big\},$$

$$[V_\theta^E, A, \Delta_f^e, \|\cdot, \cdots, \cdot\|, p]_1 = \Big\{x \in w(E) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \Big[\Big(\|q(\Delta_f^e x_k - L), z_1, \cdots, z_{n-1}\|\Big)^{p_k}\Big] = 0, \text{ uniformly in } j \text{ for some } L\Big\},$$

and

$$[V_\theta^E, A, \Delta_f^e, \|\cdot, \cdots, \cdot\|, p]_\infty = \Big\{x \in w(E) : \sup_j \sup_r \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \Big[\Big(\|q(\Delta_f^e x_k), z_1, \cdots, z_{n-1}\|\Big)^{p_k}\Big] < \infty\Big\}.$$

For $p_k = 1$, we get

$$[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|]_0 = \Big\{x \in w(E) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \Big[M_k\Big(\Big\|\frac{q(\Delta_f^e x_k)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big] = 0, \text{ uniformly in } j\Big\},$$

$$[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|]_1 = \Big\{x \in w(E) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \Big[M_k\Big(\Big\|\frac{q(\Delta_f^e x_k - L)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big] = 0, \text{ uniformly in } j \text{ for some } L\Big\},$$

and

$$[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|]_\infty = \Big\{x \in w(E) : \sup_j \sup_r \frac{1}{h_r} \sum_{k\in I_n} a_{jk} \Big[M_k\Big(\Big\|\frac{q(\Delta_f^e x_k)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big] < \infty\Big\}.$$

For $M_k(x) = x$ and $p_k = 1$, we get

$$[V_\theta^E, A, \Delta_f^e, \|\cdot, \cdots, \cdot\|]_0 = \Big\{x \in w(E) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \Big[\big\|q(\Delta_f^e x_k), z_1, \cdots, z_{n-1}\big\|\Big] = 0, \text{uniformly in } j\Big\},$$

$$[V_\theta^E, A, \Delta_f^e, \|\cdot, \cdots, \cdot\|]_1 = \Big\{x \in w(E) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \Big[\big\|q(\Delta_f^e x_k - L), z_1, \cdots, z_{n-1}\big\|\Big] = 0, \text{ uniformly in } j \text{ for some } L\Big\}$$

and

$$[V_\theta^E, A, \Delta_f^e, \|\cdot, \cdots, \cdot\|]_\infty = \Big\{x \in w(E) : \sup_j \sup_r \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \Big[\big\|q(\Delta_f^e x_k), z_1, \cdots, z_{n-1}\big\|\Big] < \infty\Big\}.$$

For $f = 1$, we find

$$[V_\theta^E, A, \Delta^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_0 = \left\{x \in w(E) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k\left(\left\|\frac{q(\Delta^e x_k)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_k}\right] = 0, \text{ uniformly in } j\right\},$$

$$[V_\theta^E, A, \Delta^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_1 = \left\{x \in w(E) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k\left(\left\|\frac{q(\Delta^e x_k - L)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_k}\right] = 0, \text{ uniformly in } j \text{ for some } L\right\},$$

and

$$[V_\theta^E, A, \Delta^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_\infty = \left\{x \in w(E) : \sup_j \sup_r \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k\left(\left\|\frac{q(\Delta^e x_k)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_k}\right] < \infty\right\}.$$

For $A = I$, where I is the Identity matrix we have

$$[V_\theta^E, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_0 = \left\{x \in w(E) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{q(\Delta_f^e x_k)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_k}\right] = 0, \text{ uniformly in } j\right\},$$

$$[V_\theta^E, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_1 = \left\{x \in w(E) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{q(\Delta_f^e x_k - L)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_k}\right] = 0, \text{ uniformly in } j \text{ for some } L\right\},$$

and

$$[V_\theta^E, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_\infty = \left\{x \in w(E) : \sup_j \sup_r \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{q(\Delta_f^e x_k)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_k}\right] < \infty\right\}.$$

In this paper, we will denote any one of the notations $0, 1$ or ∞ by X. The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $D = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main purpose of this paper is to introduce and study some classes of Lacunary sequences over n-normed spaces defined by a non-negative matrix and a sequence of Orlicz functions. We shall study some topological properties, algebraic properties and inclusion relations of these sequence spaces.

