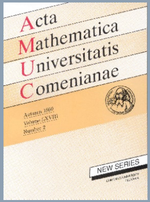

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PERTURBATION ANALYSIS OF BOUNDED HOMOGENEOUS GENERALIZED INVERSES ON BANACH SPACES

JIANBING CAO and YIFENG XUE

Abstract. Let X, Y be Banach spaces and $T\colon X \to Y$ be a bounded linear operator. In this paper, we initiate the study of the perturbation problems for bounded homogeneous generalized inverse T^h and quasi-linear projector generalized inverse T^H of T. Some applications to the representations and perturbations of the Moore-Penrose metric generalized inverse T^M of T are also given. The obtained results in this paper extend some well-known results for linear operator generalized inverses in this field.

1. Introduction

The expression and perturbation analysis of the generalized inverses (resp., the Moore-Penrose inverses) of bounded linear operators on Banach spaces (resp., Hilbert spaces) have been widely studied since Nashed's book [18] was published in 1976. Ten years ago, Chen and Xue [8] proposed a notation so-called the stable perturbation of a bounded operator instead of the rank-preserving perturbation of a matrix. Using this new notation, they established the perturbation analyses for the Moore-Penrose inverse and the least square problem on Hilbert spaces in [6, 9, 26]. Meanwhile, Castro-González and Koliha established the perturbation analysis for Drazin inverse by using of the gap-function in [4, 5, 14]. Later, some of their results were generalized by Chen and Xue [27, 28] in terms of stable perturbation.

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Throughout this paper, X, Y are always Banach spaces over real field $\mathbb{R}$ and $B(X,Y)$ is the Banach space consisting of bounded linear operators from X to Y. For $T \in B(X,Y)$, let $\mathcal{N}(T)$ (resp., $\mathcal{R}(T)$) denote the null space (resp., range) of T. It is well-known that if $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are topologically complemented in the spaces X and Y, respectively, then there exists a (projector) generalized inverse $T^+ \in B(Y,X)$ of T such that

$$TT^+T = T, \quad T^+TT^+ = T^+, \quad T^+T = I_X - P_{\mathcal{N}(T)}, \quad TT^+ = Q_{\mathcal{R}(T)},$$

where $P_{\mathcal{N}(T)}$ and $Q_{\mathcal{R}(T)}$ are the bounded linear projectors from X and Y onto $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively, (cf. [**6**, **18**, **25**]). But, in general, not every closed subspace in a Banach space is complemented. Thus the linear generalized inverse T^+ of T may not exist. In this case, we may seek other types of generalized inverses for T. Motivated by the ideas of linear generalized inverses and metric generalized inverses (cf. [**18**, **20**]), by using the so-called homogeneous (resp., quasi-linear) projector in Banach space, Wang and Li [**22**] defined the homogeneous (resp., quasi-linear) generalized inverse. Then, some further study on these types of generalized inverses in Banach space was given in [**1**, **17**]. More importantly, from results in [**17**, **20**], we know that in some reflexive Banach spaces X and Y, for an operator $T \in B(X,Y)$ there may exists a bounded quasi-linear (projector) generalized inverse of T, which is generally neither linear nor metric generalized inverse of T. So, from this point of view, it is important and necessary to study bounded homogeneous and quasi-linear (projector) generalized inverses in Banach spaces.

Since the homogeneous (or quasi-linear) projector in Banach space are no longer linear, the linear projector generalized inverse and the homogeneous (or quasi-linear) projector generalized inverse in Banach spaces are quite different. Motivated by the new perturbation results of closed linear generalized inverses [**12**], in this paper, we initiate the study of the following problems for bounded homogeneous (resp., quasi-linear projector) generalized inverse: Let $T \in B(X,Y)$ with a bounded homogeneous (resp., quasi-linear projector) generalized inverse T^h (resp., T^H), what conditions on the small perturbation δT can guarantee that the bounded homogeneous (resp., quasi-linear projector) generalized inverse $\bar{T}^h$ (resp. $\bar{T}^H$) of the perturbed operator $\bar{T} = T + \delta T$ exists? Furthermore, if it exists, when does $\bar{T}^h$ (resp., $\bar{T}^H$) have the simplest expression $(I_X + T^h \delta T)^{-1} T^h$ (resp., $(I_X + T^H \delta T)^{-1} T^H$? With the concept of the quasi-additivity and the notation of stable perturbation in [8], we present some perturbation results on homogeneous generalized inverses and quasi-linear projector generalized inverses in Banach spaces. Explicit representation and perturbation for the Moore-Penrose metric generalized inverse of the perturbed operator are also given.

2. Preliminaries

Let $T \in B(X,Y) \setminus \{0\}$. The reduced minimum module $\gamma(T)$ of T is given by

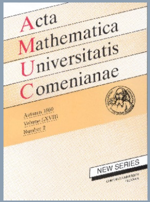
$$\gamma(T) = \inf\{\|Tx\| \mid x \in X, \operatorname{dist}(x, \mathcal{N}(T)) = 1\}, \tag{2.1}$$

where $\operatorname{dist}(x, \mathcal{N}(T)) = \inf\{\|x - z\| \mid z \in \mathcal{N}(T)\}$. It is well-known that $\mathcal{R}(T)$ is closed in Y iff $\gamma(T) > 0$ (cf. [16, 28]). From (2.1), we can obtain useful inequality as follows:

$$\|Tx\| \geq \gamma(T) \operatorname{dist}(x, \mathcal{N}(T)) \qquad \text{for all } x \in X.$$

Recall from [1, 23] that a subset D in X is called to be homogeneous if $\lambda x \in D$ whenever $x \in D$ and $\lambda \in \mathbb{R}$; a mapping $T\colon X \to Y$ is called to be a bounded homogeneous operator if T maps every bounded set in X into a bounded set in Y and $T(\lambda x) = \lambda T(x)$ for every $x \in X$ and every $\lambda \in \mathbb{R}$.

Let $H(X,Y)$ denote the set of all bounded homogeneous operators from X to Y. Equipped with the usual linear operations on $H(X,Y)$ and norm on $T \in H(X,Y)$ defined by $\|T\| = \sup\{\|Tx\| \mid \|x\| = 1, x \in X\}$, we can easily prove that $(H(X,Y), \|\cdot\|)$ is a Banach space (cf. [20, 23]).



Definition 2.1. Let M be a subset of X and $T\colon X \to Y$ be a mapping. We call T is quasi-additive on M if T satisfies

$$T(x+z) = T(x) + T(z) \qquad \text{for all } x \in X \ \text{ and } \ z \in M.$$

Now we give the concept of quasi-linear projector in Banach spaces.

