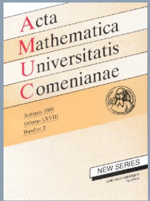


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# A NOTE ON COMPACTNESS OF TENSOR PRODUCTS

J. ZANNI AND C. S. KUBRUSLY

ABSTRACT. Compactness is preserved from a pair of operators to their tensor product. The converse was considered in [13] under some restrictions on the spectra of the operators. This paper shows that no restriction is necessary: the converse holds whenever the tensor product is nonzero.

## 1. Introduction

The tensor product of two Hilbert spaces can be constructed as the completion of the span of a collection of conjugate bilinear maps on their Cartesian product. The tensor product of two Hilbert-space operators are then naturally defined so that they can be viewed as an extention to infinite-dimensional spaces of the standard Kronecker product of matrices on finite-dimensional spaces. Important results on tensor product of (bounded linear) operators have been continuously considered in the literature (see, e.g., [1], [16], [4], [17], [3], [12], [14], [10], [8], and [7]).

In this paper we deal with preservation of compactness for tensor product of operators. Compactness for tensor product of operators has several relevant aspects from both theoretical and applied points of view (e.g., applications to partial integral operators considered in [5, 6]). The following result was established in [13]. If a tensor product $A \otimes B$ is compact and one of the operators $A$ or $B$ has a nonzero eigenvalue, then the other is compact. We improve the previous

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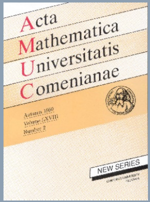
statement by dismissing the spectral condition, and showing that a nonzero tensor product $A \otimes B$ is compact if and only if $A$ and $B$ are nonzero compact operators.

## 2. Preliminaries

Notation, terminology and basic results are posed in this section. Throughout the paper, $\mathcal{H}$ and $\mathcal{K}$ are complex infinite-dimensional Hilbert spaces. Inner product in any of them will be denoted by $\langle \cdot\,;\cdot \rangle$. Let $\mathcal{B}[\mathcal{H},\mathcal{K}]$ stand for the Banach space of all bounded linear transformations of $\mathcal{H}$ into $\mathcal{K}$. A Hilbert-space operator is a bounded linear transformation of a Hilbert space into itself. Set $\mathcal{B}[\mathcal{H}] = \mathcal{B}[\mathcal{H},\mathcal{H}]$, the unital Banach algebra of all operators on $\mathcal{H}$. Let $T^* \in \mathcal{B}[\mathcal{H}]$ be the adjoint of $T \in \mathcal{B}[\mathcal{H}]$. An operator $T \in \mathcal{B}[\mathcal{H}]$ is compact if it maps bounded sequences into sequences that have a convergent subsequence (see, e.g., [11, Theorem 4.52 (d)]). For equivalent definitions see, for instance, [2, Section II.4] or [11, Section 4.9]. Thus $T \in \mathcal{B}[\mathcal{H}]$ is compact if for every bounded sequence $\{x_n\}$ of vectors $x_n$ in $\mathcal{H}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges in $\mathcal{H}$.

The concept of tensor product of two Hilbert spaces can be defined in terms of single tensors which in turn can be defined as conjugate bilinear functionals on the Cartesian product of $\mathcal{H} \times \mathcal{K}$ (see, e.g., [9]). The single tensor of $x \in \mathcal{H}$ and $y \in \mathcal{K}$ is the conjugate bilinear functional $x \otimes y \colon \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ defined by $(x \otimes y)(u,v) = \langle x\,; u\rangle\langle y\,; v\rangle$ for every $(u,v)$ in $\mathcal{H} \times \mathcal{K}$. The tensor product space $\mathcal{H} \otimes \mathcal{K}$ of $\mathcal{H}$ and $\mathcal{K}$ is the completion of the inner product space consisting of all (finite) sums of single tensors $x_i \otimes y_i$ with $x_i \in \mathcal{H}$ and $y_i \in \mathcal{K}$. The inner product on $\mathcal{H} \otimes \mathcal{K}$ is defined by

$$\Big\langle \sum_i x_i \otimes y_i\,; \sum_j w_j \otimes z_j \Big\rangle = \sum_i \sum_j \langle x_i\,; w_j\rangle\langle y_i\,; z_j\rangle$$