2. Main Results

Theorem 1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers. Then the sequence spaces $[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_X$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. Let $x = (x_k)$ and $y = (y_k) \in [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_0$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e x_k)}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right] = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e y_k)}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right] = 0.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\|\cdot, \cdots, \cdot\|$ is an n-norm on X and (M_k) is a non decreasing and convex function, by using inequality (1), we have

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e(\alpha x_k + \beta y_k))}{\rho_3}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right] \\
&\leq \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e(\alpha x_k))}{\rho_3}, z_1, \cdots, z_{n-1} \right\| \right.\right. \\
&\qquad\qquad \left.\left. + \left\| \frac{q(\Delta_f^e(\beta y_k))}{\rho_3}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right] \\
&\leq D \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \frac{1}{2^{p_k}} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e(x_k))}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right] \\
&\quad + D \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \frac{1}{2^{p_k}} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e(y_k))}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right]
\end{aligned}$$

$$\leq D \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e(x_k))}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right]$$

$$+ D \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e(y_k))}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right] = 0.$$

This proves that $[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_0$ is a linear space. Similarly we can prove that $[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_1$ and $[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_\infty$ are linear spaces. □

Theorem 2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_0 \subset [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_1 \subset [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_\infty$.*

Proof. The first inclusion is obvious. For the second inclusion, let $x \in [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_1$. Then by definition we have

$$\frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e x_k)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right]$$

$$= \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e x_k - L + L)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right]$$

$$\leq D \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e x_k - L)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right]$$

$$+ D \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(L)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right].$$

Now there exists a positive number B such that $q(L) \leq B$. Hence, we have

$$\frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e x_k)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right]$$

$$\leq \frac{D}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e x_k - L)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right]$$

$$+ D[B^H] \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k(1)^H \right].$$

Since $x \in [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_1$, we have $x \in [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_\infty$. Therefore, $[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_1 \subset [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_\infty$. This completes the proof. □

Theorem 3. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers. Then $[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_0$ is a paranormed space with*

$$g(x) = \inf \left\{ (\rho)^{\frac{p_k}{N}} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e x_k)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right]^{\frac{1}{N}} \leq 1 \right\},$$

where $0 < p_k \leq \sup p_k = H < \infty$ *and* $N = \max(1, H)$.

Proof. i) Clearly $g(x) \geq 0$ for $x = (x_k) \in [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_0$. Since $M_k(0) = 0$, we get $g(0) = 0$.
ii) $g(-x) = g(x)$.
iii) Let $x = (x_k)$ and $y = (y_k) \in [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_0$, there exists positive numbers ρ_1 and ρ_2 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e x_k)}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right] \leq 1$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e y_k)}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right] \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have

$$\begin{aligned}
&\lim_{r\to\infty} \tfrac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e (x_k + y_k))}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right] \\
&\quad = \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e (x_k + y_k))}{\rho_1 + \rho_2}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right] \\
&\quad = \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e x_k)}{\rho_1 + \rho_2}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right] \\
&\qquad + \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e y_k)}{\rho_1 + \rho_2}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right] \\
&\quad \leq \left(\frac{\rho_1}{\rho_1 + \rho_2} \right) \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e x_k)}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right] \\
&\qquad + \left(\frac{\rho_2}{\rho_1 + \rho_2} \right) \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \frac{q(\Delta_f^e y_k)}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right)^{p_k} \right] \leq 1.
\end{aligned}$$

Thus,

$g(x+y)=$

$$\inf\left\{(\rho)^{\frac{p_k}{N}}:\lim_{r\to\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}a_{jk}\left[M_k\left(\left\|\frac{q(\Delta_f^e(x_k+y_k))}{\rho},z_1,\cdots,z_{n-1}\right\|\right)^{p_k}\right]^{\frac{1}{N}}\right)\leq 1\right\}$$

$$\leq\inf\left\{(\rho_1)^{\frac{p_k}{N}}:\lim_{r\to\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}a_{jk}\left[M_k\left(\left\|\frac{q(\Delta_f^e x_k)}{\rho_1},z_1,\cdots,z_{n-1}\right\|\right)^{p_k}\right]^{\frac{1}{N}}\right)\leq 1\right\}$$

$$+\inf\left\{(\rho_2)^{\frac{p_k}{N}}:\lim_{r\to\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}a_{jk}\left[M_k\left(\left\|\frac{q(\Delta_f^e y_k)}{\rho_2},z_1,\cdots,z_{n-1}\right\|\right)^{p_k}\right]^{\frac{1}{N}}\right)\leq 1\right\}.$$

Therefore, $g(x+y)\leq g(x)+g(y)$.

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition

$$g(\lambda x)=\inf\left\{(\rho)^{\frac{p_k}{N}}:\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}a_{jk}\left[M_k\left(\left\|\frac{q(\Delta_f^e\lambda x_k)}{\rho},z_1,\cdots,z_{n-1}\right\|\right)^{p_k}\right]^{\frac{1}{N}}\leq 1\right\}$$

$$=\inf\left\{(|\lambda|t)^{\frac{p_k}{N}}:\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}a_{jk}\left[M_k\left(\left\|\frac{q(\Delta_f^e x_k)}{t},z_1,\cdots,z_{n-1}\right\|\right)^{p_k}\right]^{\frac{1}{N}}\leq 1\right\},$$

where $t=\frac{\rho}{|\lambda|}>0$. Since $|\lambda|^{p_k}\leq\max(1,|\lambda|^{\sup p_k})$, we have

$$g(\lambda x)\leq\max(1,|\lambda|^{\sup p_k})$$

$$\times\inf\left\{(t)^{\frac{p_k}{N}}:\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}a_{jk}\left[M_k\left(\left\|\frac{q(\Delta_f^e x_k)}{\rho},z_1,\cdots,z_{n-1}\right\|\right)^{p_k}\right]^{\frac{1}{N}}\leq 1\right\}.$$

