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HYPERSURFACES OF A RIEMANNIAN MANIFOLD WITH A RICCI-QUARTER SYMMETRIC METRIC CONNECTION

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Abstract. In this paper we study hypersurfaces of a Riemannian manifold endowed with a Ricci-quarter symmetric metric connection. We prove that the induced connection is also a Ricci-quarter symmetric metric connection. We consider the total geodesicness, the total umbilicity and the minimality of a hypersurface of a Riemannian manifold endowed with the Ricci-quarter symmetric metric connection. We obtain the Gauss, Weingarten and Codazzi equations with respect to the Ricci-quarter symmetric metric connection. The relation between the sectional curvatures of $\overline{M}^n$ and $M^{(n+1)}$ with respect to the Ricci-quarter symmetric metric connection has been also given.

1. Introduction

In 1975, Golab [5] introduced the notion of a quarter-symmetric linear connection in a differentiable manifold. Later Misra and Pandey [7] considered a quarter symmetric F-connection and studied some of its properties. They considered especially the case of Kaehlerian structure and introduced the notion of a Ricci-quarter symmetric metric connection. Kamilya and De [6] studied some properties of a Ricci-quarter symmetric metric connection. In [4], Quasi Einstein manifolds admitting a Ricci-quarter symmetric metric connection were considered. In 1982, Yano and Imai [10] studied some curvature conditions for quarter symmetric metric connections in Riemannian, Hermitian and Kaehlerian manifolds. In [3], De and Mondal considered hypersurfaces of Kenmotsu manifolds with a quarter symmetric non-metric connection. In [1], Ahmad, Jun and Haseeb investigated some properties of invariant hypersurfaces of almost r-paracontact Riemannian manifold endowed with a quarter symmetric metric connection.

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In the present paper, we have studied hypersurfaces of a Riemannian manifold endowed with a Ricci-quarter symmetric metric connection. The paper is organized as follows: In Section 2, we have given some properties of the Ricci-quarter symmetric metric connection; in Section 3, some necessary information about a hypersurface of a Riemannian manifold endowed with the Ricci-quarter symmetric metric connection has been given and we have proved that the induced connection is also a Ricci-quarter symmetric metric connection. We have also considered the total geodesicness, the total umbilicity and the minimality of a hypersurface of a Riemannian manifold endowed with the Ricci-quarter symmetric metric connection. In Section 4, we have obtained the Gauss, Weingarten, and Codazzi equations with respect to the Ricci-quarter symmetric metric connection. The relation between the sectional curvatures of $\overline{M}^n$ and $M^{(n+1)}$ with respect to the Ricci-quarter symmetric metric connection has been also found.

2. Preliminaries

Let M be an $(n+1)$ dimensional Riemannian manifold with a Riemannian metric g, and let ∇ be a linear connection on M. The linear connection ∇ in Riemannian manifold M is said to be *a quarter symmetric connection* if its torsion tensor T satisfies [10]

$$T(X,Y) = w(Y)LX - w(X)LY, \tag{1}$$

where w is a 1-form associated with a non-zero vector field ρ by $w(X) = g(X,\rho)$ and L is a tensor field of type $(1,1)$.

A linear connection ∇ is called *a metric connection* if [9]

$$\nabla g = 0.$$

In (1), if a tensor field L is a $(1,1)$-Ricci tensor of a Riemannian manifold M, then the linear connection ∇ of a Riemannian manifold M is called *a Ricci-quarter symmetric connection.* Such a connection together with the metric condition is said to be a *Ricci-quarter symmetric metric connection* [7].

Let ∇^* be the Levi-Civita connection in M. A Ricci-quarter symmetric metric connection ∇ on M is given by Misra and Pandey [7]

$$\nabla_X Y = \nabla^*_X Y + w(Y)LX - S^*(X,Y)\rho, \tag{2}$$

where L is a Ricci tensor of type $(1,1)$ defined by

$$S^*(X,Y) = g(LX,Y),$$

where S^* is the Ricci tensor of M.

