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ON OSCILLATION OF LIMIT FUNCTIONS

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In the paper [**1**] the set of all continuity points of the limit function of a functional sequence by using the oscillation of these functions is investigated. In the present paper we investigate the oscillation of a limit function.

Let X be a topological space and let (Y, d) be a metric space. Let $f\colon X \to Y$ be a function. The function $\omega_f\colon X \to \mathbb{R} \cup \{\infty\}$ ($\mathbb{R}$ is the set of all real numbers), given by the formula $\omega_f(x) = \inf\{d(f(U))\colon U$ is neighbourhood of $x\}$ (where $d(A) = \sup\{d(x,y)\colon x, y \in A\}$) is said to be the oscillation of the function f. It is well-known that f is continuous at x if and only if $\omega_f(x) = 0$ ([**2**]). The symbol $C(f)$ denotes the set of all continuity points of f, the letters $\mathbb{N}$ and $\mathbb{Q}$ stand for the set of all natural and rational numbers, respectively.

Let $f_n, f\colon X \to Y$ $(n = 1, 2, \dots)$ be functions. It is easy to see that if the sequence (f_n) uniformly converges to f on an open set G, then for each $x \in G$ we have

$$\lim_{n\to\infty} \omega_{f_n}(x) = \omega_f(x). \tag{1}$$

Therefore, if X is a locally compact topological space, then the uniform on compacta convergence implies (1) on X. In general the uniform on compacta convergence does not imply (1).

Example 1.1 in [**1**] shows that (1) is not true for pointwise or quasiuniform convergence. We recall that a sequence (f_n), $f_n\colon X \to Y$ quasiuniformly converges to $f\colon X \to Y$ (see [**2**]) if the sequence (f_n) pointwise converges to f and

$$\forall\, \varepsilon > 0\, \forall\, m \in \mathbb{N}\, \exists\, p \in \mathbb{N}\, \forall\, x \in X : \quad \min\{d(f_{m+1}(x), f(x)), \dots, d(f_{m+p}(x), f(x))\} < \varepsilon.$$

We shall show that for the quasiuniform convergence we have

$$\{x \in X\colon \limsup_{n\to\infty} \omega_{f_n}(x) = 0\} \subset C(f).$$

As corollary we obtain that the quasiuniform limit of continuous functions is continuous function.

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Theorem 1. *Let X be a topological space and let (Y, d) be a metric space. Let $f, f_n \colon X \to Y$ $(n = 1, 2, \dots)$ and let (f_n) converges quasiuniformly to f. Then for each $x \in X$ we have*

$$\omega_f(x) \le 2 \limsup_{n\to\infty} \omega_{f_n}(x). \tag{2}$$

Proof. Suppose that there is $x \in X$ such that $\omega_f(x) > 2 \limsup_{n\to\infty} \omega_f(x)$. Then there are α, β such that

$$2 \limsup_{n\to\infty} \omega_{f_n}(x) < \alpha < \beta < \omega_f(x). \tag{3}$$

Since $\limsup_{n\to\infty} \omega_{f_n}(x) < \frac{\alpha}{2}$, there is $n_1 \in \mathbb{N}$ such that for each $n \ge n_1$ we have $\omega_{f_n} < \frac{\alpha}{2}$. Then for each $n \ge n_1$ there is a neighbourhood U_n of x such that

$$d(f_n(U_n)) < \frac{\alpha}{2}. \tag{4}$$

Since (f_n) pointwise converges to f, there is $n_2 \in \mathbb{N}$ such that for $n \ge n_2$ we have

$$d(f_n(x), f(x)) < \frac{\beta - \alpha}{4}. \tag{5}$$

Denote $m = \max\{n_1, n_2\}$. Then there is $p \in \mathbb{N}$ such that for each $y \in X$ we have

$$\min\{d(f_{m+1}(y), f(y), \dots, d(f_{m+p}(y), f(y))\} < \frac{\beta - \alpha}{4}.$$

Denote $U = \bigcap_{i=1}^{p} U_{m+i}$. Then U is a neighbourhood of x. Let $z \in U$. Then there is $j \in \{1, 2, \dots, p\}$ such that

$$d(f_{m+j}(z), f(z)) < \frac{\beta - \alpha}{4}. \tag{6}$$

Then according to (4), (5) and (6) we obtain

$$\begin{aligned} d(f(z), f(x)) &\le d(f(z), f_{m+j}(z)) + d(f_{m+j}(z), f_{m+j}(x)) + d(f_{m+j}(x), f(x)) \\ &< \frac{\beta - \alpha}{4} + \frac{\alpha}{2} + \frac{\beta - \alpha}{4} = \frac{\beta}{2}. \end{aligned}$$

