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AMGROUPS

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ABSTRACT. Using the idea of mgroups introduced by Nazmul et al. [8], we redefine the concept of mgroups to allow flexibility of the identity element from a group X in delineating the mgroups and proved some related results.

1. Introduction

The theory of multisets is an extension of the set theory. Since inception, it has evoked a lot of research. For more details, the reader is referred to ([2],[3],[4],[5],[6],[7],[13],[14]). Theoretic study has included algebra aspect of fuzzy sets and multisets. Mordeson and Bhutani collaborated with Rosenfeld, the initiator of theory of fuzzy groups in order to estabish the algebraic structures of fuzzy sets, and worked extensively on this subject with some new results described [1]. Onasanya [11, 12] critically studied the notion and carried out some thorough reviews on fuzzy groups and anti fuzzy groups.

In [8], the underlying structure in group theory was replaced with multisets and some fundamental properties were presented. Moreover, as a suitable generalization of group theory, Awolola and Ibrahim [9], Awolola and Ejegwa [10] discussed the concept further and investigated their related properties.

In this paper, we introduce a new concept of mgroups called amgroups ($EMGs$) by redefining the concept of mgroups from a multiset space $[X]^\infty$ and obtain some related results.

2. Preliminaries

Definition 1. Let X be a set. A multiset (mset, for short) M drawn from X is represented by a count function C_M defined as $C_M : X \longrightarrow N_0 = \{0, 1, 2, \dots\}$.

For each $x \in X$, $C_M(x)$ denotes the number of occurrences of the element x in the mset M. The representation of the mset M drawn from $X = \{x_1, x_2, \dots, x_n\}$ will be as $M = [x_1, x_2, \dots, x_n]_{m_1,m_2,\dots,m_n}$ such that x_i appears m_i times $(i = 1, 2, \dots, n)$ in M.

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Let, for any positive integer n, $[X]^n$ be the set of all msets drawn from X such that no element in the mset occurs more than n times and $[X]^\infty$ be the set of all msets drawn from X such that there is no limit on the number of occurrences of an object in an mset. Therefore, $[X]^n$ and $[X]^\infty$ are referred to as mset spaces.

Definition 2. Let $M_1, M_2, M_i \in [X]^n, i \in I$. Then

(i) $M_1 \subseteq M_2 \Longleftrightarrow C_{M_1}(x) \leqslant C_{M_2}(x),\ \forall x \in X$,
(ii) $M_1 = M_2 \Longleftrightarrow C_{M_1}(x) = C_{M_2}(x),\ \forall x \in X$,
(iii) $\bigcap_{i\in I} M_i = \bigwedge_{i\in I} C_{M_i}(x),\ \forall x \in X$ (where $\bigwedge$ is the minimum operation),
(iv) $\bigcup_{i\in I} M_i = \bigvee_{i\in I} C_{M_i}(x),\ \forall x \in X$ (where $\bigvee$ is the maximum operation),
(v) $M_i^c = n - C_{M_i}(x),\ \forall x \in X,\ n \in \mathbb{Z}^+$.

Definition 3. Let X be a group. A multiset A over X is called *amgroup* if the count function of the elements of A or $C_A(x)$ satisfies the following conditions:

(i) $C_A(xy) \leqslant C_A(x) \vee C_A(y),\ \forall x, y \in X$,
(ii) $C_A(x^{-1}) = C_A(x),\ \forall x \in X$.

Example 1. Let $E = \langle a_1 | a_1^2 = 1\rangle \times \langle a_2 | a_2^2 = 1\rangle \times \dots$ be an infinite elementary abelian 2-group, $\mu \in MG(E)^\infty$ an mgroup. Then in fact $\mu \in MG(E)^{\mu(1)}$ so μ is constant on some infinite $F \subseteq E$. On the other hand, setting $E_0 = \emptyset,\ E_i = \langle a_1\rangle \times \cdots \times \langle a_i\rangle,\ \mu(a) = i + 1$ provided $a \in E_{i+1} \backslash E_i$ then $\mu \in EMG(E)^\infty$ is an amgroup with $\mu \notin MG(E)^k$ for any positive integer K. Hence, amgroups may provide valuable new tools in infinite group theory.