So, the fact that the scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. □

Theorem 4. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $f\geq 1$ then the inclusion $[V_\theta^E,A,\Delta_f^{e-1},\mathcal{M},\|\cdot,\cdots,\cdot\|]_X\subset[V_\theta^E,A,\Delta_f^e,\mathcal{M},\|\cdot,\cdots,\cdot\|]_X$ is strict. In general*

$$[V_\theta^E,A,\Delta_f^i,\mathcal{M},\|\cdot,\cdots,\cdot\|]_X\subset[V_\theta^E,A,\Delta_f^e,\mathcal{M},\|\cdot,\cdots,\cdot\|]_X$$

for all $i=1,2,...,e-1$ and the inclusion is strict.

Proof. Let $x\in[V_\theta^E,A,\Delta_f^{e-1},\mathcal{M},\|\cdot,\cdots,\cdot\|]_\infty$. Then we have

$$\sup_r\frac{1}{h_r}\sum_{k\in I_r}a_{jk}\left[M_k\left(\left\|\frac{q(\Delta_f^{e-1}x_k)}{\rho},z_1,\cdots,z_{n-1}\right\|\right)\right]<\infty.$$

By definition, we have

$$\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\left[M_k\left(\left\|\frac{q(\Delta_f^e x_k)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]$$
$$= \frac{1}{h_r}\sum_{k\in I_r} a_{jk}\left[M_k\left(\left\|\frac{q(\Delta_f^{e-1} x_k)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]$$
$$+\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\left[M_k\left(\left\|\frac{q(\Delta_f^{e-1} x_{k+1})}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]$$
$$\leq \infty.$$

Thus $[V_\theta^E, A, \Delta_f^{e-1}, \mathcal{M}, \|\cdot, \cdots, \cdot\|]_\infty \subset [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|]_\infty$.

Proceeding in this manner, we have

$$[V_\theta^E, A, \Delta_f^i, \mathcal{M}, \|\cdot, \cdots, \cdot\|]_\infty \subset [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|]_\infty$$

for all $i = 1, 2, \cdots, e-1$.

The sequence $x = (x_k) \in [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|]_\infty$ but does not belong to $[V_\theta^E, A, \Delta_f^{e-1}, \mathcal{M}, \|\cdot, \cdots, \cdot\|]_\infty$ for $E = \mathbb{C}, M(x) = x$ and $\rho = 1$.

Similarly, we can prove for the case $[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|]_0$ and $[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|]_1$ in view of the above proof. □

Corollary 1. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then*

$$[V_\theta^E, A, \Delta_f^{e-1}, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_1 \subset [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_0.$$

Theorem 5. *Let $0 < p_k \leq t_k < \infty$ for each k, $\mathcal{M} = (M_k)$ be a sequence of Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and e be a positive integer. Then we have*

$$[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, t]_\infty \subset [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_\infty.$$

Proof. Let $x = (x_k) \in [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, t]_\infty$. Then we have

$$\sup_j \sup_r \frac{1}{h_r}\sum_{k\in I_r} a_{jk}\left[M_k\left(\left\|\frac{q(\Delta_f^e x_k)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{t_k}\right] < \infty.$$

This implies that $\sup_r \frac{1}{h_r}\sum_{k\in I_r} a_{jk}\left[M_k\left(\left\|\frac{q(\Delta_f^e x_k)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right] < \epsilon \quad (0 < \epsilon < 1)$, for sufficiently large k. Hence, we have

$$\sup_r \frac{1}{h_r}\sum_{k\in I_r} a_{jk}\left[M_k\left(\left\|\frac{q(\Delta_f^e x_k)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_k}\right]$$
$$\leq \sup_r \frac{1}{h_r}\sum_{k\in I_r} a_{jk}\left[M_k\left(\left\|\frac{q(\Delta_f^e x_k)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{t_k}\right] < \infty.$$

This implies that $x = (x_k) \in [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_\infty$.

Thus $[V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, t]_\infty \subset [V_\theta^E, A, \Delta_f^e, \mathcal{M}, \|\cdot, \cdots, \cdot\|, p]_\infty$. This completes the proof. □

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Kuldip Raj
Department of Mathematics,
Shri Mata Vaishno Devi University,
Katra-182320, J & K (India)
Email address: anuchoudhary407@gmail.com

Anu Choudhary
Department of Mathematics,
Shri Mata Vaishno Devi University,
Katra-182320, J & K (India)
Email address: kuldipraj68@gmail.com