Definition 2.2 (cf. [**17**, **20**])**.** Let $P \in H(X,X)$. If $P^2 = P$, P is called a homogeneous projector. In addition, if P is also quasi-additive on $\mathcal{R}(P)$, i.e., for any $x \in X$ and any $z \in \mathcal{R}(P)$,

$$P(x+z) = P(x) + P(z) = P(x) + z,$$

then P is called a quasi-linear projector.

Clearly, from Definition 2.2, we see that the bounded linear projectors, orthogonal projectors in Hilbert spaces are all quasi-linear projectors.

Let $P \in H(X,X)$ be a quasi-linear projector. Then by [**17**, Lemma 2.5], $\mathcal{R}(P)$ is a closed linear subspace of X and $\mathcal{R}(I_X - P) = \mathcal{N}(P)$. Thus, we can define "the quasi-linearly complement" of a closed linear subspace as follows. Let V be a closed subspace of X. If there exists a bounded quasi-linear projector P on X such that $V = \mathcal{R}(P)$, then V is said to be bounded quasi-linearly complemented in X and $\mathcal{N}(P)$ is the bounded quasi-linear complement of V in X. In this case, as usual, we may write $X = V \dotplus \mathcal{N}(P)$, where $\mathcal{N}(P)$ is a homogeneous subset of X and "$\dotplus$" means that $V \cap \mathcal{N}(P) = \{0\}$ and $X = V + \mathcal{N}(P)$.

Definition 2.3. Let $T \in B(X,Y)$. If there is $T^h \in H(Y,X)$ such that

$$TT^hT = T, \qquad T^hTT^h = T^h,$$

then we call T^h is a bounded homogeneous generalized inverse of T. Furthermore, if T^h is also quasi-additive on $\mathcal{R}(T)$, i.e., for any $y \in Y$ and any $z \in \mathcal{R}(T)$, we have

$$T^h(y+z) = T^h(y) + T^h(z),$$

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then T^h is called a bounded quasi-linear generalized inverse of T.

Obviously, the concept of bounded homogeneous (or quasi-linear) generalized inverse is a generalization of bounded linear generalized inverse.

Definition 2.3 was first given in paper [1] for linear transformations and bounded linear operators. The existence of a homogeneous generalized inverse of $T\in B(X,Y)$ is also given in [1]. In the following proposition, we will give a new proof of the existence of a homogeneous generalized inverse of a bounded linear operator.

Proposition 2.4. *Let $T\in B(X,Y)\setminus\{0\}$. Then T has a homogeneous generalized inverse $T^h\in H(Y,X)$ iff $\mathcal{R}(T)$ is closed and there exist a bounded quasi-linear projector $P_{\mathcal{N}(T)}\colon X\to\mathcal{N}(T)$ and a bounded homogeneous projector $Q_{\mathcal{R}(T)}\colon Y\to\mathcal{R}(T)$.*

Proof. Suppose that there is $T^h\in H(Y,X)$ such that $TT^hT=T$ and $T^hTT^h=T^h$. Put $P_{\mathcal{N}(T)}=I_X-T^hT$ and $Q_{\mathcal{R}(T)}=TT^h$. Then $P_{\mathcal{N}(T)}\in H(X,X)$, $Q_{\mathcal{R}(T)}\in H(Y,Y)$ and

$$\begin{aligned}P^2_{\mathcal{N}(T)}&=(I_X-T^hT)(I_X-T^hT)=I_X-T^hT-T^hT(I_X-T^hT)=P_{\mathcal{N}(T)},\\ Q^2_{\mathcal{R}(T)}&=TT^hTT^h=TT^h=Q_{\mathcal{R}(T)}.\end{aligned}$$

From $TT^hT=T$ and $T^hTT^h=T^h$, we can get that $\mathcal{N}(T)=\mathcal{R}(P_{\mathcal{N}(T)})$ and $\mathcal{R}(T)=\mathcal{R}(Q_{\mathcal{R}(T)})$. Since for any $x\in X$ and any $z\in\mathcal{N}(T)$,

$$\begin{aligned}P_{\mathcal{N}(T)}(x+z)&=x+z-T^hT(x+z)=x+z-T^hTx\\ &=P_{\mathcal{N}(T)}x+z=P_{\mathcal{N}(T)}x+P_{\mathcal{N}(T)}z,\end{aligned}$$

it follows that $P_{\mathcal{N}(T)}$ is quasi-linear. Obviously, we see that $Q_{\mathcal{R}(T)}\colon Y\to\mathcal{R}(T)$ is a bounded homogeneous projector.

Now for any $x\in X$,

$$\operatorname{dist}(x,\mathcal{N}(T))\le\|x-P_{\mathcal{N}(T)}x\|=\|T^hTx\|\le\|T^h\|\|Tx\|.$$

Thus, $\gamma(T) \geq \dfrac{1}{\|T^h\|} > 0$ and hence $\mathcal{R}(T)$ is closed in Y.

Conversely, for $x \in X$, let $[x]$ stand for equivalence class of x in $X/\mathcal{N}(T)$. Define mappings $\phi\colon \mathcal{R}(I_X - P_{\mathcal{N}(T)}) \to X/\mathcal{N}(T)$ and $\hat{T}\colon X/\mathcal{N}(T) \to \mathcal{R}(T)$, respectively, by

$$\phi(x) = [x] \qquad \text{for all } x \in \mathcal{R}(I_X - P_{\mathcal{N}(T)}) \text{ and } \hat{T}([z]) = Tz \qquad \text{for all } z \in X.$$

Clearly, $\hat{T}$ is bijective. Noting that, the quotient space $X/\mathcal{N}(T)$ with the norm $\|[x]\| = \operatorname{dist}(x, \mathcal{N}(T))$ is a Banach space (cf. [25]) and $\|Tx\| \geq \gamma(T) \operatorname{dist}(x, \mathcal{N}(T))$ with $\gamma(T) > 0$ for all $x \in X$, we have $\|\hat{T}[x]\| \geq \gamma(T)\|[x]\|$ for all $x \in X$. Therefore, $\|\hat{T}^{-1}y\| \leq \dfrac{1}{\gamma(T)}\|y\|$, for all $y \in \mathcal{R}(T)$.