Therefore, for $s, t \in U$ we have

$$d(f(s), f(t)) \le d(f(s), f(x)) + d(f(x), f(t)) < \frac{\beta}{2} + \frac{\beta}{2} = \beta.$$

From this we get $\omega_f(x) < \beta$, which contradicts to [**3**]. □

Evidently, [**2**] is not true for the pointwise convergence. However, we have

Theorem 2. *Let X be a Baire space and let (Y,d) be a separable metric space. Let $f_n, f\colon X \to Y$ $(n = 1, 2, \dots)$ be functions and let (f_n) converge pointwise to f. Let the function f be locally bounded and let $M > 2$. Then $\{x \in X\colon \omega_f(x) \leq M \cdot \liminf\limits_{n\to\infty} \omega_f(x)\}$ is dense in X.*

If (Y,d) is an arbitrary metric space then $\{x \in X\colon \omega_f(x) \leq M \cdot \liminf\limits_{n\to\infty} \omega_f(x)\}$ is dense in X for each $M > 3$.

First we shall prove the following

Lemma 1. *Let X be a Baire space and let (Y,d) be a separable metric space ((Y,d) be arbitrary metric space). Let G be an open set in X, let $M > 2$ $(M > 3)$ and let $0 < S < \infty$. Let $f_n, f\colon X \to Y$ $(n = 1, 2 \dots)$ be functions and let $\lim\limits_{n\to\infty} f_n = f$. Let $\liminf\limits_{n\to\infty} \omega_{f_n}(x) \leq S$ for each $x \in G$. Then $\{x \in G\colon \omega_f(x) \geq M \cdot S\}$ is a nowhere dense set in X.*

Proof. Denote $A = \{x \in G\colon \omega_f(x) \geq M \cdot S\}$. Suppose that A is not nowhere dense in X. The there is a nonempty open set $H \subset G$ such that A is dense in H. We shall show that $H \subset A$. Let $z \in H - A$. Then $\omega_f(z) < MS$. Since ω_f is upper semi-continuous ([**2**]), there is a neighbourhood U of z such that $\omega_f(x) < MS$ for each $x \in U$, a contradiction with the density of A. Hence

$$\forall\, x \in H\colon \omega_f(X) \geq M \cdot S. \tag{7}$$

I. (Y,d) is a separable metric space and $M > 2$.

Let $\{u_1, u_2, u_3, \dots\}$ be a countable dense subset of Y. Then $Y = \bigcup\limits_{n=1}^{\infty} S(u_n, \frac{S}{24}(M-2))$ (where $S(u,\varepsilon)$ is the open sphere of radius $\varepsilon > 0$ about u). Since X is a Baire space, there is $j \in \mathbb{N}$ such that the set $H \cap f^{-1}(S(u_j, \frac{S}{24}(M-2)))$ is not of the first category. Denote

$$B = H \cap f^{-1}\left(S(u_j, \frac{S}{24}(M-2))\right),$$

$$D = H \cap f^{-1}\left(S(u_j, \frac{S}{4}(M+2))\right),$$

$$A_k = \left\{x \in H\colon \forall\, n \geq k;\ d(f_n(x), f(x)) < \frac{S}{24}(M-2)\right\}$$

for $k \in \mathbb{N}$.

Then evidently $B \subset D$, $A_k \subset A_{k+1}$ for each $k \in \mathbb{N}$ and $H = \bigcup\limits_{k=1}^{\infty} A_k$. Then there is $i \in \mathbb{N}$ such that $B \cap A_i$ is not nowhere dense. Therefore there is a nonempty open set $J \subset H$ such that $B \cap A_i$ is dense in J. Then $B \cap A_n$ is dense in J for each $n \geq i$.

Now we shall prove that $J-D$ is a nonempty set. If namely $J-D=\emptyset$, then $J\subset D$ and hence $f(J)\subset S\big(u_j,\frac{S}{4}(M+2)\big)$. From this we have $d(f(J))\le\frac{S}{2}(M+2)$ and $\omega_f(x)\le\frac{S}{2}(M+2)<MS$ for each $x\in J$, a contradiction with (7).

Let $z\in J-D$. Let $p\in\mathbb{N}$ be such that $z\in A_p$. Since $\liminf\limits_{n\to\infty}\omega_{f_n}(z)\le S<\frac{S}{8}(M+6)$, there is $m\ge\max\{i,p\}$ such that

$$\omega_{f_m}(z)<\frac{S}{8}(M+6). \tag{8}$$

Let U be arbitrary neighbourhood of z. Then there is $v\in B\cap A_m\cap U\cap J$. We have

$$\begin{aligned}\frac{S}{4}(M+2)&\le d(f(z),u_j)\\&\le d(f(z),f_m(z))+d(f_m(z),f_m(v))+d(f_m(v),f(v))+d(f(v),u_j)\\&<\frac{S}{24}(M-2)+d(f_m(z),f_m(v))+\frac{S}{24}(M-2)+\frac{S}{24}(M-2).\end{aligned}$$

From this we get $d(f_m(z),f_m(v))>\frac{S}{8}(M+6)$ and hence $d(f_m(U))>\frac{S}{8}(M+6)$. Since this is true for each neighbourhood of z, we have $\omega_{f_m}(z)\ge\frac{S}{8}(M+6)$, a contradiction with (8).