Definition 4. Let $A, B \in [X]^n$, we have the following definitions:

(i) $C_{A\circ B}(x) = \bigwedge\{C_A(y) \bigvee C_B(z) : y, z \in X,\ yz = x\}$,
(ii) $C_{A^{-1}}(x) = C_A(x^{-1})$.

We call $A \circ B$ the product of A and B and A^{-1} the inverse of A.

Definition 5. Let $A, B \in EMG(X)$. Then A is said to be a subamgroup of B if $A \subseteq B$.

Example 2. Let $X = \langle a, b | a^2 = b^2 = 1, ba = ab\rangle$, $A = [1, a, b, ab]_{1,2,4,4}$, and $B = [1, a, b, ab]_{2,3,4,4}$. Clearly, $A, B \in EMG(X)$ and $A \subseteq B$. Thus, A is a subamgroup of B.

Definition 6. Let $A \in EMG(X)$. Then A is called an abelian amgroup over X if $C_A(xy) = C_A(yx),\ \forall x, y \in X$. The set of all abelian amgroups over X is denoted by $AEMG(X)$.

3. Main results

Proposition 1. *Let $A \in EMG(X)$.*

(i) *$C_A(x^n) \leqslant C_A(x),\ \forall x \in X$.*
(ii) *If $C_A(x^{-1}) \leqslant C_A(x)$, then $C_A(x^{-1}) = C_A(x)$.*
(iii) *If $C_A(x) < C_A(y)$, for some $x, y \in X$, then $C_A(xy) = C_A(y) = C_A(yx)$.*

(iv) $C_A(xy^{-1}) = C_A(e)$ *implies* $C_A(x) = C_A(y)$.

Proof. (i) and (ii) follows immediately.

(iii): Let $C_A(x) < C_A(y)$ for some $x, y \in X$. Since $A \in EMG(X)$, then $C_A(xy) \leqslant C_A(x) \bigvee C_A(y) = C_A(y)$. Now,

$$C_A(y) = C_A(xyx^{-1}) \leqslant C_A(xy) \bigvee C_A(x) = C_A(xy)$$

since $C_A(x) < C_A(y)$ and $C_A(x) < C_A(xy)$. Therefore, $C_A(xy) = C_A(y)$. The equality $C_A(yx) = C_A(y)$ can be obtained similarly.

(iv): Given $A \in EMG(X)$ and $C_A(xy^{-1}) = C_A(e)$ $\forall x, y \in X$. Then

$$\begin{aligned} C_A(x) &= C_A(x(y^{-})y)) = C_A((xy^{-1})y) \\ &\leqslant C_A(xy^{-1}) \bigvee C_A(y) = C_A(e) \bigvee C_A(y) = C_A(y). \end{aligned}$$

Now,

$$\begin{aligned} C_A(y) &= C_A(y^{-1}) = C_A(ey^{-1}) = C_A((x^{-1}x)y^{-1}) \\ &\leqslant C_A(x^{-1}) \bigvee C_A(xy^{-1}) = C_A(x) \bigvee C_A(e) = C_A(x). \end{aligned}$$

Hence, $C_A(x) = C_A(y)$. □

Proposition 2. *Let* $A \in [X]^n$. *Then* $A \in EMG(X)$ *if and only if*

$$C_A(xy^{-1}) \leqslant C_A(x) \bigvee C_A(y)$$

holds for all $x, y \in X$.