Since $P_{\mathcal{N}(T)}$ is a quasi-linear projector, it follows that ϕ is bijective and $\phi^{-1}([x]) = (I_X - P_{\mathcal{N}(T)})x$ for all $x \in X$. Obviously, ϕ^{-1} is homogeneous and for any $z \in \mathcal{N}(T)$,

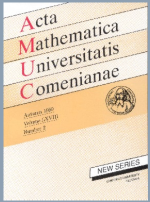
$$\|\phi^{-1}([x])\| = \|(I_X - P_{\mathcal{N}(T)})(x - z)\| \leq (1 + \|P_{\mathcal{N}(T)}\|)\|x - z\|$$

which implies that $\|\phi^{-1}\| \leq 1 + \|P_{\mathcal{N}(T)}\|$. Put $T_0 = \hat{T} \circ \phi\colon \mathcal{R}(I_X - P_{\mathcal{N}(T)}) \to \mathcal{R}(T)$. Then $T_0^{-1} = \phi^{-1} \circ \hat{T}^{-1}\colon \mathcal{R}(T) \to \mathcal{R}(I_X - P_{\mathcal{N}(T)})$ is homogeneous and bounded with $\|T_0^{-1}\| \leq \gamma(T)^{-1}(1 + \|P_{\mathcal{N}(T)}\|)$. Set $T^h = (I_X - P_{\mathcal{N}(T)})T_0^{-1}Q_{\mathcal{R}(T)}$. Then $T^h \in H(Y, X)$ and

$$TT^hT = T, \quad T^hTT^h = T^h, \quad TT^h = Q_{\mathcal{R}(T)}, \quad T^hT = I_X - P_{\mathcal{N}(T)}.$$

This finishes the proof. $\square$

Recall that a closed subspace V in X is Chebyshev if for any $x \in X$, there is a unique $x_0 \in V$ such that $\|x - x_0\| = \operatorname{dist}(x, V)$. Thus, for the closed Chebyshev space V, we can define a mapping $\pi_V : X \to V$ by $\pi_V(x) = x_0$. π_V is called the metric projector from X onto V. From [20], we know that π_V is a quasi-linear projector with $\|\pi_V\| \leq 2$. Then by Proposition 2.4, we have the following corollary.



Corollary 2.5 ([19, 20]). *Let $T \in B(X,Y) \setminus \{0\}$ with $\mathcal{R}(T)$ closed. Assume that $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are Chebyshev subspaces in X and Y, respectively. Then there is $T^h \in H(Y,X)$ such that*

$$TT^hT = T, \quad T^hTT^h = T^h, \quad TT^h = \pi_{\mathcal{R}(T)}, \quad T^hT = I_X - \pi_{\mathcal{N}(T)}. \tag{2.2}$$

The bounded homogeneous generalized inverse T^h in (2.2) is called the Moore-Penrose metric generalized inverse of T. Such T^h in (2.2) is unique and is denoted by T^M (cf. [20]).

Corollary 2.6. *Let $T \in B(X,Y) \setminus \{0\}$ such that the bounded homogeneous generalized inverse T^h exists. Assume that $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are Chebyshev subspaces in X and Y, respectively. Then $T^M = (I_X - \pi_{\mathcal{N}(T)})T^h\pi_{\mathcal{R}(T)}$.*

Proof. Since $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are Chebyshev subspaces, it follows from Corollary 2.5 that T has the unique Moore-Penrose metric generalized inverse T^M which satisfies

$$TT^MT = T, \quad T^MTT^M = T^M, \quad TT^M = \pi_{\mathcal{R}(T)}, \quad T^MT = I_X - \pi_{\mathcal{N}(T)}.$$

Set $T^\natural = (I_X - \pi_{\mathcal{N}(T)})T^h\pi_{\mathcal{R}(T)}$. Then $T^\natural = T^MTT^hTT^M = T^MTT^M = T^M$. $\square$

3. Perturbations for bounded homogeneous generalized inverse

In this section, we extend some perturbation results of linear generalized inverses to bounded homogeneous generalized inverses. We start our investigation with some lemmas which are prepared for the proof of our main results. The following result is well-known for bounded linear operators, we generalize it to the bounded homogeneous operators in the following form.

Lemma 3.1. *Let $T \in H(X,Y)$ and $S \in H(Y,X)$ such that T is quasi-additive on $\mathcal{R}(S)$ and S is quasi-additive on $\mathcal{R}(T)$, then $I_Y + TS$ is invertible in $H(Y,Y)$ if and only if $I_X + ST$ is invertible in $H(X,X)$.*

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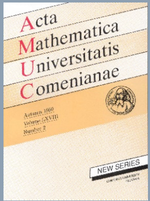


Proof. If there is a $\Phi \in H(Y,Y)$ such that $(I_Y + TS)\Phi = \Phi(I_Y + TS) = I_Y$, then

$$\begin{aligned} I_X &= I_X + ST - ST = I_X + ST - S((I_Y + TS)\Phi)T \\ &= I_X + ST - ((S + STS)\Phi)T \qquad (S \text{ quasi-additive on } \mathcal{R}(T)) \\ &= I_X + ST - ((I_X + ST)S\Phi)T \\ &= (I_X + ST)(I_X - S\Phi T) \qquad (T \text{ quasi-additive on } \mathcal{R}(S)). \end{aligned}$$

Similarly, we also have $I_X = (I_X - S\Phi T)(I_X + ST)$. Thus, $I_X + ST$ is invertible on X with $(I_X + ST)^{-1} = (I_X - S\Phi T) \in H(X,X)$.

The converse can also be proved by using the above argument. □

Lemma 3.2. *Let $T \in B(X,Y)$ such that $T^h \in H(Y,X)$ exists and let $\delta T \in B(X,Y)$ such that T^h is quasi-additive on $\mathcal{R}(\delta T)$ and $(I_X + T^h\delta T)$ is invertible in $B(X,X)$. Then $I_Y + \delta T T^h \colon Y \to Y$ is invertible in $H(Y,Y)$ and*

$$\Phi = T^h(I_Y + \delta T T^h)^{-1} = (I_X + T^h\delta T)^{-1}T^h \tag{3.1}$$

is a bounded homogeneous operator with $\mathcal{R}(\Phi) = \mathcal{R}(T^h)$ and $\mathcal{N}(\Phi) = \mathcal{N}(T^h)$.

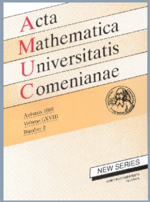
Proof. By Lemma 3.1, $I_Y + \delta T T^h \colon Y \to Y$ is invertible in $H(Y,Y)$.

Clearly, $I_X + T^h\delta T$ is a linear bounded operator and $I_Y + \delta T T^h \in H(Y,Y)$. From the equation

$$(I_X + T^h\delta T)T^h = T^h(I_Y + \delta T T^h)$$

and $T^h \in H(Y,X)$, we get that Φ is a bounded homogeneous operator. Finally, from (3.1), we can obtain that $\mathcal{R}(\Phi) = \mathcal{R}(T^h)$ and $\mathcal{N}(\Phi) = \mathcal{N}(T^h)$. □

Recall from [8] that for $T \in B(X,Y)$ with bounded linear generalized inverse $T^+ \in B(Y,X)$, we say that $\bar{T} = T + \delta T \in B(X,Y)$ is a stable perturbation of T if $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^+) = \{0\}$. Now for



$T \in B(X,Y)$ with $T^h \in H(Y,X)$, we also say that $\bar{T} = T + \delta T \in B(X,Y)$ is a stable perturbation of T if $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^h) = \{0\}$.