We denote the curvature tensor of M with respect to the Ricci-quarter symmetric metric connection ∇ by R. So we have

$$\begin{aligned} R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \\ &= R^*(X,Y)Z - M(Y,Z)LX + M(X,Z)LY \\ &\quad -S^*(Y,Z)QX + S^*(X,Z)QY + \pi(Z)\left[(\nabla_X L)(Y) - (\nabla_Y L)(X)\right] \\ &\quad - \left[(\nabla_X S^*)(Y,Z) - (\nabla_Y S^*)(X,Z)\right]\rho, \end{aligned}$$

where

$$R^*(X,Y)Z = \nabla^*_X\nabla^*_Y Z - \nabla^*_Y\nabla^*_X Z - \nabla^*_{[X,Y]}Z$$

is the curvature tensor of the manifold with respect to the Levi-Civita connection ∇^* and M is a tensor of type $(0,2)$ defined by

$$M(X,Y) = g(QX,Y) = (\nabla_X w)(Y) - w(Y)w(LX) + \frac{1}{2}w(\rho)S^*(X,Y),$$

and Q is a tensor field of type $(2,1)$ defined by [7]

$$QX = \nabla_X\rho - w(LX)\rho + \frac{1}{2}w(\rho)LX.$$

3. Hypersurfaces

Let $\overline{M}$ be an n dimensional hypersurface immersed in M by the immersion $i:\overline{M}\to M$. If B denotes the derivative of i, then any vector field $\overline{X}\in T(\overline{M})$ implies $B\overline{X}\in T(M)$. We denote the objects belonging to $\overline{M}$ by $\overline{X},\overline{L}$, etc.

Let N be an oriented unit normal vector field on $\overline{M}$. Then the induced $\overline{g}$ on $\overline{M}$ is $\overline{g}(\overline{X},\overline{Y}) = g(\overline{X},\overline{Y})$. Then we have [2]

$$g(\overline{X},N) = 0 \qquad \text{and} \qquad g(N,N) = 1.$$

Let $\overline{\nabla}^*$ be the induced connection on a hypersurface from ∇^* with respect to the unit normal N, then the Gauss equation is given by

$$\nabla^*_{\overline{X}}\overline{Y} = \overline{\nabla}^*_{\overline{X}}\overline{Y} + h(\overline{X},\overline{Y})N, \tag{3}$$

where h is the second fundamental tensor

$$h(\overline{X},\overline{Y}) = h(\overline{Y},\overline{X}) = \overline{g}(H\overline{X},\overline{Y}),$$

and H is a tensor field of type $(1,1)$ of $\overline{M}$.

If $\overline{\nabla}$ is the induced connection on the hypersurface from the Ricci-quarter symmetric metric connection ∇ with respect to the unit normal N, then we have

$$\nabla_{\overline{X}}\overline{Y} = \overline{\nabla}_{\overline{X}}\overline{Y} + m(\overline{X},\overline{Y})N. \tag{4}$$

Now every vector field X on M is decomposed as

$$X = \overline{X} + l(X)N,$$

where l is a 1-form on M. For any tangent vector field $\overline{X}$ on $\overline{M}$ and normal N we have

$$LN = \overline{N} + KN, \tag{5}$$

$$L\overline{X} = \overline{L}\ \overline{X} + b(\overline{X})N, \tag{6}$$

$$\rho = \overline{\rho} + \lambda N, \tag{7}$$

where $\overline{L}$ is a Ricci tensor field of type $(1, 1)$ on the hypersurface $\overline{M}$, b is a 1-form, K and λ are scalar functions on $\overline{M}$.

Using (2), (6), and (7), we have

$$\nabla_{\overline{X}}\overline{Y} = \nabla^*_{\overline{X}}\overline{Y} + w(\overline{Y})\left\{\overline{L}\ \overline{X} + b(\overline{X})N\right\} - S^*(\overline{X}, \overline{Y})\left\{\overline{\rho} + \lambda N\right\}, \tag{8}$$

where $w(\overline{Y}) = \overline{w}(\overline{Y})$. Using (3) and (4) in (8) yields

$$\begin{aligned}\overline{\nabla}_{\overline{X}}\overline{Y} + m(\overline{X}, \overline{Y})N =& \overline{\nabla}^*_{\overline{X}}\overline{Y} + h(\overline{X}, \overline{Y})N + w(\overline{Y})\left\{\overline{L}\ \overline{X} + b(\overline{X})N\right\} \\ &- S^*(\overline{X}, \overline{Y})\left\{\overline{\rho} + \lambda N\right\}.\end{aligned}$$