for every $\sum_i x_i \otimes y_i$ and $\sum_j w_j \otimes z_j$ in $\mathcal{H} \otimes \mathcal{K}$. The norm on $\mathcal{H} \otimes \mathcal{K}$ is the one induced by the inner product. The tensor product of two operators $T \in \mathcal{B}[\mathcal{H}]$ and $S \in \mathcal{B}[\mathcal{K}]$ is the operator $T \otimes S \colon \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ defined by

$$(T \otimes S) \sum_i x_i \otimes y_i = \sum_i T x_i \otimes S y_i \quad \text{for every} \quad \sum_i x_i \otimes y_i \in \mathcal{H} \otimes \mathcal{K},$$

which lies in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$. The proposition below summarizes some basic well-known properties of tensor products in a Hilbert space setting that will be required in the sequel (see, e.g., [9]). (Here $\|T\|$ denotes the induced uniform norm of $T$ on $\mathcal{B}[\mathcal{H}]$, and $\|T \otimes S\|$ the induced uniform norm of $T \otimes S$ on $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$.)

**Proposition 1.** *Take $\alpha, \beta \in \mathbb{C}$, $x, w \in \mathcal{H}$, $y, z \in \mathcal{K}$, $T, A \in \mathcal{B}[\mathcal{H}]$, and $S, B \in \mathcal{B}[\mathcal{K}]$ arbitrary. The following identities hold true:*

(a) $\alpha\,\beta(x \otimes y) = \alpha\, x \otimes \beta\, y, \qquad \alpha\,\beta(T \otimes S) = \alpha\, T \otimes \beta\, S,$
(b) $(x + w) \otimes (y + z) = x \otimes y + w \otimes y + x \otimes z + w \otimes z,$
$(T + A) \otimes (S + B) = T \otimes S + A \otimes S + T \otimes B + A \otimes B,$
(c) $TA \otimes SB = (T \otimes S)(A \otimes B),$
(d) $(T \otimes S)^* = (T^* \otimes S^*),$
(e) $\|x \otimes y\| = \|x\|\,\|y\|,$
$\|T \otimes S\| = \|T\|\,\|S\|.$

For further properties on tensor products, see, for instance, [9] and [15].

## 3. Tensor Product and Compact Operators

Consider the following result presented in this journal [13, Theorem 2].

**Theorem 1.** *If $A \in \mathcal{B}[\mathcal{H}]$ and $B \in \mathcal{B}[\mathcal{K}]$ are compact, then $A \otimes B \in \mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ is compact. Conversely, if $A \otimes B \in \mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ is compact and one of $A \in \mathcal{B}[\mathcal{H}]$ or $B \in \mathcal{B}[\mathcal{K}]$ has a nonzero eigenvalue, then the other is compact.*

**Remark 1.** It is worth noticing that part of the results in [13] was worked out for separable Hilbert spaces, although the preceding Theorem 1 does not require that the Hilbert spaces are separable, as well as the forthcoming Theorem 2. Indeed, the direct assertion in Theorem 1 (but not the converse) was proved in [13] by using the fact that a compact operator on a Hilbert space (not necessarily separable) is the uniform limit of finite-rank operators — see, e.g., [2, Theorem II.4.4] or [11, Problem 5.42(b)]. This, however, also holds in a Banach space with a Schauder basis (which turns out to be separable) — see, e.g., [11, Problem 4.58].

We improve the converse in Theorem 1 by showing in Theorem 2 that if $A \otimes B$ is nonzero and compact, then so are both $A$ and $B$ without any restriction on their spectra. First we need the following elementary auxiliary result which says that if a sequence of single tensors converges, then it must converge to a single tensor.

**Lemma 1.** *Take $s_n = x \otimes y_n \in \mathcal{H} \otimes \mathcal{K}$ with $x \in \mathcal{H}$ and $y_n \in \mathcal{K}$ for each positive integer $n$. If $\{s_n\}$ converges in $\mathcal{H} \otimes \mathcal{K}$ to $s \in \mathcal{H} \otimes \mathcal{K}$, then $s = x \otimes y$ for some $y \in \mathcal{K}$.*