Lemma 3.3. *Let $T \in B(X,Y)$ such that $T^h \in H(Y,X)$ exists. Suppose that $\delta T \in B(X,Y)$ such that T^h is quasi-additive on $\mathcal{R}(\delta T)$ and $I_X + T^h \delta T$ is invertible in $B(X,X)$. Put $\bar{T} = T + \delta T$. If $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^h) = \{0\}$, then*

$$\mathcal{N}(\bar{T}) = (I_X + T^h \delta T)^{-1} \mathcal{N}(T) \quad \textit{and} \quad \mathcal{R}(\bar{T}) = (I_Y + \delta T T^h) \mathcal{R}(T).$$

Proof. Set $P = (I_X + T^h \delta T)^{-1}(I_X - T^h T)$. We first show that $P^2 = P$ and $\mathcal{R}(P) = \mathcal{N}(\bar{T})$. Since $T^h T T^h = T^h$, we get $(I_X - T^h T) T^h \delta T = 0$ and then

$$(I_X - T^h T)(I_X + T^h \delta T) = I_X - T^h T, \tag{3.2}$$

and so

$$I_X - T^h T = (I_X - T^h T)(I_X + T^h \delta T)^{-1}. \tag{3.3}$$

Now, by using (3.2) and (3.3), it is easy to get $P^2 = P$.

Since T^h is quasi-additive on $\mathcal{R}(\delta T)$, we see $I_X - T^h T = (I_X + T^h \delta T) - T^h \bar{T}$. Then for any $x \in X$, we have

$$\begin{aligned} Px &= (I_X + T^h \delta T)^{-1}(I_X - T^h T)x \\ &= (I_X + T^h \delta T)^{-1}[(I_X + T^h \delta T) - T^h \bar{T}]x \\ &= x - (I_X + T^h \delta T)^{-1} T^h \bar{T} x. \end{aligned} \tag{3.4}$$

From (3.4), we get that if $x \in \mathcal{N}(\bar{T})$, then $x \in \mathcal{R}(P)$. Thus, $\mathcal{N}(\bar{T}) \subset \mathcal{R}(P)$.

Conversely, let $z \in \mathcal{R}(P)$, then $z = Pz$. From (3.4), we get $(I_X + T^h \delta T)^{-1} T^h \bar{T} x = 0$. Therefore, we have $\bar{T} x \in \mathcal{R}(\bar{T}) \cap \mathcal{N}(T^h) = \{0\}$. Thus, $x \in \mathcal{N}(\bar{T})$ and then $\mathcal{R}(P) = \mathcal{N}(\bar{T})$.

From the Definition of T^h, we have $\mathcal{N}(T) = \mathcal{R}(I_X - T^hT)$. Thus,

$$(I_X + T^h\delta T)^{-1}\mathcal{N}(T) = (I_X + T^h\delta T)^{-1}\mathcal{R}(I_X - T^hT) = \mathcal{R}(P) = \mathcal{N}(\bar{T}).$$

Now, we prove that $\mathcal{R}(\bar{T}) = (I_Y + \delta TT^h)\mathcal{R}(T)$. From $(I_Y + \delta TT^h)T = \bar{T}T^hT$, we get that $(I_Y + \delta TT^h)\mathcal{R}(T) \subset \mathcal{R}(\bar{T})$. On the other hand, since T^h is quasi-additive on $\mathcal{R}(\delta T)$ and $\mathcal{R}(P) = \mathcal{N}(\bar{T})$ for any $x \in X$ we have

$$\begin{aligned} 0 = \bar{T}Px &= \bar{T}(I_X + T^h\delta T)^{-1}(I_X - T^hT)x \\ &= \bar{T}x - \bar{T}(I_X + T^h\delta T)^{-1}(T^h\delta Tx + T^hTx) \\ &= \bar{T}x - \bar{T}(I_X + T^h\delta T)^{-1}T^h\bar{T}x = \bar{T}x - \bar{T}T^h(I_Y + \delta TT^h)^{-1}\bar{T}x \\ &= \bar{T}x - (I_Y + \delta TT^h - I_Y + TT^h)(I_Y + \delta TT^h)^{-1}\bar{T}x \\ &= (I_Y - TT^h)(I_Y + \delta TT^h)^{-1}\bar{T}x. \end{aligned} \tag{3.5}$$

Since $\mathcal{N}(I_Y - TT^h) = \mathcal{R}(T)$, it follows (3.5) that $(I_Y + \delta TT^h)^{-1}\mathcal{R}(\bar{T}) \subset \mathcal{R}(T)$, that is, $\mathcal{R}(\bar{T}) \subset (I_Y + \delta TT^h)\mathcal{R}(T)$. Consequently, $\mathcal{R}(\bar{T}) = (I_Y + \delta TT^h)\mathcal{R}(T)$. □

Now we can present the main perturbation result for bounded homogeneous generalized inverse on Banach spaces.

Theorem 3.4. *Let $T \in B(X,Y)$ such that $T^h \in H(Y,X)$ exists. Suppose that $\delta T \in B(X,Y)$ such that T^h is quasi-additive on $\mathcal{R}(\delta T)$ and $I_X + T^h\delta T$ is invertible in $B(X,X)$. Put $\bar{T} = T + \delta T$. Then the following statements are equivalent:*

(1) *$\Phi = T^h(I_Y + \delta TT^h)^{-1}$ is a bounded homogeneous generalized inverse of $\bar{T}$;*
(2) *$\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^h) = \{0\}$;*
(3) *$\mathcal{R}(\bar{T}) = (I_Y + \delta TT^h)\mathcal{R}(T)$;*
(4) *$(I_X + T^h\delta T)\mathcal{N}(\bar{T}) = \mathcal{N}(T)$;*

(5) $(I_Y + \delta T T^h)^{-1} \bar{T} \mathcal{N}(T) \subset \mathcal{R}(T)$.

Proof. We prove our theorem by showing that

$$(3) \Rightarrow (5) \Rightarrow (4) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1) \Rightarrow (3).$$

(3) $\Rightarrow$ (5) This is obvious since $(I_Y + \delta T T^h)$ is invertible and $\mathcal{N}(T) \subset X$.

(5) $\Rightarrow$ (4). Let $x \in \mathcal{N}(\bar{T})$, then we see $(I_X + T^h \delta T)x = x - T^h T x \in \mathcal{N}(T)$. Hence $(I_X + T^h \delta T)\mathcal{N}(\bar{T}) \subset \mathcal{N}(T)$. Now for any $x \in \mathcal{N}(T)$, by (5), there exists $z \in X$ such that $\bar{T}x = (I_Y + \delta T T^h) T z = \bar{T} T^h T z$. So $x - T^h T z \in \mathcal{N}(\bar{T})$, and hence

$$(I_X + T^h \delta T)(x - T^h T z) = (I_X - T^h T)(x - T^h T z) = x.$$

Consequently, $(I_X + T^h \delta T)\mathcal{N}(\bar{T}) = \mathcal{N}(T)$.