Now taking tangential and normal parts from both sides, we have

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla}^*_{\overline{X}}\overline{Y} + w(\overline{Y})\overline{L}\ \overline{X} - S^*(\overline{X}, \overline{Y})\overline{\rho} \tag{9}$$

and

$$m(\overline{X}, \overline{Y}) = h(\overline{X}, \overline{Y}) + w(\overline{Y})b(\overline{X}) - \lambda S^*(\overline{X}, \overline{Y}). \tag{10}$$

From (9), it follows that

$$\overline{T}(\overline{X}, \overline{Y}) = w(\overline{Y})\overline{L}\ \overline{X} - w(\overline{X})\overline{L}\ \overline{Y},$$

and also using (4), we have

$$\left(\nabla_{\overline{X}}g\right)(\overline{Y}, \overline{Z}) = \left(\overline{\nabla}_{\overline{X}}\overline{g}\right)(\overline{Y}, \overline{Z}).$$

Thus we get the following.

Theorem 1. *The connection induced on a hypersurface of a Riemannian manifold with a Ricci-quarter symmetric metric connection is also a Ricci-quarter symmetric metric connection.*

3.1. **Totally geodesic and totally umbilic hypersurfaces.** Let $\left\{\overline{E}_1, \ldots, \overline{E}_n\right\}$ be n orthonormal vector fields in $\overline{M}$. Then the function

$$\frac{1}{n}\sum_{i=1}^{n} h(\overline{E}_i, \overline{E}_i)$$

is the mean curvature of $\overline{M}$ with respect to the Levi-Civita connection $\overline{\nabla}^*$ and

$$\frac{1}{n}\sum_{i=1}^{n} m(\overline{E}_i, \overline{E}_i)$$

is called the mean curvature of $\overline{M}$ with respect to the Ricci-quarter symmetric metric connection $\overline{\nabla}$.

From this we have the following definitions.

Definition 1. If h vanishes, we call $\overline{M}$ a totally geodesic hypersurface of M with respect to the Levi-Civita connection $\overline{\nabla}^*$.

Definition 2. The hypersurface $\overline{M}$ is called totally umbilical with respect to the connection $\overline{\nabla}^*$ if h is proportional to the metric tensor $\overline{g}$.

If we replace h by m in the above definitions we get a totally geodesic hypersurface and a totally umbilical hypersurface with respect to the Ricci-quarter symmetric connection $\overline{\nabla}$.

Thus we get the following theorem.

Theorem 2. *In order that the mean curvature of $\overline{M}$ with respect to $\overline{\nabla}^*$ coincides with that of $\overline{M}$ with respect to $\overline{\nabla}$, it is necessary and sufficient that ρ and $L\overline{X}$ are tangent to M. Hence $\overline{M}$ is minimal with respect to the Levi-Civita connection if and only if it is minimal with respect to the Ricci-quarter symmetric metric connection.*

Proof. In view of (10), we get

$$m(\overline{E}_i, \overline{E}_i) = h(\overline{E}_i, \overline{E}_i) + w(\overline{E}_i)b(\overline{E}_i) - \lambda S^*(\overline{E}_i, \overline{E}_i),$$

summing up for $i = 1, 2, \ldots, n$ and dividing by n, we obtain that

$$\frac{1}{n}\sum_{i=1}^{n} m(\overline{E}_i, \overline{E}_i) = \frac{1}{n}\sum_{i=1}^{n} h(\overline{E}_i, \overline{E}_i) \tag{11}$$

if and only if $\lambda = 0$ and $b = 0$. Hence, from (6) and (7), it follows that

$$\rho = \overline{\rho} \qquad \text{and} \qquad L\overline{X} = \overline{L}\ \overline{X}.$$

Thus ρ and $L\overline{X}$ are in a tangent space of M. Moreover, it is clear from (11) that $\overline{M}$ is minimal with respect to the Levi-Civita connection if and only if it is minimal with respect to the Ricci-quarter symmetric metric connection. □

Theorem 3. *Let ρ and $L\overline{X}$ be tangent to M. Then the hypersurface $\overline{M}$ is totally umbilical with respect to the Levi-Civita connection $\overline{\nabla}^*$ if and only if it is totally umbilical with respect to the Ricci-quarter symmetric metric connection $\overline{\nabla}$.*