II. (Y,d) is arbitrary metric space and $M>3$.
Denote $A_k=\{x\in H\colon\forall n\ge k;\ d(f_n(x),f(x))<\frac{S}{24}(M-3)\}$ for $k\in\mathbb{N}$. Then there is $i\in\mathbb{N}$ such that A_i is not nowhere dense. Therefore there is a nonempty open set $J\subset H$ such that A_i is dense in J.

Let $x\in J$. Let $p\in\mathbb{N}$ be such that $x\in A_p$. Since $\liminf\limits_{n\to\infty}\omega_{f_n}(x)\le S<\frac{S}{12}(M+9)$, there is $m\ge\max\{i,p\}$ such that $\omega_{f_m}(x)<\frac{S}{12}(M+9)$. Therefore there is a neighbourhood U_x of x such that $d(f_m(U_x))<\frac{S}{12}(M+9)$.

Let $y\in U_x\cap A_i$. Then

$$\begin{aligned}d(f(y),f(x))&\le d(f(y),f_m(y))+d(f_m(y),f_m(x))+d(f_m(x),f(x))\\&<\frac{S}{24}(M-3)+\frac{S}{12}(M+9)+\frac{S}{24}(M-3)=\frac{S}{6}(M+3).\end{aligned}\tag{9}$$

Let $y,z\subset U_x\cap A_i$. Then similarly

$$d(f(y),f(z))<\frac{S}{6}(M+3). \tag{10}$$

Now let $r\in J$. Let $u,v\in U_r\cap J$. Then there are $s\in U_r\cap U_u\cap A_i$ and $t\in U_r\cap U_v\cap A_i$. According to (9) and (10) we have

$$d(f(u),f(v))\le d(f(u),f(s))+d(f(s),f(t))+d(f(t),f(v))<\frac{S}{2}(M+3).$$

Therefore $d(f(U_r \cap J)) \leq \frac{S}{2}(M+3)$ and hence $\omega_f(r) < M \cdot S$, a contradiction with (7). $\square$

Proof of Theorem 2. Since f is locally bounded, we have $\omega_f(x) < \infty$ for each $x \in X$. Suppose that this theorem is not true. Then there is an open nonempty set G in X such that

$$\forall\, x \in G\colon \infty > \omega_f(x) > M \cdot \liminf_{n\to\infty} \omega_{f_n}(x). \tag{11}$$

Define functions $h, g\colon G \to \mathbb{R}$ as

$$h(x) = \liminf_{n\to\infty} \omega_{f_n}(x),$$
$$g(x) = \inf\{\sup\{h(y) : y \in U\}\colon U \text{ is a neighbourhood of } x\}.$$

Then g is a nonnegative upper semi-continuous function. Let K be such that $M > K > 2$ ($M > K > 3$ if (Y,d) is not separable). Then we have $h(x) < \frac{1}{M}\omega_f(x) < \frac{1}{K}\omega_f(x)$ for each $x \in G$. We observe that

$$g(x) \leq \frac{1}{M}\omega_f(x) < \frac{1}{K}\omega_f(x).$$

Since X is a Baire space, so there is $z \in G \cap C(\omega_f)$. Let α be such that $g(z) < \alpha < \frac{1}{K}\omega_f(z)$. Then there is an open neighbourhood U of z such that

$$h(x) \leq g(x) < \alpha < \frac{1}{K}\omega_f(x) \quad \text{for each } x \in U. \tag{12}$$

Hence according to Lemma 1 the set $\{x \in U\colon \omega_f(x) \geq K\alpha\}$ is nowhere dense, a contradiction with (12). $\square$

The following example shows that the assumption "f is locally bounded" cannot be omitted.

Example 1. Let $X = Y = \mathbb{R}$, let $\mathbb{Q} = \{q_1, q_2, q_3, \dots\}$ (one-to-one sequence). Define functions $f_n, f\colon \mathbb{R} \to \mathbb{R}$ as

$$f_n(x) = \begin{cases} k, & \text{if } x = q_k \text{ and } k \leq n, \\ 0, & \text{otherwise;} \end{cases}$$

$$f(x) = \begin{cases} k, & \text{if } x = q_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\lim_{n\to\infty} f_n = f$, $\omega_f(x) = \infty$ for each $x \in X$ and $\liminf_{n\to\infty} \omega_{f_n}(x) = f(x)$ for each $x \in X$.