(4) $\Rightarrow$ (2). Let $y \in R(\bar{T}) \cap N(T^h)$, then there exists $x \in X$ such that $y = \bar{T}x$ and $T^h \bar{T} x = 0$. We can check that

$$T(I_X + T^h \delta T)x = Tx + T T^h \delta T x = Tx + T T^h \bar{T} x - T T^h T x = 0.$$

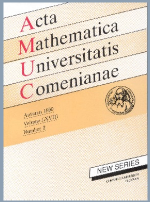
Thus, $(I_X + T^h \delta T)x \in \mathcal{N}(T)$. By (4), $x \in \mathcal{N}(\bar{T})$ and so that $y = \bar{T}x = 0$.

(2) $\Rightarrow$ (3) follows from Lemma 3.3.

(3) $\Rightarrow$ (1). From Lemma 3.2, we see that

$$\Phi = T^h (I_Y + \delta T T^h)^{-1} = (I_X + T^h \delta T)^{-1} T^h$$

is a bounded homogeneous operator with $\mathcal{R}(\Phi) = \mathcal{R}(T^h)$ and $\mathcal{N}(\Phi) = \mathcal{N}(T^h)$. Now we need to prove that $\Phi \bar{T} \Phi = \Phi$ and $\bar{T} \Phi \bar{T} = \bar{T}$. We first prove $\Phi \bar{T} \Phi = \Phi$. Since T^h is quasi-additive on



$\mathcal{R}(\delta T)$, we have $T^h\bar{T} = T^hT + T^h\delta T$. Therefore,

$$\begin{aligned}\Phi\bar{T}\Phi &= (I_X + T^h\delta T)^{-1}T^h\bar{T}(I_X + T^h\delta T)^{-1}T^h\\ &= (I_X + T^h\delta T)^{-1}[(I_X + T^h\delta T) - (I_X - T^hT)](I_X + T^h\delta T)^{-1}T^h\\ &= (I_X + T^h\delta T)^{-1}T^h - (I_X + T^h\delta T)^{-1}(I_X - T^hT)(I_X + T^h\delta T)^{-1}T^h\\ &= \Phi - (I_X + T^h\delta T)^{-1}(I_X - T^hT)T^h(I_Y + \delta TT^h)^{-1}\\ &= \Phi.\end{aligned}$$

Now we prove $\bar{T}\Phi\bar{T} = \bar{T}$. The identity $\mathcal{R}(\bar{T}) = (I_Y + \delta TT^h)\mathcal{R}(T)$ means that $(I_Y - TT^h)(I_Y + \delta TT^h)^{-1}\bar{T} = 0$. So

$$\begin{aligned}\bar{T}\Phi\bar{T} &= (T + \delta T)T^h(I_Y + \delta TT^h)^{-1}\bar{T}\\ &= (I_Y + \delta TT^h + TT^h - I_Y)(I_Y + \delta TT^h)^{-1}\bar{T}\\ &= \bar{T}.\end{aligned}$$

(1) $\Rightarrow$ (3) Since $\bar{T}\Phi\bar{T} = \bar{T}$, we have $(I_Y - TT^h)(I_Y + \delta TT^h)^{-1}\bar{T} = 0$ by the proof of (3) $\Rightarrow$ (1). Thus, $(I_Y + \delta TT^h)^{-1}\mathcal{R}(\bar{T}) \subset \mathcal{R}(T)$. From $(I_Y + \delta TT^h)T = \bar{T}T^hT$, we get $(I_Y + \delta TT^h)\mathcal{R}(T) \subset \mathcal{R}(\bar{T})$. So $(I_Y + \delta TT^h)\mathcal{R}(T) = \mathcal{R}(\bar{T})$. □

Corollary 3.5. *Let $T \in B(X, Y)$ such that $T^h \in H(Y, X)$ exists. Suppose that $\delta T \in B(X, Y)$ such that T^h is quasi-additive on $\mathcal{R}(\delta T)$ and $\|T^h\delta T\| < 1$. Put $\bar{T} = T + \delta T$. If $\mathcal{N}(T) \subset \mathcal{N}(\delta T)$ or $\mathcal{R}(\delta T) \subset \mathcal{R}(T)$, then $\bar{T}$ has a homogeneous bounded generalized inverse*

$$\bar{T}^h = T^h(I_Y + \delta TT^h)^{-1} = (I_X + T^h\delta T)^{-1}T^h.$$

Proof. If $N(T) \subset N(\delta T)$, then $N(T) \subset N(\bar{T})$. So Condition (5) of Theorem 3.4 holds. If $\mathcal{R}(\delta T) \subset R(T)$, then $R(\bar{T}) \subset \mathcal{R}(T)$. So $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^h) \subset \mathcal{R}(T) \cap \mathcal{N}(T^h) = \{0\}$ and consequently,

$\bar{T}$ has the homogeneous bounded generalized inverse $T^h(I_Y + \delta T T^h)^{-1} = (I_X + T^h \delta T)^{-1} T^h$ by Theorem 3.4. $\square$

Proposition 3.6. *Let $T \in B(X,Y)$ with $\mathcal{R}(T)$ closed. Assume that $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are Chebyshev subspaces in X and Y, respectively. Let $\delta T \in B(X,Y)$ such that T^M is quasi-additive on $\mathcal{R}(\delta T)$ and $\|T^M \delta T\| < 1$. Put $\bar{T} = T + \delta T$. Suppose that $\mathcal{N}(\bar{T})$ and $\overline{\mathcal{R}(\bar{T})}$ are Chebyshev subspaces in X and Y, respectively. If $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$, then $\mathcal{R}(\bar{T})$ is closed in Y and $\bar{T}$ has the Moore-Penrose metric generalized inverse*

$$\bar{T}^M = (I_X - \pi_{\mathcal{N}(\bar{T})})(I_X + T^M \delta T)^{-1} T^M \pi_{\mathcal{R}(\bar{T})}$$

with $\|\bar{T}^M\| \le \dfrac{2\|T^M\|}{1 - \|T^M \delta T\|}$.