Proof. The proof follows easily from (10). □

4. Gauss, Weingarten, and Codazzi equations with respect to Ricci-quarter symmetric metric connection

In this section we shall obtain the Gauss, Weingarten, and Codazzi equations with respect to the Ricci-quarter symmetric metric connection. For the Levi-Civita connection ∇^*, the Weingarten equations are given by

$$\nabla^*_{\overline{X}} N = -H\overline{X} \tag{12}$$

for any vector field in $\overline{M}$, where H has the meaning already stated. In view of the equation (2), we get

$$\nabla_{\overline{X}} N = \nabla^*_{\overline{X}} N + \lambda L\overline{X} - S^*(\overline{X},N)\rho, \tag{13}$$

where $\lambda = w(N)$. From (6), (7) and (13), it follows that

$$\nabla_{\overline{X}} N = \nabla^*_{\overline{X}} N + \lambda \overline{L}\ \overline{X} - b(\overline{X})\rho. \tag{14}$$

Thus, from (12) and (14), we get

$$\nabla_{\overline{X}} N = -H\overline{X} + \lambda \overline{L}\ \overline{X} - b(\overline{X})\rho,$$

which is the equation of Weingarten with respect to the Ricci-quarter symmetric metric connection.

Let us denote the curvature tensor of $\overline{M}$ with respect to $\overline{\nabla}$ by $\overline{R}$. Then

$$\overline{R}(\overline{X},\overline{Y})\overline{Z} = \overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z} - \overline{\nabla}_{\overline{Y}}\overline{\nabla}_{\overline{X}}\overline{Z} - \overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z}.$$

Using (4) and (14) in $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, we get

(15)

$$\begin{aligned}
&R(\overline{X},\overline{Y})\overline{Z}\\
&= \overline{R}(\overline{X},\overline{Y})\overline{Z} + m(\overline{Y},\overline{Z})\left\{-H\overline{X} + \lambda\overline{L}\ \overline{X} - b(\overline{X})\rho\right\}\\
&\quad - m(\overline{X},\overline{Z})\left\{-H\overline{Y} + \lambda\overline{L}\ \overline{Y} - b(\overline{Y})\rho\right\}\\
&\quad + \left\{\left(\overline{\nabla}_{\overline{X}} m\right)(\overline{Y},\overline{Z}) - \left(\overline{\nabla}_{\overline{Y}} m\right)(\overline{X},\overline{Z}) - m(w(\overline{Y})\overline{L}\ \overline{X} - w(\overline{X})\overline{L}\ \overline{Y},\overline{Z})\right\} N
\end{aligned}$$

From (10), it follows that

$$\begin{aligned} R(\overline{X},\overline{Y},\overline{Z},\overline{W}) = {} & \overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) \\ & + m(\overline{X},\overline{Z})m(\overline{Y},\overline{W}) - m(\overline{Y},\overline{Z})m(\overline{X},\overline{W}), \end{aligned} \tag{16}$$

where

$$R(\overline{X},\overline{Y},\overline{Z},\overline{W}) = g(R(\overline{X},\overline{Y})\overline{Z},\overline{W}),\quad \overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = g(\overline{R}(\overline{X},\overline{Y})\overline{Z},\overline{W})$$

and $\overline{W}$ is a tangent vector field on $\overline{M}$. The equation (16) is the equation of Gauss with respect to the Ricci-quarter symmetric metric connection.

From (15), the normal component of $R(\overline{X},\overline{Y})\overline{Z}$ is given by

(17)

$$\left(R(\overline{X},\overline{Y})\overline{Z}\right)^{\perp} = \left(\overline{\nabla}_{\overline{X}} m\right)(\overline{Y},\overline{Z}) - \left(\overline{\nabla}_{\overline{Y}} m\right)(\overline{X},\overline{Z}) - m(w(\overline{Y})\overline{L}\ \overline{X} - w(\overline{X})\overline{L}\ \overline{Y},\overline{Z}).$$

The equation (17) is the equation of Codazzi with respect to the Ricci-quarter symmetric metric connection.