Next example shows that Theorem 2 (and Lemma 1) does not hold for $M = 2$.

Example 2. Let $X = Y = \mathbb{R}$, $\mathbb{Q} = \{q_1, q_2, \dots\}$ (one-to-one sequence) and $\mathbb{Q} = A \cup B$, where A and B are dense disjoint sets. Define $f_n, f\colon \mathbb{R} \to \mathbb{R}$ as

$$f_n(x) = \begin{cases} 1 - \frac{1}{k}, & \text{if } x = q_k,\ x \in A \text{ and } k \le n, \\ \frac{1}{k} - 1, & \text{if } x = q_k,\ x \in B \text{ and } k \le n, \\ 0, & \text{otherwise;} \end{cases}$$

$$f(x) = \begin{cases} 1 - \frac{1}{k}, & \text{if } x = q_k \text{ and } x \in A, \\ \frac{1}{k} - 1, & \text{if } x = q_k \text{ and } x \in B, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\lim\limits_{n\to\infty} f_n = f$, $\omega_f(x) = 2$ for each $x \in X$ and

$$\liminf_{n\to\infty} \omega_{f_n}(x) = \begin{cases} 1 - \frac{1}{k}, & \text{if } x = q_k, \\ 0, & \text{otherwise.} \end{cases}$$

This example shows also that the set $\{x \in X : \omega_f(x) \le M \cdot \liminf\limits_{n\to\infty} \omega_{f_n}(x)\}$ (where $M > 2$) may be not residual.

Since every function $f\colon \mathbb{Q} \to \mathbb{R}$ is in the first Baire class the assumption on X to be a Baire space (in Theorem 2) cannot be omitted. Next example shows that Theorem 2 does not hold for arbitrary metric space (Y, d) with $M > 2$.

Example 3. Let $\{B_n\colon n \in \mathbb{N}\}$ be a countable base in $\mathbb{R}$. We choose two different points $a_1, b_1 \in B_1$ and for $n > 1$ we choose two different points $a_n, b_n \in B_n - \{a_1, b_1, \dots, a_{n-1}, b_{n-1}\}$. Denote $P = \mathbb{R} - \{a_1, b_1, \dots, a_n, b_n, \dots\}$.

Let $X = \mathbb{R}$ with the usual topology and let $Y = \mathbb{R}$ with the following metric d:

$$d(y,x) = d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x = a_n,\ y \in P \text{ and } |y - a_n| \le |y - b_n|, \\ & \text{or } x = b_n,\ y \in P \text{ and } |y - b_n| < |y - a_n|, \\ & \text{or } x, y \in P,\ x \ne y, \\ 3, & \text{if } x = a_n,\ y = b_n, \\ 2, & \text{otherwise.} \end{cases}$$

Further denote for $n \in \mathbb{N}$

$$U_n = (a_n - \frac{|a_n - b_n|}{2}, a_n + \frac{|a_n - b_n|}{2}),$$
$$V_n = (b_n - \frac{|a_n - b_n|}{2}, b_n + \frac{|a_n - b_n|}{2}),$$

and for $j > k$ denote

$$D_k^j = \begin{cases} U_k, & \text{if } a_j \in U_k, \\ V_k, & \text{if } a_j \in V_k, \\ \mathbb{R}, & \text{if } a_j \notin U_k \cup V_k; \end{cases}$$

$$C_k^j = \begin{cases} U_k, & \text{if } b_j \in U_k, \\ V_k, & \text{if } b_j \in V_k, \\ \mathbb{R}, & \text{if } b_j \notin U_k \cup V_k. \end{cases}$$

Now for $j > n$ choose $p_n^j \in P \cap D_1^j \cap D_2^j \cap \cdots \cap D_n^j$ and $q_n^j \in P \cap C_1^j \cap C_2^j \cap \cdots \cap C_n^j$.

Define $f_n, f\colon X \to Y$ as

$$f_n(x) = \begin{cases} x, & \text{if } x \in P \cup \{a_1, b_1, \dots, a_n, b_n\}, \\ p_n^j, & \text{if } x = a_j \text{ and } j > n, \\ q_n^j, & \text{if } x = b_j \text{ and } j > n; \end{cases}$$

$$f(x) = x \quad \text{for each } x \in X.$$

Then for each $x \in X$ we have $\lim_{n\to\infty} f_n(x) = f(x)$, $\omega_f(x) = 3$ and $\liminf_{n\to\infty} \omega_{f_n}(x) = 1$.

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