Proof. Let $A \in EMG(X)$. Then

$$C_A(xy^{-1}) \leqslant C_A(x) \bigvee C_A(y^{-1}) = C_A(x) \bigvee C_A(y), \ \forall x, y \in X.$$

Conversely, let the given condition be satisfied, i.e.,

$$C_A(xy^{-1}) \leqslant C_A(x) \bigvee C_A(y).$$

Now,

$$C_A(e) = C_A(xx^{-1}) \leqslant C_A(x) \bigvee C_A(x) = C_A(x)$$

and

$$C_A(x^{-1}) = C_A(ex^{-1}) \leqslant C_A(e) \bigvee C_A(x) = C_A(x).$$

Hence $C_A(xy) = C_A(x(y^{-1})^{-1}) \leqslant C_A(x) \bigvee C_A(y^{-1}) = C_A(x) \bigvee C_A(y)$, which completes the proof. □

Proposition 3. *Let* $A \in [X]^n$. *Then* $A \in EMG(X)$ *if and only if* $A \leqslant A \circ A$ *and* $A^{-1} = A$.

Proof. Let $x, y \in X$. Since $A \in EMG(X)$, then $C_A(xy) \leqslant C_A(x) \bigvee C_A(y)$ and hence $C_{A \circ A}(z) = \bigwedge_{z=xy} \{C_A(x) \bigvee C_A(y)\} \geqslant \bigwedge_{z=xy} C_A(xy) = C_A(z)$. Therefore, $A \leqslant A \circ A$. On the other hand, $A \in EMG(X)$ thus $C_A(x^{-1}) = C_A(x)$, $\forall x \in X$. But by definition, $C_A(x^{-1}) = C_{A^{-1}}(x)$. Therefore, $A^{-1} = A$.

Conversely, let the given conditions be satisfied. If $A = A \circ A$ and $A^{-1} = A$, then it is sufficient to prove $A \in EMG(X)$. Now

$$C_{A \circ A}(z) = \bigwedge_{z=xy} \{C_A(x) \bigvee C_A(y)\} \leqslant C_A(x) \bigvee C_A(y), \ \forall x, y \in X,$$

hence $C_A(xy) \leqslant C_A(x) \bigvee C_A(y), xy = z$. Since the equations $C_A(x) = C_{A^{-1}}(x)$ and $C_{A^{-1}}(x) = C_A(x^{-1})$ hold, it follows that $C_A(x^{-1}) = C_A(x), \ \forall x \in X$. Therefore, $A \in EMG(X)$. $\square$

Proposition 4. *Let $A, B \in EMG(X)$. Then $A \cup B \in EMG(X)$.*

Proof. Let $x, y \in A \cup B \in EMG(X)$. Hence $x, y \in A$ or $x, y \in B$. Therefore, $C_A(xy) \leqslant C_A(x) \vee C_A(y)$ or $C_B(xy) \leqslant C_B(x) \vee C_B(y)$. Now

$$\begin{aligned} C_{A \cup B}(xy) = C_A(xy) \bigvee C_B(xy) &\leqslant \Big[C_A(x) \bigvee C_A(y)\Big] \bigvee \Big[C_B(x) \bigvee C_B(y)\Big] \\ &= \Big[C_A(x) \bigvee C_B(x)\Big] \bigvee \Big[C_A(y) \bigvee C_B(y)\Big] \\ &= C_{A \cup B}(x) \bigvee C_{A \cup B}(y) \end{aligned}$$

and $C_{A \cup B}(x^{-1}) = C_A(x^{-1}) \bigvee C_B(x^{-1}) = C_A(x) \bigvee C_B(x) = C_{A \cup B}(x)$. Therefore, $A \cup B \in EMG(X)$. $\square$

Remark 1. If $\{A_i\}_{i \in I}$ is a family of amgroups, then $\bigcap_{i \in I} A_i$ need not be an amgroup over X.

Remark 2. If $A \in EMG(X)$, then A^c need not be an $EMG(X)$. However, $A^c \in EMG(X)$ if and only if $C_A(x) = C_A(e), \ \forall x \in X$.