Theorem 4. *A totally umbilical hypersurface $\overline{M}$ of M with vanishing curvature tensor with respect to the Ricci-quarter symmetric metric connection is of constant curvature.*

Proof. Since $\overline{M}$ is a totally umbilical hypersurface, $m = k\overline{g}$ where k is a scalar. If we put $R = 0$ and $m = k\overline{g}$ in the equation (16), we obtain

$$\overline{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = k^2 \left[\overline{g}(\overline{Y}, \overline{Z})\overline{g}(\overline{X}, \overline{W}) - \overline{g}(\overline{X}, \overline{Z})\overline{g}(\overline{Y}, \overline{W})\right].$$

Hence such a hypersuface $\overline{M}$ is of constant curvature. □

We assume that M is a space of constant curvature and $\overline{M}$ is a conformally flat hypersurface. Since M is of constant curvature, hence it is an Einstein manifold and conformally flat.

Theorem 5. [8] *If V_n is a conformally flat hypersurface of a conformally flat space V_{n+1} and V_n is a quasi-umbilical hypersurface, that is, there exists a non-zero vector field v_i such that the second fundamental tensor h_{ji} is given in the form $h_{ji} = \alpha g_{ji} + \beta v_j v_i$ for some functions α and β on V_n, where α is differentiable.*

Using the above theorem, we have

$$h(\overline{X}, \overline{Y}) = \alpha g(\overline{X}, \overline{Y}) + \beta w(\overline{X}) w(\overline{Y}), \tag{18}$$

where α and β are some functions on $\overline{M}$ such that α is differentiable. From (10) and (18), it follows that

$$m(\overline{X}, \overline{Y}) = \gamma g(\overline{X}, \overline{Y}) + \beta w(\overline{X}) w(\overline{Y}) + w(\overline{Y}) b(\overline{X}),$$

where $\gamma = \alpha - \frac{r}{n+1}\lambda$ if and only if $b = 0$. Then we get

$$m(\overline{X}, \overline{Y}) = \gamma g(\overline{X}, \overline{Y}) + \beta w(\overline{X}) w(\overline{Y}).$$

Thus we obtain the following.

Theorem 6. *Let $\overline{M}$ be a quasi-umbilical hypersurface of a conformally flat manifold M with respect to the Levi-Civita connection ∇^*. Then $\overline{M}$ is a quasi-umbilical hypersurface of a conformally flat manifold M with respect to the Ricci-quarter symmetric metric connection $\overline{\nabla}$ if and only if $L\overline{X}$ is tangent to M.*

Now let $\overline{X}$ and $\overline{Y}$ be orthogonal unit tangent vector fields on $\overline{M}$ and π be a subspace of the tangent space spanned by the orthonormal base $\{\overline{X}, \overline{Y}\}$. Then in view of (16) we can write

$$R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}) = \overline{R}(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}) + m(\overline{X}, \overline{Y}) m(\overline{Y}, \overline{X}) - m(\overline{Y}, \overline{Y}) m(\overline{X}, \overline{X}).$$

Let $K(\pi)$ and $\overline{K}(\pi)$ be the sectional curvatures of M and $\overline{M}$ at a point $p \in M$, respectively, with respect to the Ricci-quarter symmetric metric connection. Then we get

$$K(\pi) = \overline{K}(\pi) + m(\overline{X}, \overline{Y}) m(\overline{Y}, \overline{X}) - m(\overline{Y}, \overline{Y}) m(\overline{X}, \overline{X}). \tag{19}$$

Let γ be a geosedic in M which lies in $\overline{M}$ and $\overline{T}$ be a unit tangent vector field of γ in $\overline{M}$. Then $h(\overline{T}, \overline{T}) = 0$ and from (10), it follows that

$$m(\overline{T}, \overline{T}) = w(\overline{T}) b(\overline{T}) - \lambda S^*(\overline{T}, \overline{T}).$$

Let π be the subspace of the tangent space spanned by $\overline{X}$ and $\overline{T}$, and let ρ and $L\overline{X}$ be tangent to M. Then from (10), it follows that $m(\overline{T},\overline{T}) = 0$. Thus using (19), we have

$$K(\pi) = \overline{K}(\pi) + m(\overline{X},\overline{T})m(\overline{T},\overline{X}).$$

Hence we have the following theorem.

Theorem 7. *Let γ be a geosedic in M which lies in $\overline{M}$ and $\overline{T}$ be a unit tangent vector field of γ in $\overline{M}$. Let π be a subspace of the tangent space spanned by $\overline{X}$ and $\overline{T}$. If ρ and $L\overline{X}$ are tangent to M, then $\overline{K}(\pi) \leq K(\pi)$ along γ.*

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