Proof. T^M exists by Corollary 2.5. Since $T^M \delta T$ is $\mathbb{R}$-linear and $\|T^M \delta T\| < 1$, we have $I_X + T^M \delta T$ is invertible in $B(X,X)$. By Theorem 3.4 and Proposition 2.4, $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$ implies that $\mathcal{R}(\bar{T})$ is closed and $\bar{T}$ has a bounded homogeneous generalized inverse $\bar{T}^h = (I_X + T^M \delta T)^{-1} T^M$. Then by Corollary 2.6, $\bar{T}^M$ has the form

$$\bar{T}^M = (I_X - \pi_{\mathcal{N}(\bar{T})})(I_X + T^M \delta T)^{-1} T^M \pi_{\mathcal{R}(\bar{T})}.$$

Note that $\|x - \pi_{\mathcal{N}(\bar{T})} x\| = \operatorname{dist}(x, \mathcal{N}(\bar{T})) \le \|x\|$ for all $x \in X$. So $\|I_X - \pi_{\mathcal{N}(\bar{T})}\| \le 1$. Therefore,

$$\|\bar{T}^M\| \le \|I_X - \pi_{\mathcal{N}(\bar{T})}\| \|(I_X + T^M \delta T)^{-1} T^M\| \|\pi_{\mathcal{R}(\bar{T})}\| \le \frac{2\|T^M\|}{1 - \|T^M \delta T\|}.$$

This completes the proof. $\square$

4. Perturbation for quasi-linear projector generalized inverse

It is well-known that the range of a bounded qausi-linear projector on a Banach space is closed (see [17, Lemma 2.5]). Thus, from Definition 2.3 and the proof of Proposition 2.4, the following result is obvious.

Proposition 4.1. *Let $T \in B(X,Y) \setminus \{0\}$. Then T has a bounded quasi-linear generalized inverse $T^h \in H(Y,X)$ iff there exist a bounded linear projector $P_{\mathcal{N}(T)} \colon X \to \mathcal{N}(T)$ and a bounded quasi-linear projector $Q_{\mathcal{R}(T)} \colon Y \to \mathcal{R}(T)$.*

Motivated by Proposition 4.1, related results in [1, 17, 22] and the definition of oblique projections of generalized inverses on Banach spaces (see [18, 25]), we introduce the notion of quasi-linear projector generalized inverse of a bounded linear operator on Banach spaces as follows.

Definition 4.2. Let $T \in B(X,Y)$. Let $T^H \in H(Y,X)$ be a bounded homogeneous operator. If there exist a bounded linear projector $P_{\mathcal{N}(T)}$ from X onto $\mathcal{N}(T)$ and a bounded quasi-linear projector $Q_{\mathcal{R}(T)}$ from Y onto $\mathcal{R}(T)$, respectively, such that

(1) $TT^HT = T$;
(2) $T^HTT^H = T^H$;
(3) $T^HT = I_X - P_{\mathcal{N}(T)}$;
(4) $TT^H = Q_{\mathcal{R}(T)}$;

then T^H is called a quasi-linear projector generalized inverse of T.

For $T \in B(X,Y)$, if T^H exists, then from Proposition 4.1 and Definition 2.3, we see that $\mathcal{R}(T)$ is closed and T^H is quasi-additive on $\mathcal{R}(T)$. In this case, we may call T^H is a quasi-linear operator. Choose $\delta T \in B(X,Y)$ such that T^H is also quasi-additive on $\mathcal{R}(\delta T)$, then $I_X + T^H\delta T$ is a bounded linear operator and $I_Y + \delta TT^H$ is a bounded linear operator on $\mathcal{R}(\bar{T})$.

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Lemma 4.3. *Let $T \in B(X,Y)$ such that T^H exists and let $\delta T \in B(X,Y)$ such that T^H is quasi-additive on $\mathcal{R}(\delta T)$. Put $\bar{T} = T + \delta T$. Assumes that $X = \mathcal{N}(\bar{T}) \dot{+} \mathcal{R}(T^H)$ and $Y = \mathcal{R}(\bar{T}) \dot{+} \mathcal{N}(T^H)$. Then*

(1) *$I_X + T^H \delta T \colon X \to X$ is a invertible bounded linear operator;*
(2) *$I_Y + \delta T T^H \colon Y \to Y$ is a invertible quasi-linear operator;*
(3) *$\Upsilon = T^H (I_Y + \delta T T^H)^{-1} = (I_X + T^H \delta T)^{-1} T^H$ is a bounded homogeneous operator.*

Proof. Since $I_X + T^H \delta T \in B(X,X)$, we only need to show that $\mathcal{N}(I_X + T^H \delta T) = \{0\}$ and $\mathcal{R}(I_X + T^H \delta T) = X$ under the assumptions.

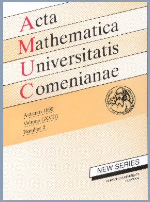
We first show that $\mathcal{N}(I_X + T^H \delta T) = \{0\}$. Let $x \in \mathcal{N}(I_X + T^H \delta T)$, then

$$(I_X + T^H \delta T)x = (I_X - T^H T)x + T^H \bar{T} x = 0$$

since T^H is quasi-linear. Thus $(I_X - T^H T)x = 0 = T^H \bar{T} x$, and hence $\bar{T} x \in \mathcal{R}(\bar{T}) \cap \mathcal{N}(T^H)$. Noting that $Y = \mathcal{R}(\bar{T}) \dot{+} \mathcal{N}(T^H)$, we have $\bar{T} x = 0$, and hence $x \in \mathcal{R}(T^H) \cap \mathcal{N}(\bar{T})$. From $X = \mathcal{N}(\bar{T}) \dot{+} \mathcal{R}(T^H)$, we get that $x = 0$.

Now, we prove that $\mathcal{R}(I_X + T^H \delta T) = X$. Let $x \in X$ and put $x_1 = (I_X - T^H T)x$, $x_2 = T^H T x$. Since $Y = \mathcal{R}(\bar{T}) \dot{+} \mathcal{N}(T^H)$, we have $\mathcal{R}(T^H) = T^H \mathcal{R}(\bar{T})$. Therefore, from $X = \mathcal{N}(\bar{T}) \dot{+} \mathcal{R}(T^H)$, we get that $\mathcal{R}(T^H) = T^H \mathcal{R}(\bar{T}) = T^H \bar{T} \mathcal{R}(T^H)$. Consequently, there is $z \in Y$ such that $T^H(T x_2 - \bar{T} x_1) = T^H \bar{T} T^H z$. Set $y = x_1 + T^H z \in X$. Noting that T^H is quasi-additive on $\mathcal{R}(T)$ and $\mathcal{R}(\delta T)$, respectively. we have

$$\begin{aligned}
(I_X + T^H \delta T)y &= (I_X - T^H T + T^H \bar{T})(x_1 + T^H z) \\
&= x_1 + T^H \bar{T} x_1 + T^H \bar{T} T^H z \\
&= x_1 + T^H \bar{T} x_1 + T^H (T x_2 - \bar{T} x_1) \\
&= x.
\end{aligned}$$



Therefore, $X = \mathcal{R}(I_X + T^H\delta T)$.