Proposition 5. *Let $A \in EMG(X)$ and $x \in X$. Then $C_A(xy) = C_A(y)$ $\forall y \in X$ if and only if $C_A(x) = C_A(e)$.*

Proof. If $C_A(xy) = C_A(y)$ $\forall y \in X$, then $y = e$.

Conversely, assume $C_A(x) = C_A(e)$. Then $C_A(xy) \leqslant C_A(x) \bigvee C_A(y) = C_A(y)$ and on the other hand, $C_A(y) \leqslant C_A(x^{-1}) \bigvee C_A(xy) = C_A(xy)$. $\square$

Proposition 6. *Let $A \in EMG(X)$. Then the non-empty sets defined as*

$$A^n = \{x \in X : C_A(x) \leqslant n, \ n \in \mathbb{Z}^+\}$$

and

$$A_* = \{x \in X : C_A(x) = C_A(e)\}$$

are subgroups of X.

Proof. Let $x, y \in A^n$. It implies that $C_A(x) \leqslant n$ and $C_A(y) \leqslant n$. Then $C_A(xy^{-1}) \leqslant [C_A(x) \bigvee C_A(y)] \leqslant n$ and hence if $x, y \in A^n$, then $xy^{-1} \in A^n$. Hence $A^n, \ n \in \mathbb{Z}^+$ are subgroups of X.

Again, let $x, y \in A_*$. Then $C_A(x) = C_A(y) = C_A(e)$. Now, $C_A(xy^{-1}) \leqslant [C_A(x) \bigvee C_A(y)] = [C_A(e) \bigvee C_A(e)] = C_A(e)$. But $C_A(e) \leqslant C_A(xy^{-1})$, i.e., $C_A(xy^{-1}) = C_A(e)$. Therefore, $xy^{-1} \in A_*$. Hence A_* is a subgroup of X. $\square$

Proposition 7. *Let $A \in EMG(X)$. Then the following assertions are equivalent:*

(i) $C_A(xy) = C_A(yx),\ \forall x, y \in X,$

(ii) $C_A(xyx^{-1}) = C_A(y),\ \forall x, y \in X,$

(iii) $C_A(xyx^{-1}) \leqslant C_A(y),\ \forall x, y \in X,$

(iv) $C_A(xyx^{-1}) \geqslant C_A(y),\ \forall x, y \in X.$

Proof. $(i) \Rightarrow (ii)$: Let $x, y \in X$. Then $C_A(x^{-1}xy) = C_A(ey) = C_A(y)$.

$(ii) \Rightarrow (iii)$ Trivial.

$(iii) \Rightarrow (iv)$: $C_A(xyx^{-1}) \geqslant C_A\left(x^{-1}\left[xyx^{-1}\right](x^{-1})^{-1}\right) = C_A(y)$.

$(iv) \Rightarrow (i)$: Let $x, y \in X$. Then $C_A(xy) = C_A(x\,[yx]\,x^{-1}) \geqslant C_A(yx) = C_A(y\,[xy]\,y^{-1}) \geqslant C_A(xy)$. Hence, $C_A(xy) = C_A(yx)$.

Thus the above assertions are equivalent. □

Proposition 8. *Let $A \in AEMG(X)$. Then $A_*, A^n,\ n \in \mathbb{Z}^+$ are normal subgroups of X.*

Proof. (i): Assume $C_A(e) = 1$. Then $A_* = A^1$. Hence, it is not difficult to see that A_* is a normal subgroup of X.

(ii): Let $x \in X$ and $y \in A^n$, then $C_A(y) \leqslant n$. Since $A \in AEMG(X)$, then $C_A(xy) = C_A(yx)\ \forall x, y \in X$. By Proposition 7, $C_A(xyx^{-1}) = C_A(y)$ and this implies $C_A(xyx^{-1}) = C_A(y) \leqslant n$. Thus, $xyx^{-1} \in A^n$ is a normal subgroup of X. □

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