*Proof.* Let $\{s_n\}$ be a sequence of vectors in $\mathcal{H} \otimes \mathcal{K}$ as in the lemma statement. If $\{s_n\}$ converges in $\mathcal{H} \otimes \mathcal{K}$, then $\{s_n\}$ is a Cauchy sequence,

$$\|s_m - s_n\| \to 0 \quad \text{as} \quad m, n \to \infty.$$

Since $s_m - s_n = x \otimes y_m - x \otimes y_n = x \otimes (y_n - y_m)$, and since $\|x \otimes (y_n - y_m)\| = \|x\| \, \|y_n - y_m\|$, it follows that

$$\|x\| \, \|y_n - y_m\| \to 0 \quad \text{as} \quad m, n \to \infty.$$

Thus $\{y_n\}$ is a Cauchy sequence in $\mathcal{K}$. Since $\mathcal{K}$ is a Hilbert space, $\{y_n\}$ converges in $\mathcal{K}$ to, say, $y \in \mathcal{K}$. Since $\{s_n\}$ converges, let $s$ be its limit in $\mathcal{H} \otimes \mathcal{K}$, so that (by continuity of the conjugate bilinear functional)

$$s = \lim_n s_n = \lim_n (x \otimes y_n) = x \otimes \lim_n y_n = x \otimes y. \qquad \square$$

**Theorem 2.** *If $A \otimes B$ is a nonzero compact tensor product in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$, then both $A \in \mathcal{B}[\mathcal{H}]$ and $B \in \mathcal{B}[\mathcal{K}]$ are nonzero compact operators.*

*Proof.* Suppose $A \otimes B \neq O$ in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$. Thus $A \neq O$ and $B \neq O$ in $\mathcal{B}[\mathcal{H}]$ and $\mathcal{B}[\mathcal{K}]$. Since $A \neq O$, there exists $x \neq 0$ in $\mathcal{H}$ such that $\|Ax\| \neq 0$. Take an arbitrary bounded sequence $\{y_n\}$ of vectors $y_n$ in $\mathcal{K}$, and let $\{s_n\}$ be a sequence of single tensors $s_n$ in $\mathcal{H} \otimes \mathcal{K}$ given by $s_n = x \otimes y_n$ for each positive integer $n$. Note that $\{s_n\}$ is bounded because $\{y_n\}$ is bounded. In fact,

$$\sup_n \|s_n\| = \sup_n \|x \otimes y_n\| = \sup_n \|x\|\,\|y_n\| = \|x\| \sup_n \|y_n\| < \infty.$$

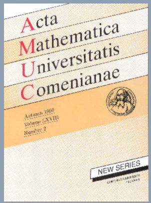
Thus, if $A \otimes B$ is compact, then there exists a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $\{(A \otimes B)s_{n_k}\}$ converges in $\mathcal{H} \otimes \mathcal{K}$. Hence, by Lemma 1,

$$(A \otimes B)s_{n_k} = (A \otimes B)(x \otimes y_{n_k}) = Ax \otimes By_{n_k} \to Ax \otimes z$$

for some $z \in \mathcal{K}$ (not necessarily in the range of $B$, since the range of $B$ is not necessarily closed in $\mathcal{K}$). Then, by Proposition 1(a, b, e),

$$\begin{aligned} \|Ax\|\|By_{n_k} - z\| &= \|Ax \otimes (By_{n_k} - z)\| = \|Ax \otimes By_{n_k} - Ax \otimes z)\| \\ &= \|(A \otimes B)(x \otimes y_{n_k}) - Ax \otimes z\| = \|(A \otimes B)s_{n_k} - Ax \otimes z\| \to 0. \end{aligned}$$

Therefore, since $\|Ax\| \neq 0$, it follows that $By_{n_k} \to z$. Outcome: Recalling that $\{y_n\}$ was taken to be an arbitrary bounded sequence, then we can conclude that for every bounded sequence $\{y_n\}$ of vectors $y_n$ in $\mathcal{K}$, there exists a subsequence $\{y_{n_k}\}$ of it such that $\{By_{n_k}\}$ converges in $\mathcal{K}$. This means that $B$ is compact. Symmetrically, if $A \otimes B$ is nonzero and compact, then $A$ is compact. $\square$




Theorems 1 and 2 together yield a complete characterization of compactness preservation for tensor products.

**Corollary 1.** *A tensor product $A \otimes B \in \mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ is nonzero and compact if and only if $A \in \mathcal{B}[\mathcal{H}]$ and $B \in \mathcal{B}[\mathcal{K}]$ are both nonzero and compact.*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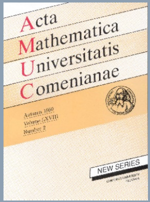


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J. Zanni, Catholic University of Rio de Janeiro, 22453-900, Rio de Janeiro, RJ, Brazil,
*e-mail*: `jzanni@gmail.com`

C. S. Kubrusly, Catholic University of Rio de Janeiro, 22453-900, Rio de Janeiro, RJ, Brazil,
*e-mail*: `carlos@ele.puc-rio.br`

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