As in Lemma 3.2, we have $\Upsilon = T^H(I_Y + \delta T T^H)^{-1} = (I_X + T^H\delta T)^{-1}T^H$ is a bounded homogeneous operator. $\square$

Theorem 4.4. *Let $T \in B(X,Y)$ such that T^H exists and let $\delta T \in B(X,Y)$ such that T^H is quasi-additive on $\mathcal{R}(\delta T)$. Put $\bar{T} = T + \delta T$. Then the following statements are equivalent:*

1. *$I_X + T^H\delta T$ is invertible in $B(X,X)$ and $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^H) = \{0\}$;*
2. *$I_X + T^H\delta T$ is invertible in $B(X,X)$ and $\Upsilon = T^H(I_Y + \delta T T^H)^{-1} = (I_X + T^H\delta T)^{-1}T^H$ is a quasi-linear projector generalized inverse of $\bar{T}$;*
3. *$X = \mathcal{N}(\bar{T}) \dotplus \mathcal{R}(T^H)$ and $Y = \mathcal{R}(\bar{T}) \dotplus \mathcal{N}(T^H)$, i.e., $\mathcal{N}(\bar{T})$ is topological complemented in X and $\mathcal{R}(\bar{T})$ is quasi-linearly complemented in Y.*

Proof. (1) $\Rightarrow$ (2) By Theorem 3.4, $\Upsilon = T^H(I_Y + \delta T T^H)^{-1} = (I_X + T^H\delta T)^{-1}T^H$ is a bounded homogeneous generalized inverse of T. Let $y \in Y$ and $z \in \mathcal{R}(\bar{T})$. Then $z = Tx + \delta T x$ for some $x \in X$. Since T^H is quasi-additive on $\mathcal{R}(T)$ and $\mathcal{R}(\delta T)$, it follows that

$$T^H(y+z) = T^H(y + Tx + \delta T x) = T^H(y) + T^H(Tx) + T^H(\delta T x) = T^H y + T^H z,$$

i.e., T^H is quasi-additive on $\mathcal{R}(\bar{T})$, and hence Υ is quasi-linear. Set

$$\bar{P} = (I_X + T^H\delta T)^{-1}(I_X - T^H T), \qquad \bar{Q} = \bar{T}(I_X + T^H\delta T)^{-1}T^H.$$

Then, by the proof of Lemma 3.3, $\bar{P} \in H(X,X)$ is a projector with $\mathcal{R}(\bar{P}) = \mathcal{N}(\bar{T})$. Note that $(I_X + T^H\delta T)^{-1}$ and $I_X - T^H T$ are all linear. So $\bar{P}$ is linear. Furthermore,

$$\begin{aligned}\Upsilon\bar{T} &= (I_X + T^H\delta T)^{-1}T^H(T + \delta T)\\ &= (I_X + T^H\delta T)^{-1}(I_X + T^H\delta T + T^H T - I_X)\\ &= I_X - \bar{P}.\end{aligned}$$

Since T^H is quasi-additive on $\mathcal{R}(\bar{T})$, it follows that $\bar{Q} = \bar{T}(I + T^H\delta T)^{-1}T^H = \bar{T}\Upsilon$ is quasi-linear and bounded with $\mathcal{R}(\bar{Q}) \subset \mathcal{R}(\bar{T})$. Note that

$$\begin{aligned}\bar{Q} &= \bar{T}T^H(I_Y + \delta TT^H)^{-1} = (I_Y + \delta TT^H + TT^H - I_Y)(I_Y + \delta TT^H)^{-1}\\ &= I_Y - (I_Y - TT^H)(I_Y + \delta TT^H)^{-1}.\end{aligned}$$

According to Lemma 3.3, $(I_Y + \delta TT^H)^{-1}\mathcal{R}(\bar{T}) = \mathcal{R}(T)$, so we have $\mathcal{R}(\bar{T}) = \bar{Q}(\mathcal{R}(\bar{T})) \subset \mathcal{R}(\bar{Q})$. Thus, $\mathcal{R}(\bar{Q}) = \mathcal{R}(\bar{T})$. From $\Upsilon\bar{T} = I_X - \bar{P}$ and $\mathcal{R}(\bar{P}) = \mathcal{N}(\bar{T})$, we see that $\Upsilon\bar{T}\Upsilon = \Upsilon$. Then we have

$$\bar{Q}^2 = \bar{T}(I_X + T^H\delta T)^{-1}T^H\bar{T}(I_X + T^H\delta T)^{-1}T^H = \bar{T}\Upsilon\bar{T}\Upsilon = \bar{Q}.$$

Therefore, by Definition 4.2, we get $\bar{T}^H = \Upsilon$.

(2) $\Rightarrow$ (3) From $\bar{T}^H = T^H(I_Y + \delta TT^h)^{-1} = (I_X + T^H\delta T)^{-1}T^H$, we obtain that $\mathcal{R}(\bar{T}^H) = \mathcal{R}(T^H)$ and $\mathcal{N}(\bar{T}^H) = \mathcal{N}(T^H)$. From $\bar{T}\bar{T}^H\bar{T} = \bar{T}$ and $\bar{T}^H\bar{T}\bar{T}^H = \bar{T}^H$, we get that

$$\begin{aligned}&\mathcal{R}(I_X - \bar{T}^H\bar{T}) = \mathcal{N}(\bar{T}), \qquad \mathcal{R}(\bar{T}^H\bar{T}) = \mathcal{R}(\bar{T}^H),\\ &\mathcal{R}(\bar{T}\bar{T}^H) = \mathcal{R}(\bar{T}), \qquad \mathcal{R}(I_Y - \bar{T}\bar{T}^H) = \mathcal{N}(\bar{T}^H).\end{aligned}$$

Thus $\mathcal{R}(\bar{T}^H) = \mathcal{R}(T^H)$ and $\mathcal{R}(I_Y - \bar{T}\bar{T}^H) = \mathcal{N}(T^H)$. Therefore,

$$\begin{aligned}X &= \mathcal{R}(I_X - \bar{T}^H\bar{T}) \dotplus \mathcal{R}(\bar{T}^H\bar{T}) = \mathcal{N}(\bar{T}) \dotplus \mathcal{R}(T^H),\\ Y &= \mathcal{R}(\bar{T}\bar{T}^H) \dotplus \mathcal{R}(I_Y - \bar{T}\bar{T}^H) = \mathcal{R}(\bar{T}) \dotplus \mathcal{N}(T^H).\end{aligned}$$

(3) $\Rightarrow$ (1) By Lemma 4.3, $I_X + T^H\delta T$ is invertible in $H(X, X)$. Now from $Y = \mathcal{R}(\bar{T}) \dotplus \mathcal{N}(T^H)$, we get $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^H) = \{0\}$. □

Lemma 4.5 ([2]). *Let $A \in B(X, X)$. Suppose that there exist two constants $\lambda_1, \lambda_2 \in [0, 1)$ such that*

$$\|Ax\| \le \lambda_1\|x\| + \lambda_2\|(I_X + A)x\| \qquad \textit{for all } x \in X.$$

Then $I_X + A\colon X \to X$ is bijective. Moreover, for any $x \in X$,

$$\frac{1-\lambda_1}{1+\lambda_2}\|x\| \le \|(I_X + A)x\| \le \frac{1+\lambda_1}{1-\lambda_2}\|x\|,$$
$$\frac{1-\lambda_2}{1+\lambda_1}\|x\| \le \|(I_X + A)^{-1}x\| \le \frac{1+\lambda_2}{1-\lambda_1}\|x\|.$$

Let $T \in B(X,Y)$ such that T^H exists. Let $\delta T \in B(X,Y)$ such that T^H is quasi-additive on $\mathcal{R}(\delta T)$ and satisfies

$$\|T^H \delta T x\| \le \lambda_1 \|x\| + \lambda_2 \|(I_X + T^H \delta T)x\| \quad \text{for all} \quad x \in X, \tag{4.1}$$

where $\lambda_1, \lambda_2 \in [0,1)$.

Corollary 4.6. *Let $T \in B(X,Y)$ such that T^H exists. Suppose that $\delta T \in B(X,Y)$ such that T^H is quasi-additive on $\mathcal{R}(\delta T)$ and satisfies (4.1). Put $\bar{T} = T + \delta T$. Then $I_X + T^H \delta T$ is invertible in $H(X,X)$ and $\bar{T}^H = (I_X + T^H \delta T)^{-1} T^H$ is well-defined with*

$$\frac{\|\bar{T}^H - T^H\|}{\|T^H\|} \le \frac{(2+\lambda_1)(1+\lambda_2)}{(1-\lambda_1)(1-\lambda_2)}.$$

Proof. By using Lemma 4.5, we get that $I_X + T^H \delta T$ is invertible in $H(X,X)$ and

$$\|(I_X + T^H \delta T)^{-1}\| \le \frac{1+\lambda_2}{1-\lambda_1}, \qquad \|I_X + T^H \delta T\| \le \frac{1+\lambda_1}{1-\lambda_2}. \tag{4.2}$$

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From Theorem 4.4, we see $\bar{T}^H = T^H(I_Y + \delta T T^H)^{-1} = (I_X + T^H\delta T)^{-1}T^H$ is well-defined. Now we can compute

$$
\begin{aligned}
\frac{\|\bar{T}^H - T^H\|}{\|T^H\|} &\leq \frac{\|(I_X + T^H\delta T)^{-1}T^H - T^H\|}{\|T^H\|}\\
&\leq \frac{\|(I_X + T^H\delta T)^{-1}[I_X - (I_X + T^H\delta T)]T^H\|}{\|T^H\|}\\
&\leq \|(I_X + T^H\delta T)^{-1}\|\|T^H\delta T\|.
\end{aligned}
\tag{4.3}
$$

Since $\lambda_2 \in [0,1)$, then from the second inequality in (4.2), we get that $\|T^H\delta T\| \leq \dfrac{2+\lambda_1}{1-\lambda_2}$. Now, by using (4.3) and (4.2), we can obtain

$$
\frac{\|\bar{T}^H - T^H\|}{\|T^H\|} \leq \frac{(2+\lambda_1)(1+\lambda_2)}{(1-\lambda_1)(1-\lambda_2)}.
$$

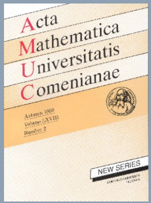
This completes the proof. □

Corollary 4.7. *Let $T \in B(X,Y)$ with $\mathcal{R}(T)$ closed. Assume that $\mathcal{R}(T)$ and $\mathcal{N}(T)$ are Chebyshev subspaces in Y and X, respectively. Let $\delta T \in B(X,Y)$ such that $\mathcal{R}(\delta T) \subset \mathcal{R}(T)$, $\mathcal{N}(T) \subset \mathcal{N}(\delta T)$ and $\|T^M\delta T\| < 1$. Put $\bar{T} = T + \delta T$. If T^M is quasi-additive on $\mathcal{R}(T)$, then $\bar{T}^M = T^M(I_Y + \delta T T^M)^{-1} = (I_X + T^M\delta T)^{-1}T^M$ with*

$$
\frac{\|\bar{T}^M - T^M\|}{\|T^M\|} \leq \frac{\|T^M\delta T\|}{1 - \|T^M\delta T\|}.
$$

Proof. From $\mathcal{R}(\delta T) \subset \mathcal{R}(T)$ and $\mathcal{N}(T) \subset \mathcal{N}(\delta T)$, we get that $\pi_{\mathcal{R}(T)}\delta T = \delta T$ and $\delta T\pi_{\mathcal{N}(T)} = 0$, that is, $TT^M\delta T = \delta T = \delta T T^M T$. Consequently,

$$
\bar{T} = T + \delta T = T(I_X + T^M\delta T) = (I_Y + \delta T T^M)T \tag{4.4}
$$



Since T^M is quasi-additive on $\mathcal{R}(T)$ and $\|T^M\delta T\| < 1$, we get that $I_X + T^M\delta T$ and $I_Y + \delta T T^M$ are all invertible in $H(X, X)$. So from (**??**), we have $\mathcal{R}(\bar{T}) = \mathcal{R}(T)$ and $\mathcal{N}(\bar{T}) = \mathcal{N}(T)$, and hence $\bar{T}^H = T^M(I_Y + \delta T T^M)^{-1} = (I_X + T^M\delta T)^{-1}T^M$ by Theorem 4.4. Finally, by Corollary 2.6,

$$\begin{aligned}\bar{T}^M &= (I_X - \pi_{\mathcal{N}(\bar{T})})\bar{T}^H\pi_{\mathcal{R}(\bar{T})} = (I_X - \pi_{\mathcal{N}(T)})T^M(I_Y + \delta T T^M)^{-1}\pi_{\mathcal{R}(T)}\\ &= (I_X + T^M\delta T)^{-1}T^M\pi_{\mathcal{R}(T)} = (I_X + T^M\delta T)^{-1}T^M = T^M(I_Y + \delta T T^M)^{-1}\end{aligned}$$

and then

$$\|\bar{T}^M - T^M\| \le \|(I_X - T^M\delta T)^{-1} - I_X\|\|T^M\| \le \frac{\|T^M\delta T\|\|T^M\|}{1 - \|T^M\delta T\|}.$$

The proof is completed. □

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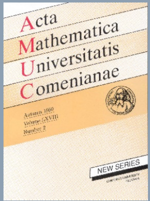

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